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HIERARCHICAL DIFFERENTIAL GAMES BETWEEN MANUFACTURER AND RETAILER

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Abstract. A two-dimensional microeconomic model with three bounded controls is created and investigated. The model describes a manufacturer producing a consumer good and a retailer that buys this product in order to resell it for a profit. Two types of differential hierarchical games will be applied in order to model the interactions between the manufacturer and retailer. We will consider the difficult case in which the maximum of the objective functions can be reached only on the boundary of the admissible set. Optimal strategies for manufacturer and retailer in both games will be found. The object of our interest is the investigation of the vertical integration of retail and industrial groups. We will determine the conditions of interaction that produce a stable and maximally effective structure over given planning periods.

1. **Introduction.** One of the defining characteristics of the modern economy is its globalization ([1]). Although large companies often dominate the market by their actions alone, we should not neglect the importance of interactions between large and small companies, such as a large manufacturer seeking to exploit a market through cooperation with smaller retailers. Likewise, a large, powerful retailer (e.g., Wal-Mart or Home Depot) may seek out manufacturers including small businesses, to produce new products or to increase supply capacities for an item in exceeding demand. The fact that these relationships are not rare suggests that a comparative analysis of the form of business integration may give greater insight into their relative significance in the global economy.

Manufacturer-retailer interactions are often modeled as hierarchical games or "leader-follower" games, in which one player (the "leader") has the privilege to play first and announces his decision ahead of the other players. The theoretical literature has examined some of these games ([2]-[4]). However, what unites all theses papers is that either the considered games are static or models are simplistic. Additionally, published results on hierarchical games ([5]-[8]) tend to be limited to the case where the control is not bounded so that a Stackelberg equilibrium is not difficult to find. If restrictions on the control exist, then the condition of the maximum is supplied by control, which can be reached inside of the admissible region ([9]). Furthermore, although the analytical tools of differential games have

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been influenced by those of optimal control ([5],[9]), it is difficult to apply these tools to complex dynamic systems. Most published works deal with only one-dimensional models ([7],[9],[10]-[12]).

In this work we consider an economic control model describing the vertical integration of a manufacturer and retailer as a power structure. We will answer the following questions:

1) What benefit does the Leader gain? When the manufacturer or retailer leads, will the leader be the more powerful player and gain greater profits?

2) When a manufacturer or a retailer leads, how does optimal policy for each player compare with the case of the Manufacturer-Stackelberg and the Retailer-Stackelberg games?

Our paper is organized as follows. In Section 2, we will describe our model and properties of its state variables, we also mention that two types of Leader-Follower games will be solved. In Section 3, we state Game 1 (Manufacturer-Stackelberg Leader) and demonstrate its solution. In Section 4, we state Game 2 (Retailer-Stackelberg Leader) and formulate its solution by a perturbation method and Pontryagin Maximum Principle for hybrid control system. In Subsection 4.1, the first part of the Game 2 (Problem 2.1) is solved and in Subsections 4.2 and 4.3, its second part (Problem 2.2) is completed. Our optimal solutions are obtained analytically. However, in order to parametrically explore Game 1 versus Game 2 we use numerical simulation in Section 5. Finally, Section 6 contains our conclusions.

2. **Hierarchical Games between Manufacturer and Retailer.** Let us consider a manufacturer producing some perishable consumer good, the demand for which always exists, and a retailer, which buys the good in order to resell it. We will model interactions between manufacturer and retailer as follows:

$$\begin{cases} \dot{x}(t) = -v(t)x(t) + u(t), \\ \dot{y}(t) = -w(t)y(t) + v(t)x(t),\ t \in [0,T], \\ x(0) = x_0,\ y(0) = y_0;\ x_0, y_0 > 0. \end{cases} \tag{1}$$

Here $x(t)$ is the inventory of the manufacturer, $y(t)$ is the inventory of the retailer, $u(t)$ is the volume of the total production for the manufacturer, $v(t)$ is the rate of purchasing the good, and $w(t)$ the rate of reselling the good to the third party for the retailer. Here $u(t)$, and $v(t)$, $w(t)$ are control parameters of the manufacturer and retailer, respectively.

Let $D_M(T)$ be the control set, such that $D_M(T)$ is the set of all Lebesgue measurable functions $u(t)$, satisfying the inequalities, $0 \leq u(t) \leq u_0$, for almost all $t \in [0,T]$. Let $D_R(T)$ be the control set, such that $D_R(T)$ is the set of all pair Lebesgue measurable functions $(v(t), w(t))$, satisfying the inequalities, $0 \leq v(t) \leq v_0$ and $0 \leq w(t) \leq w_0$, for almost all $t \in [0,T]$.

The goal of the manufacturer is to maximize the following objective function of profit,

$$J_M = \int_0^T (p_1 v(t)x(t) - p_0 u(t))dt \to \max_{u(\cdot)\in D_M(T)}. \tag{2}$$

The goal of the retailer is to maximize his profit given by

$$J_R = \int_0^T (p_2 w(t)y(t) - p_1 v(t)x(t))dt \to \max_{(v(\cdot),w(\cdot))\in D_R(T)}. \tag{3}$$

Here p_0 is the unit cost of the product, p_1 is the price at which the manufacturer sells his product to the retailer, p_2 is the price at which the retailer resells his product to his customers.

The following statement describes the properties of variables $x(t)$ and $y(t)$ for the system (1).

