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H^2-SOLUTIONS FOR SOME ELLIPTIC EQUATIONS WITH NONLINEAR BOUNDARY CONDITIONS

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Abstract. The following elliptic equation with nonlinear boundary condition is considered: $-\Delta u + bu = f(x)$ in Ω, $-\frac{\partial u}{\partial n} = \beta(u) - g(u)$ on $\partial\Omega$, where $b \geq 0$, $f \in L^2(\Omega)$, $\beta(u)$ is a monotone increasing function on $\mathbb{R}^1$ and $g(u)$ is its small perturbation. It is shown that this problem admits a solution u belonging to $H^2(\Omega)$ under suitable conditions on β and g. The method of our proof relies on some approximation procedures and the classical but new arguments for H^2-estimates near the boundary which can work under (non-monotone) nonlinear boundary conditions.

1. **Introduction.** Consider the following heat equation with nonlinear boundary condition:

$$\begin{cases} u_t = \Delta u & \text{in } \Omega \times (0,T), \\ -\dfrac{\partial u}{\partial n} = \beta(u) & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega. \end{cases}$$

The typical example of this kind of nonlinear flux condition on the boundary is derived from the so-called Stefan-Boltzmann's radiation law, which says that the heat energy radiation from the surface of the body J is given by $J = \sigma(T^4 - T_s^4)$, where $\sigma > 0$ is a physical constant, T is the surface temperature and T_s is the outside temperature. Thus Stefan-Boltzmann's law gives an example, where $\beta(u)$ is a monotone increasing function. In this case, the unique solvability for this parabolic equation is completely covered by the abstract theory by H.Brézis [1].
However, if we consider the case where the heat flux radiated from the surface is reflected by its surrounding materials, then we must consider also the absorption effect. For such a case, $\beta(u)$ could not be a monotone increasing function, but monotone increasing with small perturbation. In fact, such a kind of non-monotone radiation-absorption model are already proposed from the view point of engineering (see e.g. [3]). In this paper, we are concerned with such a non-monotone radiation-absorption model. Here we consider the following elliptic equation with the following

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nonlinear boundary condition:

$$\begin{cases} -\Delta u + bu = f(x) & \text{in } \Omega, \\ -\dfrac{\partial u}{\partial n} = \beta(u) - g(u) & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $b \geq 0$ and $\Omega \subset \mathbb{R}^N$ is a bounded open set with smooth boundary $\partial\Omega$.

2. **Assumptions and main result.** We impose the following condition on β and g.

(β1) $\beta(u)$ is a continuous and monotone increasing function defined on $\mathbb{R}^1$.

(β2) $\lim_{|u|\to\infty} \beta(u)/u = \infty$.

(g1) g is a locally Lipschitz continuous function on $\mathbb{R}^1$, and there exist $k \in (0,1)$, $C_1 > 0$ such that

$$|g'(u)| \leq k\beta'(u) + C_1 \qquad \text{a.e. } u \in \mathbb{R}. \tag{2}$$

Here (g1) implies that $g(u)$ can be regarded as a small perturbation for the leading term $\beta(u)$, which will play a crucial role in the later arguments. To state our main theorem, we set

$$D(j) = \left\{ u \in H^1(\Omega) : \int_{\partial\Omega} j(u) dS < \infty \right\}, \qquad j(u) = \int_0^u \beta(s)\, ds.$$

Theorem 2.1. *Assume* (β1), (β2) *and* (g1). *Then there exists* $C > 0$ *such that for any* $f \in L^2(\Omega)$, *there exists at least a solution* $u \in H^2(\Omega) \cap D(j)$ *of* (1) *satisfying*

$$\|u\|^2_{H^2(\Omega)} + \|j(u)\|_{L^1(\partial\Omega)} \leq C(1 + \|f\|^2_{L^2(\Omega)}). \tag{3}$$

When β is a (possibly multi-valued) maximal monotone operator, the following result similar to Theorem 2.1 holds true.

