DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Supplement 2009

Website: www.aimSciences.org

pp. **315–321**

STABILITY PROPERTIES AND EXISTENCE OF ALMOST PERIODIC SOLUTIONS OF VOLTERRA DIFFERENCE EQUATIONS

Yoshihiro Hamaya

Department of Information Science, Okayama University of Science
1-1 Ridai-cho, Okayama 700-0005, Japan

Abstract. In order to obtain the existence of periodic and almost periodic solutions of Volterra difference equation: $x(n+1) = f(n, x(n)) + \sum_{s=-\infty}^{n} F(n, s, x(n+s), x(n))$, we consider certain two stability properties, which are referred to as (K, ρ)-weakly uniformly asymptotically stable and (K, ρ)-uniformly asymptotically stable.

1. **Introduction.** For the ordinary differential equations and functional differential equations, the existence of almost periodic solutions of almost periodic systems has been studied by many authors. One of the most popular method is to assume the certain stability properties [3,7,8,12]. Song and Tian [10] have shown the existence of periodic and almost periodic solutions for nonlinear Volterra difference equations by means of (K, ρ)-stability conditions. Their results are to extend results in Hamaya [4] to discrete Volterra equations. In order to obtain the existence theorem for almost periodic solutions in ordinary differential equations, Sell [9] introduced a new stability concept which is referred to as the weakly uniformly asymptotically stable. This stability property is weaker than uniformly asymptotically stable (cf.[12]). Recently [11] has studied that the existence of almost periodic solutions of ordinary difference equation by using globally quasi-uniformly asymptotically stable.
In this paper, we shall discuss the relationship between weakly uniformly asymptotic stability and uniformly asymptotic stability in periodic and almost periodic Volterra difference equations, and show that for periodic Volterra difference equations, (K, ρ)-weakly uniformly asymptotic stability and (K, ρ)-uniformly asymptotic stability are equivalent. Finally, we obtain the existence of almost periodic solutions in Volterra difference equations by using this (K, ρ)-weakly uniformly asymptotically stable.

Let R^m denote Euclidean m-space, Z is the set of integers, Z^+ is the set of nonnegative integers and $|\cdot|$ will denote the Euclidean norm in R^m. For any interval $I \subset Z := (-\infty, \infty)$, we denote by $BS(I)$ the set of all bounded functions mapping I into R^m, and set $|\phi|_I = \sup\{|\phi(s)| : s \in I\}$.
Now, for any function $x : (-\infty, a) \to R^m$ and $n < a$, define a function $x_n : Z^- =$

2000 *Mathematics Subject Classification.* Primary: 39A11, 39A12.
Key words and phrases. Volterra difference equations, Almost periodic solutions and WUA stable.

$(-\infty, 0] \to R^m$ by $x_n(s) = x(n+s)$ for $s \in Z^-$. Let BS be a real linear space of functions mapping Z^- into R^m with sup-norm:

$$BS = \{\phi| \ \phi : Z^- \to R^m \quad \text{with} \quad |\phi|_\infty = \sup_{s \in Z^-} |\phi(s)| < \infty\}.$$

We introduce an almost periodic function $f(n, x) : Z \times D \to R^m$, where D is an open set in R^m.

Definition 1.1. $f(n, x)$ is said to be almost periodic in n uniformly for $x \in D$, if for any $\epsilon > 0$ and any compact set K in D, there exists a positive integer $L^*(\epsilon, K)$ such that any interval of length $L^*(\epsilon, K)$ contains an integer τ for which

$$|f(n+\tau, x) - f(n, x)| \leq \epsilon$$

for all $n \in Z$ and all $x \in K$. Such a number τ in above inquality is called an ϵ-translation number of $f(n, x)$.

In order to formulate a property of almost periodic functions, which is equivalent to the above difinition, we discuss the concept of the normality of almost periodic functions. Namely, Let $f(n, x)$ be almost periodic in n uniformly for $x \in D$. Then, for any sequence $\{h_k^{'}\} \subset Z$, there exists a subsequence $\{h_k\}$ of $\{h_k^{'}\}$ and function $g(n, x)$ such that

$$f(n + h_k, x) \to g(n, x) \tag{1}$$

uniformly on $Z \times K$ as $k \to \infty$, where K is a compact set in D. There are many properties of the discrete almost periodic functions [1,2], which are corresponding properties of the continuous almost periodic functions $f(t, x) \in C(R \times D, R^m)$ [cf.3,12]. We shall denote by $T(f)$ the function space consisting of all translates of f, that is, $f_\tau \in T(f)$, where