Lemma 2.1. *Let $u(\cdot) \in D_M(T)$ and $(v(\cdot), w(\cdot)) \in D_R(T)$ be some control functions. Then the solutions $x(t)$, $y(t)$ of the system (1) satisfy the inequalities $x(t) > 0$, $y(t) > 0$ for all $t \in [0, T]$.*

Interactions between manufacturer and retailer will be considered as a hierarchical differential game. We want to find such conditions for the dynamic game which provide the maximal possible profits for each participant, satisfying the following restrictions:

$$J_M > p_0 x_0, \; J_R > p_1 y_0. \tag{4}$$

If inequalities (4) hold, then manufacturer is producing the consumer good and retailer is buying and reselling it.

Next, we will consider two different hierarchical "Leader-Follower" Games:

(Game 1) The manufacturer is a "leader" and the retailer is a "follower",

(Game 2) The retailer is a "leader" and the manufacturer is a "follower".

Although Game 1 and Game 2 are described by the same system of differential equations (1) and functionals (2), (3) their solutions may be significantly different.

3. **Game 1 : Manufacturer - Stackelberg Game.** In this game, a manufacturer playing the Stackelberg leader chooses his production activity over the entire time period $[0, T]$ in order to maximize his profit, conditioned on the equilibrium reaction function of the retailer.

For the Game 1 the Main Problem is formulated as follows:

for the manufacturer: choose such control $u(t)$, $t \in [0, T]$, that maximizes profit J_M.

for the retailer: choose such controls $v(t)$ and $w(t)$, $t \in [0, T]$, that maximize profit J_R.

Therefore, for the Game 1 we have the following scenario:

1) A manufacturer "proposes" control function $u(t) \neq 0$ which determines his production plan over the entire planning period $[0, T]$.

2) A retailer "responds" to the proposed controls $u(t)$ by choosing control functions $v(t) = v(t, u(\cdot))$ and $w(t) = w(t, u(\cdot))$, which will maximize objective function J_R.

3) After finding the functions $v(t) = v(t, u(\cdot))$, $w(t) = w(t, u(\cdot))$, that maximize J_R, find corresponding control $u(t) = u(t, v(\cdot, u(\cdot)), w(\cdot, u(\cdot)))$, which maximize J_M.

Therefore, the Main Problem can be reduced to the consecutive solution of the following auxiliary optimal control problems:

Problem 1.1. Given function $u(t) \neq 0$, find such control functions $v(t) = v(t, u(\cdot))$, $w(t) = w(t, u(\cdot))$, $t \in [0, T]$ that maximize profit J_R.

Problem 1.2. Using the solution to Problem 1.1 functions $v(t) = v(t, u(\cdot))$, $w(t) = w(t, u(\cdot))$, find on the interval $[0, T]$ such a control function

$$u(t) = u(t, v(\cdot, u(\cdot)), w(\cdot, u(\cdot))),$$

at which J_M is maximized.

The Main Problem of the Game 1 was solved in [13]. It's solution may be summarized as follows.

For the optimal behavior of the retailer we have the formulas:

$$w_*(t) = w_0,\ t \in [0,T], \tag{5}$$

$$v_*(t) = \begin{cases} v_0, & 0 \le t \le \theta_*, \\ 0, & \theta_* < t \le T, \end{cases} \tag{6}$$

where

$$\theta_*^1 = \theta_* = T - \frac{1}{w_0}\ln\left(\frac{p_2}{p_2 - p_1}\right) \tag{7}$$

is the moment of switching of the control function $v_*(t)$.

For the manufacturer's optimal policy we have the formulas:

$$u_*(t) = \begin{cases} u_0, & 0 \le t \le \tau_*, \\ 0, & \tau_* < t \le T, \end{cases} \tag{8}$$

where

$$\tau_*^1 = \tau_* = T - \frac{1}{w_0}\ln\left(\frac{p_2}{p_2 - p_1}\right) - \frac{1}{v_0}\ln\left(\frac{p_1}{p_1 - p_0}\right) \tag{9}$$

is the moment of switching of the control function $u_*(t)$. From formulas (7),(9) it follows, that $\tau_* < \theta_*$.

This solution of the Game 1 was obtained under the following conditions:

$$p_1 - p_0 > p_1 e^{-v_0\theta_*},\ p_2 - p_1 > p_2 e^{-w_0 T},$$

where the value θ_* is defined by (7).

The economic interpretation of the results will be given in section 6.

4. **Game 2 : Retailer - Stackelberg Game.** While the manufacturer's and retailer's objective functions given by formulas (2) and (3) remain the same as in Game 1, the nature of the competition changes. The retailer being the Stackelberg leader chooses the rate of purchasing and reselling while the manufacturer plays the follower, choosing his rate of production conditioned on the retailer's choice. In actual practice this can model a powerful retailer, such as Wal-Mart, which dictates his policy to a small manufacturer (small business).

For the Game 2 the Main Problem is formulated as follows:

for the manufacturer: choose such control $u(t)$, $t \in [0,T]$, that maximizes profit J_M.

for the retailer: choose such controls $v(t)$ and $w(t)$, $t \in [0,T]$, that maximize profit J_R.

Therefore, for the Game 2 we have the following scenario:

1) A retailer "proposes" control functions $v(t), w(t) \neq 0$, which determine his plan for purchase and sales over the entire planning period $[0,T]$.