Definition 2.2. Let $j : \mathbb{R} \to (-\infty, \infty]$ be a lower semi-continuous proper convex function. The subdifferential operator ∂j is defined by

$$\partial j(u) = \{\xi \in \mathbb{R}; j(v) - j(u) \geq \xi(v-u)\ \forall v \in \mathbb{R}\}.$$

It is well known that the subdifferential operator becomes maximal monotone.

Corollary 1. *Let* $j : \mathbb{R} \to (-\infty, \infty]$ *be a lower semi-continuous proper convex function and* $\beta = \partial j$. *Assume that* $0 \in \beta(0)$, $b > 0$ *and*

(g0)$'$ *g is a monotone increasing function with* $g(0) = 0$.

(g1)$'$ *g is locally Lipschitz continuous and there exist* $\delta \in (0,1)$, $s_0 > 0$ *such that*

$$\Big(v_1 - g(u_1) - \big(v_2 - g(u_2)\big)\Big)(u_1 - u_2) \geq \delta(v_1 - v_2)(u_1 - u_2) \tag{4}$$

for any $|u_1|, |u_2| \geq s_0$ *and* $v_i \in \beta(u_i)$ $(i = 1, 2)$.

Then the assertion of Theorem 2.1 hold true.

Remark 1. When $g(u) \equiv 0$, it is well known that for any $f \in L^2(\Omega)$ there exists a unique solution $u \in H^2(\Omega)$ of (1) (see e.g. H. Brézis [1]).
However for our case, the uniqueness does not hold in general. In fact, if we take $\beta(u) = |u|^{q-2}u$ $(q > 2)$, $g(u) = \alpha u$, $f \equiv 0$ and $\alpha > 0$ large enough, then the uniqueness does not hold. To show this, it suffices to see that the associated functional attains the strictly negative global minimum on $H^1(\Omega)$ for sufficiently large $\alpha > 0$, which assures the existence of nontrivial solution of (1).

Remark 2. Theorem 2.1 assures only the existence of solution satisfying the elliptic estimates (3), but does not give any information about elliptic estimates for any given weak solutions of (1). However if we impose the additional condition:

(β3) $\exists C_2 > 0$ such that $\quad u\beta(u) \leq C_2 j(u) \quad$ for all $u \in \mathbb{R}$,

then we can show that any weak solution $u \in H^1(\Omega) \cap D(j)$ of (1) should belong to $H^2(\Omega)$ and satisfy (3).

Remark 3. The following quasilinear elliptic problem can be treated within our framework with the assertion $u \in H^2$ replaced by $\nabla u \in W^{1,p'}$ $(p' = p/(p-1))$.

$$(\mathrm{L_p}) \begin{cases} -\Delta_p u + bu = f, & (f \in W^{1,p'}(\Omega)) \\ -|\nabla u|^{p-2} \frac{\partial u}{\partial n} = \beta(u) - g(u). \end{cases}$$

3. **Outline of proof. Step1: Approximation problem**

Our functional $I(u)$ associated with (1) is given by

$$I(u) = \int_\Omega \frac{1}{2}\left(|\nabla u|^2 + bu^2\right) dx + \int_{\partial\Omega} (j(u) - G(u))\, dS - \int_\Omega f(x) u \; dx,$$

where $j(u) = \int_0^u \beta(s) ds$, $G(u) = \int_0^u g(s) ds$. But this functional may not be defined on $H^1(\Omega)$ in general, since the term $j(u)$ and $G(u)$ may not be integrable for all $u \in H^1(\Omega)$. So, we introduce the following approximations $\beta_\ell(\cdot)$ and $g_\ell(\cdot)$ for $\beta(\cdot)$ and $g(\cdot)$ respectively.