$$f_\tau(n, x) = f(n + \tau, x), \qquad \tau \in Z \tag{2}$$

Let $H(f)$ denote the uniform closure of $T(f)$ in the sense of (2). $H(f)$ is called the hull of f. In particular, we denote by $\Omega(f)$ the set of all limit functions $g \in H(f)$ such that for some sequence $\{n_k\}$, $n_k \to \infty$ as $k \to \infty$ and $f(n + n_k, x) \to g(n, x)$ uniformly on $Z \times S$ for any compact subset S in R^m. By (1), if $f : Z \times D \to R^m$ is almost periodic in n uniformly for $x \in D$, so is a function in $\Omega(f)$. The following concept of asymptotic almost periodicity was introduced by Frechet in the case of continuous function (cf.[3,12]).

Definition 1.2. $u(n)$ is said to be asymptotically almost periodic if it is a sum of a almost periodic function $p(n)$ and a function $q(n)$ defined on $I^* = [0, \infty)$ which tends to zero as $n \to \infty$, that is,

$$u(n) = p(n) + q(n).$$

$u(n)$ is asymptotically almost periodic if and only if for any sequence $\{n_k\}$ such that $n_k \to \infty$ as $k \to \infty$ there exists a subsequence $\{n_k\}$ for which $u(n + n_k)$ converges uniformly on $a \leq n < \infty$.

2. **Preliminaries.** We consider a system of Volterra difference equation

$$x(n+1) = f(n, x(n)) + \sum_{s=-\infty}^{0} F(n, s, x(n+s), x(n)), \tag{3}$$

where $f : Z \times R^m \to R^m$ is continuous at second variable $x \in R^m$ and $F(n, s, x, y)$ is defined for $n \in Z$, $s \in (-\infty, 0]$, $x \in R^m$ and $y \in R^m$, and continuous for $x \in R^m$ and $y \in R^m$.

We impose the following assumptions on Eq.(3):

(H1) $f(n, x)$ and $F(n, n+s, x, y)$ are ω-periodic function, that there is an $\omega > 0$ such that $f(n+\omega, x) = f(n, x)$ for all $n \in Z, x \in R^m$, and $F(n+\omega, s, x, y) = F(n, s, x, y)$ for all $n \in Z, s \leq 0, x \in R^m$ and $y \in R^m$.
(H2) $f(n, x)$ is an almost periodic in n uniformly for $x \in R^m$, and $F(n, s, x, y)$ is almost periodic in n uniformly for $(s, x, y) \in K^*$, that is for any $\epsilon > 0$ and any compact set K^*, there exists an integer $L^* = L^*(\epsilon, K) > 0$ such that any interval of length L^* contains a τ for which

$$|F(n+\tau, s, x, y) - F(n, s, x, y)| \leq \epsilon$$

for all $n \in Z$ and all $(s, x, y) \in K^*$.
(H3) For any $\epsilon > 0$ and any $r > 0$, there exists an $S = S(\epsilon, r) > 0$ such that

$$\sum_{s=-\infty}^{-S} |F(n, s, x(n+s), x(n))| \leq \epsilon \tag{4}$$

for all $n \in Z$, whenever $|x(\sigma)| \leq r$ for all $\sigma \leq n$.
(H4) Eq.(3) has a bounded solution $u(n)$ defined on $[0, \infty)$ which passes through $(0, u_0)$, that is $\sup_{n \geq 0} |u(n)| < \infty$ and $u_0 \in BS$.

Now we introduce ρ-stability properties with respect to the compact set K and the metric ρ.

Let K be the compact set in R^m such that $u(n) \in K$ for all $n \in Z$, where $u(n) = \phi^0(n)$ for $n \leq 0$. For any $\theta, \psi \in BS$, we set

$$\rho(\theta, \psi) \quad = \quad \sum_{j=1}^{\infty} \rho_j(\theta, \psi)/[2^j(1 + \rho_j(\theta, \psi))],$$

where

$$\rho_j(\theta, \psi) \quad = \quad \sup_{-j \leq s \leq 0} |\theta(s) - \psi(s)|,$$

Clearly, $\rho(\theta_n, \theta) \to 0$ as $n \to \infty$ if and only if $\theta_n(s) \to \theta(s)$ uniformly on any compact subset of $(-\infty, 0]$ as $n \to \infty$.
We denote by (BS, ρ) the space of bounded functions $\phi : (-\infty, 0] \to R^m$ with metric ρ.