2) A manufacturer "responds" to the proposed controls $v(t)$, $w(t)$ by choosing control functions $u(t) = u(t, v(\cdot), w(\cdot))$, which will maximize objective function J_M.

3) After finding the function $u(t) = u(t, v(\cdot), w(\cdot))$, that maximizes J_M, find corresponding controls $v(t) = v(t, u(\cdot, v(\cdot), w(\cdot)))$, $w(t) = w(t, u(\cdot, v(\cdot), w(\cdot)))$, which maximize J_R.

Therefore, the Main Problem can be reduced to consecutive solution of the following auxiliary optimal control problems:

Problem 2.1. Given functions $v(t), w(t) \neq 0$, find such control function $u(t) = u(t, v(\cdot), w(\cdot))$, $t \in [0,T]$, that maximizes profit J_M.

Problem 2.2. Using the solution to Problem 2.1, function $u(t) = u(t, v(\cdot), w(\cdot))$, find on the interval $[0, T]$ such control functions $v(t) = v(t, u(\cdot, v(\cdot), w(\cdot)))$, $w(t) = w(t, u(\cdot, v(\cdot), w(\cdot)))$ by which J_R is maximized.

Upon solving Problem 2.2 there is no possibility of time inconsistency. The retailer may not apply at any moment of time $\widehat{t} \in (0, T)$ controls $\widehat{v}(t)$, $\widehat{w}(t)$, different from controls $v(t)$, $w(t)$, selected at moment $t = 0$.

Note that certain difficulties arise in solving the Main Problem. In the solution of Problem 2.1, function $u(t)$ depends on the function $v(t)$ in such a way that it is impossible to use the classical necessary optimality conditions ([14]-[16]) in the solution of Problem 2.2. In order to solve the Main Problem we will use the method of perturbations. More precisely, let us consider the problem (1)-(3) perturbed by a small parameter $\sigma > 0$, which will change the definition of the set $D_R(T)$. The new control set $D_R^\sigma(T)$ is the set of all pair Lebesgue measurable functions, $(v(t), w(t))$, satisfying the inequalities, $\sigma \le v(t) \le v_0$ and $0 \le w(t) \le w_0$, for almost all $t \in [0, T]$.

4.1. **Solution to the Perturbed Problem 2.1.** Let some functions $v(t)$ and $w(t)$ from the control set $D_R^\sigma(T)$ be given. We consider the optimal control problem:

$$\begin{cases} \dot{x}(t) = -v(t)x(t) + u(t),\ t \in [0, T], \\ x(0) = x_0,\ x_0 > 0, \end{cases} \tag{10}$$

$$J_M = \int_0^T (p_1 v(t)x(t) - p_0 u(t))dt \to \max_{u(\cdot) \in D_M(T)}. \tag{11}$$

The existence of the optimal control $u_*^\sigma(t)$, $t \in [0, T]$ for the problem (10),(11) follows from [14]. Moreover, from the restriction (4) for the functional J_M the inequality $p_1 > p_0$ follows.

In order to solve the optimal control problem (10), (11) we will apply the Pontryagin Maximum Principle ([15]). For the optimal control, $u_*^\sigma(t)$, there exists a non-trivial solution, $\psi(t)$, of the adjoint system,

$$\begin{cases} \dot{\psi}(t) = v(t)(\psi(t) - p_1), \\ \psi(T) = 0, \end{cases} \tag{12}$$

for which

$$u_*^\sigma(t) = \begin{cases} u_0 & ,\ L_u(t) > 0, \\ \forall u \in [0, u_0] & ,\ L_u(t) = 0, \\ 0 & ,\ L_u(t) < 0, \end{cases} \tag{13}$$

where $L_u(t) = \psi(t) - p_0$ is the switching function.

From (12) for the switching function $L_u(t)$ we have the Cauchy problem:

$$\begin{cases} \dot{L}_u(t) = v(t)L_u(t) - (p_1 - p_0)v(t), \\ L_u(T) = -p_0. \end{cases} \tag{14}$$

From the analysis of the Cauchy problem (14) we conclude the following statement.

Lemma 4.1. *The switching function $L_u(t)$ has at most one zero on the interval $(0, T)$.*

Then, from the Cauchy problem (14) we find the formula for the switching function $L_u(t)$ as

$$L_u(t) = (p_1 - p_0) - p_1 e^{-\int_t^T v(\xi)d\xi},\ t \in [0, T].$$

Since $\dot{L}_u(t) = -p_1 v(t) e^{-\int_t^T v(\xi)d\xi}$, then the switching function $L_u(t)$ is decreasing. Therefore, if $L_u(0) \le 0$, then function $L_u(t)$ has no zeros on the interval $(0, T]$. If $L_u(0) > 0$, then there is precisely one zero, τ_*^σ, on the interval $(0, T)$. Therefore, we have following cases:

Case 1. If $(p_1 - p_0) - p_1 e^{-\int_0^T v(\xi)d\xi} \le 0$, then $u_*^\sigma(t) = 0$, $t \in [0, T]$.