$$\beta_\ell(u) = \begin{cases} \beta(\ell) + (u - \ell) & u > \ell, \\ \beta(u) & |u| \leq \ell, \\ \beta(-\ell) + (u + \ell) & u < -\ell, \end{cases} \qquad g_\ell(u) = \begin{cases} g(\ell) & u > \ell, \\ g(u) & |u| \leq \ell, \\ g(-\ell) & u < -\ell. \end{cases}$$

Then we consider following approximation problems.

$$\begin{cases} -\Delta u + bu = f & \text{in } \Omega, \\ -\dfrac{\partial u}{\partial n} = \beta_\ell(u) - g_\ell(u) & \text{on } \partial\Omega. \end{cases} \tag{5}$$

By the trace embedding theorem, the following functional I_ℓ for the approximation problem (5) is well defined on H^1.

$$I_\ell(u) = \int_\Omega \frac{1}{2}\left(|\nabla u|^2 + bu^2\right) dx + \int_{\partial\Omega} (j_\ell(u) - G_\ell(u))\, dS - \int_\Omega fu \; dx,$$

where $j_\ell(u) = \int_0^u \beta_\ell(s) ds$, $G_\ell(u) = \int_0^u g_\ell(s) ds$. By ($\beta$2), (g1) and the mean value theorem for the monotone function, we can verify that

$$j_\ell(u) - G_\ell(u) \geq \frac{1-k}{2} j_\ell(u) - C \geq \frac{1-k}{4} u^2 - C'. \tag{6}$$

By (6), we find that I_ℓ is bounded below and coercive on $H^1(\Omega)$. Furthermore, by using the compactness of the embedding $H^1(\Omega) \subset L^2(\partial\Omega)$, we can easily see that I_ℓ is lower semi-continuous in $H^1(\Omega)$. Hence, there exists a global minimizer u_ℓ of I_ℓ and this u_ℓ gives a solution of (5).

Step2: A priori estimates

Since u_ℓ is a global minimizer of I_ℓ,

$$\int_\Omega \frac{1}{2}\left(|\nabla u_\ell|^2 + bu_\ell^2\right) dx + \int_{\partial\Omega} (j_\ell(u_\ell) - G_\ell(u_\ell))\, dS - \int_\Omega f u_\ell dx = I_\ell(u_\ell) \leq I_\ell(0) = I(0).$$

Hence, by (6) we get

$$\|u_\ell\|^2_{H^1(\Omega)} + \|j_\ell(u_\ell)\|_{L^1(\partial\Omega)} \le C\left(1 + \|f\|^2_{L^2(\Omega)}\right). \tag{7}$$

To establish the following H^2-estimate is a key step of our proof.

Lemma 3.1. *There exists a positive constant C independent of ℓ such that*

$$\|u_\ell\|_{H^2(\Omega)} \le C\left(\|u_\ell\|_{H^1(\Omega)} + \|f\|_{L^2(\Omega)}\right). \tag{8}$$

Proof. Since the interior estimate is not affected by the nonlinear boundary conditions and can be derived from the standard argument, we give the estimates only near the boundary. By the standard way, we use a family of local charts covering $\partial\Omega$. Let $x_0 \in \partial\Omega$ and U is a neighborhood of x_0, and let $H : Q_+ \to \Omega \cap U$ be the transformation mapping with $Q_+ = \{y = (y', y_N); |y'| < 1,\ 0 < y_N < 1\}$ and $Q_0 = \{y = (y', 0); |y'| < 1\}$. We define $\tilde{u}_\ell = u_\ell \circ H$, $\tilde{f} = f \circ H$. In the new coordinate with variable y, $\tilde{u}_\ell(y) \in H^1(Q_+)$ satisfies

$$\sum_{i,j=1}^{N} \int_{Q_+} a_{ij}(y) \frac{\partial \tilde{u}_\ell}{\partial y_i} \frac{\partial \phi}{\partial y_j} J(y)dy + \int_{Q_+} b\tilde{u}_\ell \phi J(y)dy + \int_{Q_0} (\beta_\ell(\tilde{u}_\ell) - g_\ell(\tilde{u}_\ell))\phi\sigma(y')dy' = \int_{Q_+} \tilde{f}\phi J(y)dy, \tag{9}$$