In what follows, we need the following 11-definitions of stability.

Definition 2.1. The bounded solution $u(n)$ of Eq.(3) is said to be;

(i) (K, ρ)-stable (in short, (K, ρ)-S) if for any $\epsilon > 0$ there exists a $\delta(n_0, \epsilon) > 0$ such that if $n_0 \geq 0$, $\rho(x_{n_0}, u_{n_0}) < \delta(n_0, \epsilon)$, then $\rho(x_n, u_n) < \epsilon$ for all $n \geq n_0$, where $x(n)$ is a solution of (3) through (n_0, ϕ) such that $x_{n_0}(s) = \phi(s) \in K$ for all $s \leq 0$.

(ii) (K, ρ)-uniformly stable (in short, (K, ρ)-US) if for any $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that if $n_0 \geq 0$, $\rho(x_{n_0}, u_{n_0}) < \delta(\epsilon)$, then $\rho(x_n, u_n) < \epsilon$ for all $n \geq n_0$, where $x(n)$ is a solution of (3) through (n_0, ϕ) such that $x_{n_0}(s) = \phi(s) \in K$ for all $s \leq 0$.

(iii) (K, ρ)-equi asymptotically stable (in short, (K, ρ)-EAS) if it is (K, ρ)-S and for any $\epsilon > 0$ there exists a $\delta_0(n_0) > 0$ and a $T(n_0, \epsilon) > 0$ such that if $n_0 \geq 0$, $\rho(x_{n_0}, u_{n_0}) < \delta_0(n_0)$, then $\rho(x_n, u_n) < \epsilon$ for all $n \geq n_0 + T(n_0, \epsilon)$, where $x(n)$ is a solution of (3) through (n_0, ϕ) such that $x_{n_0}(s) = \phi(s) \in K$ for all $s \leq 0$.

(iv) (K, ρ)-weakly uniformly asymptotically stable (in short, (K, ρ)-WUAS) if it is (K, ρ)-US and there exists a $\delta_0 > 0$ such that if $n_0 \geq 0$, $\rho(x_{n_0}, u_{n_0}) < \delta_0$, then $\rho(x_n, u_n) \to 0$ as $n \to \infty$, where $x(n)$ is a solution of (3) through (n_0, ϕ) such that $x_{n_0}(s) = \phi(s) \in K$ for all $s \leq 0$.

(v) (K, ρ)-uniformly asymptotically stable (in short, (K, ρ)-UAS) if it is (K, ρ)-US and is (K, ρ)-quasi uniformly asymptotically stable, that is, if the δ_0 and the T in above (iii) are independent of n_0: (for any $\epsilon > 0$ there exists a $\delta_0 > 0$ and a $T(\epsilon) > 0$ such that if $n_0 \geq 0$, $\rho(x_{n_0}, u_{n_0}) < \delta_0$, then $\rho(x_n, u_n) < \epsilon$ for all $n \geq n_0 + T(\epsilon)$, where $x(n)$ is a solution of (3) through (n_0, ϕ) such that $x_{n_0}(s) = \phi(s) \in K$ for all $s \leq 0$.)

(vi) (K, ρ)-globally equi asymptotically stable (in short, (K, ρ)-GEAS) if it is (K, ρ)-S and for any $\epsilon > 0$, any $\alpha > 0$ there exists a $T(n_0, \epsilon, \alpha) > 0$ such that if $n_0 \geq 0$, $\rho(x_{n_0}, u_{n_0}) < \alpha$, then $\rho(x_n, u_n) < \epsilon$ for all $n \geq n_0 + T(n_0, \epsilon, \alpha)$, where $x(n)$ is a solution of (3) through (n_0, ϕ) such that $x_{n_0}(s) = \phi(s) \in K$ for all $s \leq 0$.

(vii) (K, ρ)-globally weakly uniformly asymptotically stable (in short, (K, ρ)-GWUAS) if it is (K, ρ)-US and $\rho(x_n, u_n) \to 0$ as $n \to \infty$, where $x(n)$ is a solution of (3) through (n_0, ϕ) such that $x_{n_0}(s) = \phi(s) \in K$ for all $s \leq 0$.