Case 2. If $(p_1 - p_0) - p_1 e^{-\int_0^T v(\xi)d\xi} > 0$, then

$$u_*^\sigma(t) = \begin{cases} u_0, & 0 \le t \le \tau_*^\sigma, \\ 0, & \tau_*^\sigma < t \le T. \end{cases} \tag{15}$$

Let us consider the Case 1. From the system (10) for the control $u_*^\sigma(t) = 0$, $t \in [0, T]$ we find the solution

$$x_*^\sigma(t) = x_0 e^{-\int_0^t v(\xi)d\xi}.$$

Next, we will estimate the value of the objective function J_M. Substituting the last expression to the formula (11) and evaluating the obtained integral, we have the chain of expressions:

$$J_M = p_1 \int_0^T v(t) x_*^\sigma(t) dt = p_1 x_0 \left(1 - e^{-\int_0^T v(\xi)d\xi}\right) \le p_0 x_0.$$

We can see that in the Case 1, the value of the objective function J_M satisfies the inequality $J_M \le p_0 x_0$ which contradicts (4). Therefore, we will not consider this situation further.

Further we will consider only the Case 2. If the inequality

$$(p_1 - p_0) > p_1 e^{-\int_0^T v(\xi)d\xi}, \tag{16}$$

is valid, then the optimal control $u_*^\sigma(t)$ satisfies relationship (15), where for the moment of switching $\tau_*^\sigma \in (0, T)$ we have the equality

$$\ln \frac{p_1}{p_1 - p_0} = \int_{\tau_*^\sigma}^T v(\xi) d\xi. \tag{17}$$

Therefore, for the considered problem (10),(11) we have the optimal control $u_*^\sigma(t)$, which satisfies the relationship (15), where the moment of switching τ_*^σ is given by (17). Moreover, from the formulas (15),(17) it follows that $u_*^\sigma(t) = u_*^\sigma(t, v(\cdot))$.

Next, for the optimal control $u_*^\sigma(t)$ we will estimate the value of the objective function J_M. Let $x_*^\sigma(t)$ be the optimal trajectory, corresponding to optimal control $u_*^\sigma(t)$, $t \in [0, T]$. We obtain the expression

$$J_M = \int_0^{\tau_*^\sigma} (p_1 v(t) x_*^\sigma(t) - p_0 u_0) dt + \int_{\tau_*^\sigma}^T p_1 v(t) x_*^\sigma(t) dt.$$

Substituting the expression $v(t) x_*^\sigma(t)$ from the differential equation (10) to the first and the second integrals, we have:

$$J_M = \int_0^{\tau_*^\sigma} \left((p_1 - p_0) u_0 - p_1 \dot{x}_*^\sigma(t)\right) dt - p_1 \int_{\tau_*^\sigma}^T \dot{x}_*^\sigma(t) dt =$$

$$= (p_1 - p_0) u_0 \tau_*^\sigma - p_1 (x_*^\sigma(\tau_*^\sigma) - x_0) - p_1 (x_*^\sigma(T) - x_*^\sigma(\tau_*^\sigma)). \tag{18}$$

From the differential equation (10) we obtain the following equality

$$x_*^\sigma(T) = x_*^\sigma(\tau_*^\sigma) e^{-\int_{\tau_*^\sigma}^T v(\xi)d\xi}. \tag{19}$$

Substituting (17),(19) into the formula (18) and transforming it, we have the chain of expressions:

$$J_M = p_0x_0 + (p_1 - p_0)(u_0\tau_*^\sigma - (x_*^\sigma(\tau_*^\sigma) - x_0)) =$$

$$= p_0x_0 + (p_1 - p_0)\int_0^{\tau_*^\sigma} v(t)x_*^\sigma(t))dt > p_0x_0.$$

We obtained the inequality $J_M > p_0x_0$ from (4). Solution to the Perturbed Problem 2.1 is complete.

4.2. **Solution to the Perturbed Problem 2.2.** Let us consider the optimal control problem:

$$\begin{cases} \dot{x}(t) = -v(t)x(t) + u_*^\sigma(t), \\ \dot{y}(t) = -w(t)y(t) + v(t)x(t),\ t \in [0,T], \\ x(0) = x_0,\ y(0) = y_0;\ x_0, y_0 > 0, \end{cases} \tag{20}$$

$$J_R = \int_0^T (p_2w(t)y(t) - p_1v(t)x(t))dt \to \max_{(v(\cdot),w(\cdot))\in D_R^\sigma(T)}. \tag{21}$$

Additionally we note, that the control function $v(t)$ satisfies relationship (16). Besides, the control $u_*^\sigma(t)$ has type (15) and the moment of switching $\tau_*^\sigma \in (0,T)$ satisfies the equality (17).

Let the inequality

$$v_0T > \ln\frac{p_1}{p_1 - p_0} \tag{22}$$

be valid.

The considered optimal control problem (20),(21) with the restriction (22) is that of a hybrid system. On the interval $[0,\tau_*^\sigma]$ the system (20) can be written as

$$\begin{cases} \dot{x}(t) = -v(t)x(t) + u_0, \\ \dot{y}(t) = -w(t)y(t) + v(t)x(t), \end{cases}$$

and on the segment $[\tau_*^\sigma, T]$ as

$$\begin{cases} \dot{x}(t) = -v(t)x(t), \\ \dot{y}(t) = -w(t)y(t) + v(t)x(t), \end{cases}$$

where $\tau_*^\sigma \in (0,T)$ is non-fixed moment of "sewing" of two systems above.

Using arguments similar to [16], the existence of the optimal controls $v_*^\sigma(t)$, $w_*^\sigma(t)$, the value of $\tau_*^\sigma \in (0,T)$ and corresponding to them optimal trajectories $x_*^\sigma(t)$, $y_*^\sigma(t)$, $t \in [0,T]$ can be shown.