for any $\phi \in \{\phi \in C^1(\overline{Q_+}); \operatorname{supp} \phi \subset Q_+ \cup Q_0\}$, where $J(y)$ is the absolute value of Jacobian, $\sigma(y')$ is the surface element, and $a_{ij}(y)$ is a coefficient satisfying the uniformly elliptic condition. We test (9) by the following function ϕ given by

$$\phi = D_{-h}(\theta^2 D_h \tilde{u}_\ell)\frac{1}{\sigma(y')},$$

where $D_h\tilde{u} = \frac{1}{|h|}(\tau_h\tilde{u} - \tilde{u})$, $\tau_h\tilde{u}(y) = \tilde{u}(y+h)$, h is a vector with last component $h_N = 0$ and θ is a smooth function composing the partition of unity.
Put $\tilde{v}_\ell = \theta\tilde{u}_\ell$. Since $a_{ij}(y)$ satisfies the uniformly elliptic condition, each term in (9) can be estimated by the same arguments as in [1].

$$\begin{aligned}
\text{(the first term)} &\ \ge\ a_0\|D_h\nabla\tilde{v}_\ell\|^2_{L^2} - C\|D_h\nabla\tilde{v}_\ell\|_{L^2}\|\tilde{u}_\ell\|_{H^1} - C\|\tilde{u}_\ell\|^2_{H^1},\\
\text{(the second term)} &\ \le\ C\|\tilde{u}_\ell\|^2_{H^1},\\
\text{(the fourth term)} &\ \le\ C\|\tilde{f}\|_{L^2}(\|D_h\nabla\tilde{v}_\ell\|_{L^2} + \|\tilde{u}_\ell\|_{H^1}).
\end{aligned}$$

The estimate for the third term is crucial. Since β_ℓ and g_ℓ satisfy (g1) with the same constant $k \in (0,1)$ and $C_1 > 0$,

$$\begin{aligned}
\text{(the third term)} &= \int_{Q_0} D_h(\beta_\ell(\tilde{u}_\ell) - g_\ell(\tilde{u}_\ell))\theta^2 D_h\tilde{u}_\ell\ dy',\\
&\ge \int_{Q_0} (D_h\tilde{u}_\ell)\theta^2|h|^{-1}\int_{\tilde{u}_\ell}^{(\tau_h\tilde{u}_\ell)} (\beta'_\ell(s) - g'_\ell(s))\,ds dy',\\
&\ge -C_1\int_{Q_0}(D_h\tilde{u}_\ell)^2\theta^2 dy',\\
&\ge -C\left(\int_{Q_0}(D_h\tilde{v}_\ell)^2 dy' + \int_{Q_0}\tilde{u}_\ell^2\ dy'\right),\\
&\ge -\epsilon\|D_h\tilde{v}_\ell\|^2_{H^1(Q_+)} - C_\epsilon\|D_h\tilde{v}_\ell\|^2_{L^2(Q_+)} - C\|\tilde{u}_\ell\|^2_{H^1(Q_+)}.
\end{aligned}$$

In the first inequality, we used the mean value theorem for the monotone function and the last inequality is deduced from the embedding $\|v\|^2_{L^2(Q_0)} \le \epsilon\|v\|^2_{H^1(Q_+)} + C\epsilon^{-1}\|v\|_{L^2(Q_+)}$. Consequently combining these estimates for the terms in (9), we get

$$\left\|\frac{\partial^2 \tilde{v}_\ell}{\partial y_i \partial y_j}\right\|_{L^2(Q_+)} \le C\left(\|\tilde{u}_\ell\|_{H^1(Q_+)} + \|\tilde{f}\|_{L_2(Q_+)}\right),$$