(viii) (K, ρ)-globally uniformly asymptotically stable (in short, (K, ρ)-GUAS) if it is (K, ρ)-US and is (K, ρ)-globally quasi uniformly asymptotically stable, that is, if the T in above (vi) are independent of n_0: (for any $\epsilon > 0$ there exists a $T(\epsilon, \alpha) > 0$ such that if $n_0 \geq 0$, $\rho(x_{n_0}, u_{n_0}) < \alpha$, then $\rho(x_n, u_n) < \epsilon$ for all $n \geq n_0 + T(\epsilon, \alpha)$, where $x(n)$ is a solution of (3) through (n_0, ϕ) such that $x_{n_0}(s) = \phi(s) \in K$ for all $s \leq 0$.)

(ix) (K, ρ)-eventually totally stable (in short, (K, ρ)-ETS) if for any $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ and $\alpha(\epsilon)$ such that if $n_0 \geq \alpha(\epsilon)$, $\rho(x_{n_0}, u_{n_0}) < \delta(\epsilon)$ and $h \in BS([n_0, \infty))$ which satisfies $|h|_{[n_0,\infty)} < \delta(\epsilon)$, then $\rho(x_n, u_n) < \epsilon$ for all $n \geq n_0$, where $x(n)$ is a solution of

$$x(n+1) = f(n, x(n)) + \sum_{s=-\infty}^{0} F(n, s, x(n+s), x(n)) + h(n),$$

through (n_0, ϕ) such that $x_{n_0}(s) = \phi(s) \in K$ for all $s \leq 0$. If we can choose $\alpha(\epsilon) \equiv 0$, then $u(n)$ is said to be (K, ρ)-totally stable (in short, (K, ρ)-TS). In the case where $h(n) \equiv 0$, this gives the definition of the (K, ρ)-US of $u(n)$.

(x) (K, ρ)-attracting in $\Omega(f, F)$ (in short, (K, ρ)-A in $\Omega(f, F)$) if there exists a $\delta_0 > 0$ such that if $n_0 \geq 0$ and any $(v, g, G) \in \Omega(u, f, F)$, $\rho(x_{n_0}, v_{n_0}) < \delta_0$, then $\rho(x_n, v_n) \to 0$ as $n \to \infty$, where $x(n)$ is a solution of

$$x(n+1) = g(n, x(n)) + \sum_{s=-\infty}^{0} G(n, s, x(n+s), x(n)), \tag{5}$$

through (n_0, ψ) such that $x_{n_0}(s) = \psi(s) \in K$ for all $s \leq 0$.

(xi) (K, ρ)-weakly uniformly asymptotically stable in $\Omega(f, F)$ (in short, (K, ρ)-WUAS in $\Omega(f, F)$) if it is (K, ρ)-US in $\Omega(f, F)$, that is if for any *epshilon* > 0 there exists a $\delta(\epsilon) > 0$ such that if $n_0 \geq 0$ and any $(v, g, G) \in \Omega(u, f, F)$, $\rho(x_{n_0}, v_{n_0}) < \delta(\epsilon)$, then $\rho(x_n, v_n) < \epsilon$ for all $n \leq n_0$, where $x(n)$ is a solution of (5) through (n_0, ψ) such that $x_{n_0}(s) = \psi(s) \in K$ for all $s \leq 0$, and (K, ρ)-A in $\Omega(f, F)$.

For (iv) and (v) in the above Definition 3, actually, the (K, ρ)-WUAS is weaker than the (K, ρ)-UAS as [11, Example 3.1] shows.

3. **Stability of Bounded Solutions in Periodic and Almost Periodic Systems.**

We now hold the following theorems 1,2,3,4 and 5 from [12, Theorems], so we will omit the proof of theorems.

Theorem 3.1. *Under the assumptions (H3) and (H4), if the bounded solution $u(n)$ of Eq.(3) is (K, ρ)-WUAS, then it is (K, ρ)-EAS.*

For the periodic system, we have the following theorem.

Theorem 3.2. *Under the assumptions (H1),(H3) and (H4), if the bounded solution $u(n)$ of Eq.(3) is (K, ρ)-WUAS, then it is (K, ρ)-UAS.*

For the almost periodic system (3), we have the following theorem.

Theorem 3.3. *Under the above assumptions (H2),(H3) and (H4), if the zero solution $u(n) \equiv 0$ of Eq.(3) is (K, ρ)-WUAS then it is (K, ρ)-UAS.*

The following theorems can be proved by the same arguments as in the proof of Theorem 1 and 2.