We apply the Pontryagin Maximum Principle for the hybrid control systems ([17],[18]) to the problem (20),(21). It follows that for the optimal controls $v_*^\sigma(t)$, $w_*^\sigma(t)$ and the value $\tau_*^\sigma \in (0,T)$ there exist non-trivial solutions $\varphi_1^\sigma(t)$, $\varphi_2^\sigma(t)$ of the adjoint system

$$\begin{cases} \dot{\varphi}_1^\sigma(t) = -v_*^\sigma(t)(\varphi_2^\sigma(t) - \varphi_1^\sigma(t) - p_1), \\ \dot{\varphi}_2^\sigma(t) = -w_*^\sigma(t)(p_2 - \varphi_2^\sigma(t)),\ t \in [0,T], \\ \varphi_1^\sigma(T) = 0,\ \varphi_2^\sigma(T) = 0, \end{cases} \tag{23}$$

for which for all $t \in [0,T]$

$$w_*^\sigma(t) = \begin{cases} w_0, & \text{if } L_w^\sigma(t) > 0, \\ \forall w \in [0,w_0], & \text{if } L_w^\sigma(t) = 0, \\ 0, & \text{if } L_w^\sigma(t) < 0, \end{cases} \tag{24}$$

and for $t \in [0, \tau_*^\sigma]$

$$v_*^\sigma(t) = \begin{cases} v_0, & \text{if } L_v^\sigma(t) > 0, \\ \forall v \in [\sigma, v_0], & \text{if } L_v^\sigma(t) = 0, \\ \sigma, & \text{if } L_v^\sigma(t) < 0, \end{cases} \tag{25}$$

and for $t \in [\tau_*^\sigma, T]$

$$v_*^\sigma(t) = \begin{cases} v_0, & \text{if } G_v^\sigma(t) > 0, \\ \forall v \in [\sigma, v_0], & \text{if } G_v^\sigma(t) = 0, \\ \sigma, & \text{if } G_v^\sigma(t) < 0. \end{cases} \tag{26}$$

Here the functions $L_w^\sigma(t) = p_2 - \varphi_2^\sigma(t)$, $L_v^\sigma(t) = \varphi_2^\sigma(t) - \varphi_1^\sigma(t) - p_1$, $G_v^\sigma(t) = x_*^\sigma(t)L_v^\sigma(t) + \eta^\sigma$, $t \in [0, T]$, are the switching functions. The value η^σ is defined from the condition of continuity of the functions $v_*^\sigma(t)G_v^\sigma(t)$ and $u_0\varphi_1^\sigma(t) + v_*^\sigma(t)x_*^\sigma(t)L_v^\sigma(t)$ at the point $t = \tau_*^\sigma$:

$$u_0\varphi_1^\sigma(t) + v_*^\sigma(t)x_*^\sigma(t)L_v^\sigma(t)|_{t=\tau_*^\sigma-0} = v_*^\sigma(t)G_v^\sigma(t)|_{t=\tau_*^\sigma+0}. \tag{27}$$

Relationships (23)-(27) form the perturbed boundary value problem for the Maximum Principle for the optimal control problem (20),(21). Using arguments similar to [19],[20], it can be shown that this boundary value problem has a unique solution which provides global maximum to the objective function J_R. Therefore, finding for each parameter $\sigma > 0$ the unique solution of the perturbed boundary value problem for the Maximum Principle (23)-(27), we simultaneously define the solution $v_*^\sigma(t)$, $w_*^\sigma(t)$, τ_*^σ, $x_*^\sigma(t)$, $y_*^\sigma(t)$, $t \in [0, T]$ of the considered optimal control problem (20),(21).

Now we will investigate properties of the controls $v_*^\sigma(t)$, $w_*^\sigma(t)$, $t \in [0, T]$. In addition to the restriction (22) we will assume that the parameter σ satisfies the inequality

$$\sigma T \le \ln \frac{p_1}{p_1 - p_0}. \tag{28}$$

Let us consider for the switching functions $L_v^\sigma(t)$, $L_w^\sigma(t)$ corresponding Cauchy problems:

$$\begin{cases} \dot{L}_w^\sigma(t) = w_*^\sigma(t)L_w^\sigma(t), \\ L_w^\sigma(T) = p_2, \end{cases} \tag{29}$$

and

$$\begin{cases} \dot{L}_v^\sigma(t) = v_*^\sigma(t)L_v^\sigma(t) - w_*^\sigma(t)L_w^\sigma(t), \\ L_v^\sigma(T) = -p_1. \end{cases} \tag{30}$$

From the Cauchy problem (29) it immediately follows that $L_w^\sigma(t) > 0$ for all $t \in [0, T]$. Then from (24) we obtain that the optimal control $w_*^\sigma(t)$ has the following type

$$w_*^\sigma(t) = w_0, \ t \in [0, T]. \tag{31}$$

From (30) by analogy with Lemma 4.1 it is easy to show the validity of the following statement.

Lemma 4.2. *The switching function $L_v^\sigma(t)$ has at most one zero on the interval $(0, T)$.*

Next, we consider on the segment $[\tau_*^\sigma, T]$ the switching function $G_v^\sigma(t)$. For this function the following statement is true.

Lemma 4.3. *The inequality*

$$G_v^\sigma(\tau_*^\sigma) > 0 \tag{32}$$

is valid.