for $(i,j) \ne (N,N)$. To obtain the estimate for $\frac{\partial^2 \tilde{v}_\ell}{\partial y_N^2}$, going back to (9) and choosing $\phi = (\theta\psi)/(a_{NN}J)$, we obtain

$$\begin{aligned}
&{}_{D'(Q_+)}\langle -\frac{\partial^2 \tilde{v}_\ell}{\partial y_N^2}, \psi\rangle_{D(Q_+)} = \int_{Q_+} \frac{\partial \tilde{v}_\ell}{\partial y_N}\frac{\partial \psi}{\partial y_N} dy \\
&\le C\left(\sum_{(i,j)\ne(N,N)} \left\|\frac{\partial^2 \tilde{v}_\ell}{\partial y_i \partial y_j}\right\|_{L^2(Q_+)} + \|\tilde{u}_\ell\|_{H^1(Q_+)} + \|\tilde{f}\|_{L_2(Q_+)}\right)\|\psi\|_{L^2(Q_+)},
\end{aligned}$$

for any $\psi \in C_c^\infty(Q_+)$. Thus H^2-estimate for $\tilde{v}_\ell$ is derived. This estimate leads to the estimate (8). □

Step3: Convergence to the original problem

By virtue of (7) and (8), $\{u_\ell\}_{\ell\in\mathbb{N}}$ is bounded in $H^2(\Omega)$. Then there exists $u \in H^2(\Omega)$ and a subsequence of $\{u_\ell\}_{\ell\in\mathbb{N}}$ denoted again by $\{u_\ell\}_{\ell\in\mathbb{N}}$ such that

$$\begin{aligned}
u_\ell &\rightharpoonup u && \text{weakly in } H^2(\Omega),\\
u_\ell(x) &\to u(x) && \text{strongly in } L^2(\partial\Omega) \text{ and a.e. } \partial\Omega,\\
\tfrac{\partial u_\ell}{\partial n} &\to \tfrac{\partial u}{\partial n} && \text{strongly in } L^2(\partial\Omega).
\end{aligned}$$

Hence, by Theorem IV. 9 of [2], there exists a nonnegative function $\kappa(x) \in L^2(\partial\Omega)$ such that $|\frac{\partial u_\ell}{\partial n}| = |\beta_\ell(u_\ell) - g_\ell(u_\ell)| \le \kappa(x)$ a.e. $\partial\Omega$. By (β1) and (g1), we can easily verify that

$$(1-\kappa)|\beta_\ell(u) - \beta(0)| \le |\beta_\ell(u) - g_\ell(u)| + C_1|u| + |g(0)| + |\beta(0)|.$$

Hence,we can also find another function $\tilde{\kappa}(x) \in L^2(\partial\Omega)$ such that $|\beta_\ell(u_\ell)|+|g_\ell(u_\ell)| \le \tilde{\kappa}(x)$ a.e. $\partial\Omega$. Since $\beta_\ell(u_\ell(x)) \to \beta(u(x))$ and $g_\ell(u_\ell(x)) \to g(u(x))$ a.e. $\partial\Omega$, by Lebesgue's dominant convergence theorem, we get

$$\begin{aligned}
\beta_\ell(u_\ell) &\to \beta(u) \quad \text{in } L^2(\partial\Omega),\\
g_\ell(u_\ell) &\to g(u) \quad \text{in } L^2(\partial\Omega).
\end{aligned}$$

Taking the limit $\ell \to \infty$ in (7), we can also verify $u \in D(j)$. Thus this $u \in H^2(\Omega) \cap D(j)$ gives a solution of original problem (1).