Theorem 3.4. *Under the assumptionss (H3) and (H4), if the bounded solution $u(n)$ of Eq.(3) is (K, ρ)-GWUAS, then it is (K, ρ)-GEAS.*

Theorem 3.5. *Assume conditions (H1),(H3) and (H4). Then the solution $u(n)$ of Eq.(3) is (K, ρ)-GWUAS implies the solution $u(n)$ of Eq.(3) is (K, ρ)-GUAS.*

For the ordinary differential equation, it is well known that a example in [12, pp 81] is of a scalar almost periodic equation such that the zero solution is GWUAS but is not GUAS.

Theorem 3.6. *Under the assumption (H2),(H3) and (H4), if the solution $u(n)$ of Eq.(3) is (K, ρ)-WUAS in $\Omega(f,F)$, then the solution $u(n)$ of Eq.(3) is (K, ρ)-ETS. Moreover, if $u(n)$ is the unique solution of Eq.(3) through (0, ϕ^0), then $u(n)$ is (K, ρ)-TS.*

Proof. Suppose that $u(n)$ is not (K, ρ)-TS. Then there exists a small $\epsilon > 0$, sequence $\{\epsilon_k\}, 0 < \epsilon_k < \epsilon$ and $\epsilon_k \to 0$ as $k \to \infty$, $\{s_k\}$, $\{n_k\}$, $\{h_k\}$ and $\{x^k\}$ such that $s_k \to \infty$ as $k \to \infty$, $0 < s_k+1 < n_k$, $h_k : \mathbf{Z} \to R^m$ is bounded function satisfying $|h_k(n)| < \epsilon_k$ for $n \geq s_k$ and that $\rho(u_{s_k}, x^k_{s_k}) < \epsilon_k$,

$$\rho(u_{n_k}, x^k_{n_k}) = \epsilon \quad \text{and} \quad \rho(u_n, x^k_n) < \epsilon \quad [s_k, n_k), \tag{6}$$

where $x^k(n)$ is a solution of

$$x(n+1) = f(n, x(n)) + \sum_{s=-\infty}^{0} F(n, s, x(n+s), x(n)) + h_k(n)$$

such that $x^k_{s_k}(s) \in$ K for all $s \leq 0$ and satisfying $x^k(n) \in$ K on $\mathbf{Z}^+$. We can assume that $\epsilon < \delta_0$ where δ_0 is the number for (K, ρ)-A in $\Omega(f,F)$ of Definition 3. Moreover, by (6) there exists a sequence $\{\tau_k\}$ such that $s_k < \tau_k < n_k$,

$$\rho(u_{\tau_k}, x^k_{\tau_k}) = \delta(\epsilon/2)/2 \tag{7}$$

and

$$\delta(\epsilon/2)/2 \leq \rho(u_n, x^k_n) \leq \epsilon \quad \text{for} \quad n \in [\tau_k, n_k], \tag{8}$$

where $\delta(\cdot)$ is the number for (K, ρ)-US in $\Omega(f,F)$. Now we may assume that $u(n+\tau_k) \to v(n)$ as $k \to \infty$ on each bounded subset of $\mathbf{Z}$ for a function v, and for the sequence $\{\tau_k\}$ such that $\tau_k \to \infty$ as $k \to \infty$. For this sequence $\{\tau_k\}$, taking a subsequence if necessary, there exists a $(v, g, G) \in \Omega(u, f, F)$. Moreover, we may assume that $x^k(n+\tau_k) \to z(n)$ as $k \to \infty$ uniformly on any bounded subset of $\mathbf{Z}$ for function z, since the sequence $\{x^k(n+\tau_k)\}$ is uniformly bounded on $\mathbf{Z}$. Because, if we set $y^k(n) = x^k(n+\tau_k)$, then $y^k(n)$ is defined on $n \geq n_0 + \tau_k$ and $y^k(n)$ is a solution of

$$x(n+1) = f(n+\tau_k, x(n)) + \sum_{s=-\infty}^{0} F(n+\tau_k, s, x(n+s), x(n)) + h_k(n+\tau_k)$$

such that $y^k_0(s) = x^k_{\tau_k}(s) \in$ K for all $s \leq 0$. Then we can show that, taking a subsequence if necessary, $y^k(n)$ converges to a solution $z(n)$ of