Proof. Assume the contradiction, i.e. $G_v^\sigma(\tau_*^\sigma) \le 0$. Since the inequality $\dot{G}_v^\sigma(t) = -w_*^\sigma(t)x_*^\sigma(t)L_w^\sigma(t) < 0$ holds for all $t \in [\tau_*^\sigma, T]$, then the function $G_v^\sigma(t)$ is decreasing on this segment. Then the inequality $G_v^\sigma(t) < 0$ is valid for all $t \in (\tau_*^\sigma, T]$. From (26) it immediately follows that $v_*^\sigma(t) = \sigma$, $t \in [\tau_*^\sigma, T]$. This relationship together with the inequality (28) leads to the fact that the equality (17) is false. Therefore, our assumption was wrong. The validity of inequality (32) is established. □

Now we consider the switching function $L_v^\sigma(t)$. For this function the following statement is true.

Lemma 4.4. *The inequality*

$$L_v^\sigma(\tau_*^\sigma) > 0 \tag{33}$$

is valid.

Proof. Again, assume the contradiction, i.e. $L_v^\sigma(\tau_*^\sigma) \le 0$. From Cauchy problem (30) we obtain the formula

$$L_v^\sigma(\tau_*^\sigma) = -(p_1 - p_0) + p_2 w_0 \int_{\tau_*^\sigma}^T e^{-w_0(T-s)} \cdot e^{-\int_{\tau_*^\sigma}^s v_*^\sigma(\xi)d\xi} ds,$$

from which it follows that

$$p_2 \le \frac{p_1 - p_0}{w_0 \int_{\tau_*^\sigma}^T e^{-w_0(T-s)} \cdot e^{-\int_{\tau_*^\sigma}^s v_*^\sigma(\xi)d\xi} ds}. \tag{34}$$

From the condition (27) and Lemma 4.3 it follows the relationship

$$u_0 \varphi_1^\sigma(\tau_*^\sigma) = v_0 G_v^\sigma(\tau_*^\sigma) - v_*^\sigma(\tau_*^\sigma - 0)x_*^\sigma(\tau_*^\sigma)L_v^\sigma(\tau_*^\sigma).$$

Considering the assumption, the inequality (32) and Lemma 2.1, from the last relationship we conclude that $\varphi_1^\sigma(\tau_*^\sigma) > 0$. This inequality can be rewritten as

$$p_2 \left(\left(1 - e^{-w_0(T-\tau_*^\sigma)}\right) - w_0 \int_{\tau_*^\sigma}^T e^{-w_0(T-s)} \cdot e^{-\int_{\tau_*^\sigma}^s v_*^\sigma(\xi)d\xi} ds \right) > p_0. \tag{35}$$

We have the equality

$$\left(1 - e^{-w_0(T-\tau_*^\sigma)}\right) - w_0 \int_{\tau_*^\sigma}^T e^{-w_0(T-s)} \cdot e^{-\int_{\tau_*^\sigma}^s v_*^\sigma(\xi)d\xi} ds =$$

$$= w_0 \int_{\tau_*^\sigma}^T e^{-w_0(T-s)} \cdot \left(1 - e^{-\int_{\tau_*^\sigma}^s v_*^\sigma(\xi)d\xi}\right) ds.$$

Therefore, the expression inside the parentheses of the inequality (35) is positive. Then we have the relationship

$$p_2 > \frac{p_0}{\left(1 - e^{-w_0(T-\tau_*^\sigma)}\right) - w_0 \int_{\tau_*^\sigma}^T e^{-w_0(T-s)} \cdot e^{-\int_{\tau_*^\sigma}^s v_*^\sigma(\xi)d\xi} ds}. \tag{36}$$

Comparing the inequalities (34),(36) we obtain the relationship

$$(p_1 - p_0)\left(1 - e^{-w_0(T-\tau_*^\sigma)}\right) - p_1 w_0 \int_{\tau_*^\sigma}^T e^{-w_0(T-s)} \cdot e^{-\int_{\tau_*^\sigma}^s v_*^\sigma(\xi)d\xi} ds > 0.$$

From this inequality it follows a contradictory chain of expressions:

$$0 < (p_1 - p_0)w_0 \int_{\tau_*^\sigma}^{T} e^{-w_0(T-s)}ds - p_1 w_0 \int_{\tau_*^\sigma}^{T} e^{-w_0(T-s)} \cdot e^{-\int_{\tau_*^\sigma}^{s} v_*^\sigma(\xi)d\xi} ds =$$

$$= (p_1 - p_0)w_0 \int_{\tau_*^\sigma}^{T} e^{-w_0(T-s)} \cdot \left(1 - e^{\int_s^T v_*^\sigma(\xi)d\xi}\right) ds < 0.$$

Therefore, our assumption was wrong. The validity of inequality (33) is established. □

From the properties of the switching functions $L_v^\sigma(t)$, $G_v^\sigma(t)$, inequalities (32),(33) and relationships (25),(26) it follows that the optimal control $v_*^\sigma(t)$ has the type

$$v_*^\sigma(t) = \begin{cases} v_0, & 0 \le t \le \theta_*^\sigma, \\ \sigma, & \theta_*^\sigma < t \le T, \end{cases} \tag{37}$$

where $\theta_*^\sigma \in (\tau_*^\sigma, T]$ is the moment of switching. Solution to the Perturbed Problem 2.2 is complete.