3.1. **proof of Corollary 1.** For the case where β is maximal monotone, instead of β_ℓ, we use Yosida approximation for β, namely

$$\beta_\lambda = \frac{1}{\lambda}(\mathrm{I} - J_\lambda), \qquad J_\lambda = (\mathrm{I} + \lambda\beta)^{-1},$$

for $\lambda > 0$. First we assume g is a Lipschitz continuous function on $\mathbb{R}^1$. We approximate g as follows,

$$g_\lambda = g \circ J_\lambda.$$

By assumption (g0)' and (g1)', we can verify the following two inequalities:

$$\big(\beta_\lambda(u) - g_\lambda(u)\big)u \ge \delta\beta_\lambda(u)u - C, \tag{10}$$

$$\Big(\beta_\lambda(u_1) - g_\lambda(u_1) - \big(\beta_\lambda(u_2) - g_\lambda(u_2)\big)\Big)(u_1 - u_2) \geq - \max_{|s|\leq s_0} |g'(s)|(u_1 - u_2)^2, \quad (11)$$

for any $u, u_1, u_2 \in \mathbb{R}$, $\lambda > 0$. Since we assume that $b > 0$, by the quite same arguments as in the proof of Theorem 2.1, for any $\lambda > 0$ there exists $u_\lambda \in H^1(\Omega)$ satisfying

$$\begin{cases} -\Delta u_\lambda + bu_\lambda = f(x), \\ -\dfrac{\partial u_\lambda}{\partial n} = \beta_\lambda(u_\lambda) - g_\lambda(u_\lambda). \end{cases}$$

In the proof of Theorem 2.1, (g1) is always used to estimate the difference between $\{\beta(u_1) - g(u_1)\}$ and $\{\beta(u_2) - g(u_2)\}$. Hence the arguments in the proof of Theorem 2.1 can also work with (g1) replaced by (11). Thus we can obtain the following H^2-estimates.

$$\|u_\lambda\|^2_{H^2(\Omega)} + \|j_\lambda(u_\lambda)\|_{L^1(\partial\Omega)} \leq C(1 + \|f\|^2_{L^2(\Omega)}), \quad (12)$$

where $j_\lambda(u) = \int_0^u \beta_\lambda(s)ds$. Consequently there exists a subsequence $\{u_{\lambda_\ell}\}_{\ell\in\mathbb{N}}$ ($\lambda_\ell \to 0$ as $\ell \to \infty$) such that

$$\begin{aligned} u_{\lambda_\ell} &\rightharpoonup u_0 && \text{weakly in } H^2(\Omega), \\ u_{\lambda_\ell}(x) &\to u_0(x) && \text{strongly in } L^2(\partial\Omega) \text{ and a.e. } x \in \partial\Omega, \\ \tfrac{\partial u_{\lambda_\ell}}{\partial n} &\to \tfrac{\partial u_0}{\partial n} && \text{strongly in } L^2(\partial\Omega). \end{aligned}$$

Hence, there exists a nonnegative function $\kappa(x) \in L^2(\partial\Omega)$ such that $\left|\frac{\partial u_{\lambda_\ell}}{\partial n}(x)\right| \leq \kappa(x)$. By (10),

$$\delta|\beta_{\lambda_\ell}(u_{\lambda_\ell})| \leq |\beta_{\lambda_\ell}(u_{\lambda_\ell}) - g_{\lambda_\ell}(u_{\lambda_\ell})| + C' \leq \kappa(x) + C'. \quad (13)$$

By the definition of β_λ,

$$\delta|u_{\lambda_\ell} - J_{\lambda_\ell}u_{\lambda_\ell}| \leq \lambda_\ell\left(\kappa(x) + C'\right),$$

which implies

$$J_{\lambda_\ell}u_{\lambda_\ell} \to u_0 \quad \text{in } L^2(\partial\Omega).$$

By the boundary condition and (13), we can verify that

$$|g_{\lambda_\ell}(u_{\lambda_\ell})| \leq |\beta_{\lambda_\ell}(u_{\lambda_\ell})| + \left|\frac{\partial u_{\lambda_\ell}}{\partial n}\right| \leq C''(\kappa(x) + 1).$$