$$x(n+1) = g(n, x(n)) + \sum_{s=-\infty}^{0} G(n, s, x(n+s), x(n))$$

such that $z_0(s) \in$ K for $s \leq 0$, by the same argument for $\sum$-calculations with condition (H3) as in the proof of Theorem 3. Then, $z \in K$. Now, suppose that $n_k - \tau_k \to \infty$ as $k \to \infty$. Letting $k \to \infty$ in (8), we have $\delta(\epsilon/2)/2 \leq \rho(v_n, z_n) \leq \epsilon$ on $n \geq 0$. Since $\epsilon < \delta_0$ and $u(n)$ is (K, ρ)-A in $\Omega(f,F)$, we have $\delta(\epsilon/2)/2 \leq \rho(v_n, z_n) \to 0$ as $n \to \infty$, which is a contradiction. Thus $n_k - r_k \not\to \infty$ as $k \to \infty$. Taking a subsequence again if necessary, we can assum that $n_k - r_k \to r < \infty$ as $k \to \infty$ and that for the sequence $\{\tau_k\}$ there exists a $(\hat{v}, \hat{g}, \hat{G}) \in \Omega(u, f, F)$. Letting $k \to \infty$ in (7) and (6), we have $\rho(v_0, z_0) = \delta(\epsilon/2)/2 < \delta(\epsilon/2)$ and $\rho(v_n, z_n) = \epsilon$. But $u(n)$ is

(K, ρ)-US in $\Omega(f, F)$. Thus $\rho(v_0, z_0) < \delta(\epsilon/2)$ yields $\rho(v_n, z_n) < \epsilon/2$ for all $n \geq 0$, which contradicts $\rho(v_n, z_n) = \epsilon$. This shows that $u(n)$ is (K, ρ)-TS. □

We have the following existence theorem of an almost periodic solution for Eq.(3).

Theorem 3.7. *Under the assumption (H2),(H3) and (H4), if the solution $u(n)$ of Eq.(3) is (K, ρ)-WUAS in $\Omega(f, F)$ and $u(n)$ is the unique solution of Eq.(3) through (0, ϕ^0), then the Eq.(3) has an almost periodic solution.*

Proof. From Theorem 6, the unique solution $u(n)$ of Eq.(3) through (0, u_0) is (K, ρ)-TS. Thus, by Theorem 1 and 2 in [6], we have an almost periodic solution. □

REFERENCES

[1] R. P. Agarwal, *Difference Equations and Inequalities 2nd Edition*, Dekker, 2000.
[2] C. Corduneanu, *Almost periodic discrete processes*, Libertas Mathematica, 2 (1982) 159-169.
[3] A. M. Fink, *Almost Periodic Differential Equations*, Lecture Notes in Mathematics 377, Springer-Verlag, 1974.
[4] Y. Hamaya, *Periodic solutions of nonlinear integrodifferential equations*, Tohoku Math. J., 41 (1989), 105-116.
[5] Y. Hamaya, *Total stability property in limiting equations of integrodifferential equation*, Funkcialaj Ekvacioj, 33 (1990), 345-362.
[6] Y. Hamaya, *Existence of almost periodic solution in a difference equation with infinite delay*, Journal of Difference Equations and Applications, 9 (2003), 227-237.
[7] Y. Hino, S. Murakami and T. Naito, *Functional Differential Equations with Infinite Delay*, Lecture Notes in Mathematics, Springer-Verlag 1473, 1991.
[8] J. Kato, A. A Martynyuk and A. A Shestakov, *Stability of Motion of Nonautonomous Systems (Method of Limiting Equations)*, Gordon and Breach Publishers, 1996.
[9] G. R. Sell, Periodic solutions and asymptotic stability, Journal of Differential Equations, 2 (1966) 143-157.
[10] Y. Song and H. Tian, *Periodic and almost periodic solutions of nonlinear Volterra difference equations with unbounded delay*, Journal of Computational and Applied Mathematics, 205 (2007) 859-870.
[11] Y. Xia and S. S. Cheng, *Quasi-uniformly asymptotic stability and existence of almost periodic solutions of difference equations with applications in population dynamic systems*, Journal of Difference Equations and Applications, 14 (2008) 59-81.
[12] T. Yoshizawa, *Stability Theory and the Existence of Periodic Solutions and Almost Periodic Solutions*, Applied Mathematical Sciences 14, Springer-Verlag, 1975.

Received July 2008; revised April 2009.

E-mail address: hamaya@mis.ous.ac.jp