4.3. **Solution to the Game 2.** Let us do the limiting transition $\sigma \to +0$ for the perturbed boundary value problem for the Maximum Principle (23)-(27). The limiting functions $v_*^0(t)$, $w_*^0(t)$, $x_*^0(t)$, $y_*^0(t)$, $t \in [0, T]$ and limiting value $\tau_*^0 \in (0, T)$ under this transition will be controls, trajectories and the moment of "sewing" of the non-perturbed boundary value problem for the Maximum Principle. They satisfy the relationships (23)-(27) in which the value σ is replaced by 0. This boundary value problem has a unique solution, which provides global maximum to the objective function J_R. Therefore, the controls $v_*^0(t)$, $w_*^0(t)$, value $\tau_*^0 \in (0, T)$ and trajectories $x_*^0(t)$, $y_*^0(t)$, are unique solution of the non-perturbed optimal control problem (20),(21). Since for switching functions $L_v^0(t)$, $G_v^0(t)$, $L_w^0(t)$ arguments are valid, which are similar to arguments from subsection 4.2, then we have the statements similar to Lemma 4.3 and Lemma 4.4 and for the controls $v_*^0(t)$, $w_*^0(t)$ the formulas (31),(37) are valid.

The function $u_*^0(t)$ that forms the solution of the non-perturbed Problem 2.1 is defined by relationship (15), in which σ is replaced by 0 as well.

In further considerations we will omit the symbol ^{0}.

From the formulas (15),(17),(37) we conclude that the moments of switching $\tau_*^2 = \tau_* \in (0, T)$, $\theta_*^2 = \theta_* \in (\tau_*, T]$ of the optimal controls $u_*(t)$, $v_*(t)$ are connected by equality

$$\theta_* - \tau_* = \frac{1}{v_0} \ln \frac{p_1}{p_1 - p_0}. \tag{38}$$

For the value θ_* we have the implicit equation $G_v(\theta_*) = 0$.

Next, for the optimal controls $v_*(t)$, $w_*(t)$ and value τ_* we will estimate the value of the objective function J_R. Let the functions $x_*(t)$, $y_*(t)$ be the corresponding optimal trajectories. We will find the conditions for which the inequality from (4) will be valid. From the equations of the system (1) we express the relationships:

$$v_*(t)x_*(t) = u_*(t) - \dot{x}_*(t),\ w_*(t)y_*(t) = u_*(t) - \dot{x}_*(t) - \dot{y}_*(t),\ t \in [0, T]. \tag{39}$$

Substituting these equalities into the objective function J_R and taking into account the formula (15) after corresponding transformations we have the equality

$$J_R = (p_2 - p_1)u_0\tau_* - (p_2 - p_1)(x_*(T) - x_0) - p_2(y_*(T) - y_0).$$

Using the relationships (39) to the last expression we can rewrite it as

$$J_R - p_1 y_0 = (p_2 - p_1) w_0 \int_0^{\tau_*} y_*(s) ds + \tag{40}$$
$$+ (p_2 - p_1)(x_*(\tau_*) + y_*(\tau_*)) + (p_1 x_*(T) - p_2(x_*(T) + y_*(T))).$$

Separately we transform the last component. Taking into account the solutions of Cauchy problems (20),(29),(30) we have the equality

$$p_1 x_*(T) - p_2(x_*(T) + y_*(T)) = -L_v(\tau_*) x_*(\tau_*) - L_w(\tau_*)(x_*(\tau_*) + y_*(\tau_*)).$$

Substituting this expression into the formula (40) and transforming it we obtain the following equality

$$J_R - p_1 y_0 = (p_2 - p_1) w_0 \int_0^{\tau_*} y_*(s) ds + \varphi_1(\tau_*)(x_*(\tau_*) + y_*(\tau_*)) + L_v(\tau_*) y_*(\tau_*). \tag{41}$$

We note that from Lemma 2.1 and Lemma 4.4 it follows positivity of the third component. Then the inequality $J_R > p_1 y_0$ from (4) will be valid if the inequalities:

$$p_2 \geq p_1, \ \varphi_1(\tau_*) \geq 0 \tag{42}$$

will be hold. We check these inequalities for the optimal control $v_*(t)$, $t \in [0, T]$ from (37).

From the relationship (38) for value $\theta_* \in (\tau_*, T]$ we have the restriction

$$\frac{1}{v_0} \ln \frac{p_1}{p_1 - p_0} < \theta_* \leq T. \tag{43}$$

From the Cauchy problem (30) and the definition of switching function $L_v(t)$ we find formula for the value $\varphi_1(\tau_*)$. Substituting into this expression the formula (38) and transforming the obtained relationship we rewrite the formula for $\varphi_1(\tau_*)$ as

$$\varphi_1(\tau_*) = p_2 \left(\frac{p_0}{p_1} - v_0 e^{-w_0(T-\theta_*)} \left(1 - \frac{p_0}{p_1}\right)^{\frac{w_0}{v_0}} \int_0^{-\frac{1}{v_0} \ln\left(1 - \frac{p_0}{p_1}\right)} e^{(w_0 - v_0)s} ds \right) - p_0.$$

The expression inside parentheses is positive. For this it is sufficient to consider on the interval $[0, 1)$ the function

$$F(z) = z - v_0 e^{-w_0(T-\theta_*)} (1 - z)^{\frac{w_0}{v_0}} \int_0^{-\frac{1}{v_0} \ln(1-z)} e^{(w_0 - v_0)s} ds.$$

Evaluating derivative $\dot{F}(z)$ and determining it's sign we can see that the function $F(z)$ takes only positive values.