Thus by Lebesgue's dominant convergence lemma,

$$g_{\lambda_\ell}(u_{\lambda_\ell}) \to g(u_0) \quad \text{in } L^2(\partial\Omega).$$

Hence,

$$\beta_{\lambda_\ell}(u_{\lambda_\ell}) \to -\frac{\partial u_0}{\partial n} + g(u_0) \quad \text{in } L^2(\partial\Omega).$$

Since $\beta_{\lambda_\ell}(u_{\lambda_\ell}) \in \beta(J_{\lambda_\ell}u_{\lambda_\ell})$ and $J_{\lambda_\ell}u_{\lambda_\ell} \to u_0$ in $L^2(\partial\Omega)$, the demiclosedness of β in $L^2(\partial\Omega)$ assures that $-\frac{\partial u_0}{\partial n} + g(u_0) \in \beta(u_0)$, whence follows that u_0 gives the desired solutions. Moreover, taking the limit $\lambda \to 0$ in (12), we can obtain

$$\|u_0\|^2_{H^2(\Omega)} + \|j(u_0)\|_{L^1(\partial\Omega)} \leq C(1 + \|f\|^2_{L^2(\Omega)}). \quad (14)$$

For the case where g is not Lipschitz continuous, for each $\ell \geq s_0$, we approximate g as follows,

$$g_\ell(u) = \begin{cases} g(\ell) & \text{if } u > \ell, \\ g(u) & \text{if } |u| \leq \ell, \\ g(-\ell) & \text{if } u < -\ell. \end{cases}$$

Then g_ℓ is Lipschitz continuous function on $\mathbb{R}$ and satisfies (g1)' with same $\delta \in (0,1)$ and $s_0 > 0$ for all $\ell \geq s_0$. Hence (10) and (11) holds true for all $\ell \geq s_0$. Hence, by the previous arguments, there exist $u_\ell \in H^2(\Omega) \cap D(j)$ satisfying

$$\begin{cases} -\Delta u_\ell + b u_\ell = f(x), \\ -\dfrac{\partial u_\ell}{\partial n} \in \beta(u_\ell) - g_\ell(u_\ell), \end{cases}$$

and

$$\|u_\ell\|^2_{H^2(\Omega)} + \|j(u_\ell)\|_{L^1(\partial\Omega)} \leq C(1 + \|f\|^2_{L^2(\Omega)}),$$

where C does not depend on ℓ. Hence there exists a subsequence $\{u_\ell\}_{\ell \in \mathbb{N}}$ such that

$$\begin{aligned} u_\ell &\rightharpoonup u && \text{weakly in } H^2(\Omega), \\ u_\ell(x) &\to u(x) && \text{a.e. } \partial\Omega, \\ \tfrac{\partial u_\ell}{\partial n} &\to \tfrac{\partial u}{\partial n} && \text{strongly in } L^2(\partial\Omega). \end{aligned}$$

Assumptions (g0)' and (g1)' imply

$$\delta |g_\ell(u_\ell(x))| \leq (1-\delta) \left| \frac{\partial u_\ell}{\partial n}(x) \right| + C.$$

Hence, by quite the same arguments as above, we can verify that

$$\begin{aligned} g_\ell(u_\ell) &\to g(u) && \text{strongly in } L^2(\partial\Omega). \\ -\frac{\partial u_\ell}{\partial n} + g_\ell(u_\ell) &\to -\frac{\partial u}{\partial n} + g(u) && \text{strongly in } L^2(\partial\Omega) \end{aligned}$$

Since $-\frac{\partial u_\ell}{\partial n} + g_\ell(u_\ell) \in \beta(u_\ell)$ and $u_\ell \to u$ in $L^2(\partial\Omega)$, by the same reason as before, we can assure that $-\frac{\partial u}{\partial n}(x) \in \beta(u(x)) - g(u(x))$ a.e. $\partial\Omega$. Thus this u gives the desired solution.

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