Then we rewrite the second inequality from (42) as

$$p_2 \geq \frac{p_0}{\frac{p_0}{p_1} - v_0 e^{-w_0(T-\theta_*)} \left(1 - \frac{p_0}{p_1}\right)^{\frac{w_0}{v_0}} \int_0^{-\frac{1}{v_0} \ln\left(1 - \frac{p_0}{p_1}\right)} e^{(w_0 - v_0)s} ds}. \tag{44}$$

Finding the minimum value of the expression on the right handside of inequality (44) by variable $\theta_* \in (\tau_*, T]$, we determine the condition

$$p_2 \geq \frac{p_0}{\frac{p_0}{p_1} - v_0 \left(1 - \frac{p_0}{p_1}\right)^{\frac{w_0}{v_0}} \int_0^{-\frac{1}{v_0} \ln\left(1 - \frac{p_0}{p_1}\right)} e^{(w_0 - v_0)s} ds}. \tag{45}$$

If it is valid then the expression on the right handside of the formula (41) is positive. Moreover, we can see that the expression on the right handside of inequality (45) is greater than value p_1. Then the first inequality from (42) holds. Therefore, the value of objective function J_R satisfies the required inequality from (4).

Therefore, if inequalities (22),(45) hold, then relationships (15),(31),(37) form the optimal solution of the Game 2.

5. **Computer Simulations.** In order to gain further insight into the properties of the two games, we simulated the manufacturer-retailer interactions for different parametric values. First, we select the planning period to be $T = 5$ days, a manufacturer wholesale price $p_1 = \$2$, retailer's price $p_2 = \$4$, the unit cost to be $p_0 = \$1$, and the maximum production rate $u_0 = 1000$ units per day. Note that $p_2 - p_1 = \$2$ and $p_1 - p_0 = \$1$. If the manufacturer is the leader, then the optimal profit for the retailer $J_R = \$4742.68$, is greater than that of the manufacturer $J_M = \$2662.38$. In order to maximize profits, the manufacturer must sell his goods, during the first 3.5 days and then stop production while the retailer must buy the goods during the first 4.3 days and then resell it to the consumer at \$4 per unit. See Figure 1.

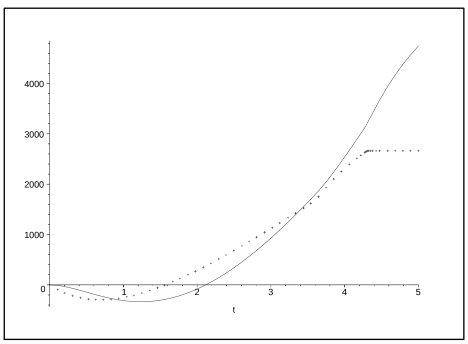


Figure 1

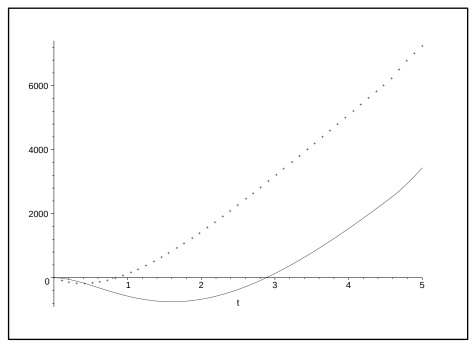


Figure 2

On the graphs the manufacturer's cumulative profit by current time t, $J_M(t)$ is shown dashed and the cumulative retailer's profit $J_R(t)$ is shown as solid, $t \in [0, T]$. These functions are calculated by formulas (2),(3) for controls $u_*(t)$, $v_*(t)$, $w_*(t)$, $t \in [0, T]$ for Game 1 and Game 2, respectively. The maximum payoffs of manufacturer and retailer are $J_M(T)$ and $J_R(T)$.

If the retailer is the leader, then the maximum profit for the retailer under the optimal policy is $J_R = \$4440.98$ and that of the manufacturer is $J_M = \$3341.19$. In both cases the retailer profits are more than the manufacturer.

Keep the planning period at $T = 5$ days and select the parameter values as $p_0 = \$1$, $p_1 = \$3$, and $p_2 = \$5$, so that $p_2 - p_1 = p_1 - p_0 = \$2$. If the manufacturer is the leader, then $J_M = \$5437.52$ and $J_R = \$4397.27$. In order to obtain the maximum profit of \$5437.52, the manufacturer must produce and sell his goods during 3.5 days and then stop the production. The retailer would buy the goods at the highest rate during 4 days to obtain the maximum possible profit, \$4397.27 which is yet less than that of the manufacturer. If the retailer is the leader, then $J_M = \$7240.08$ and $J_R = \$3429.48$. See Figure 2.

6. **Conclusions.** In our work we created a two-dimensional control model to describe vertical integration of manufacturer and retailer as a power structure. We investigated the type of optimal controls by means of switching functions. This allows us to reduce a very complex boundary value problem for the Maximum Principle to one of finite dimensional optimization. While most papers published on

differential games do not have any restrictions on control, by allowing the controls to be bounded we extracted a class of control models which describes economic situation with greater realism.

We summarize the answers to the questions posed in the introduction:

1. What benefit does the Leader gain? When the manufacturer or retailer leads, will the leader be the more power player and receive higher profit?

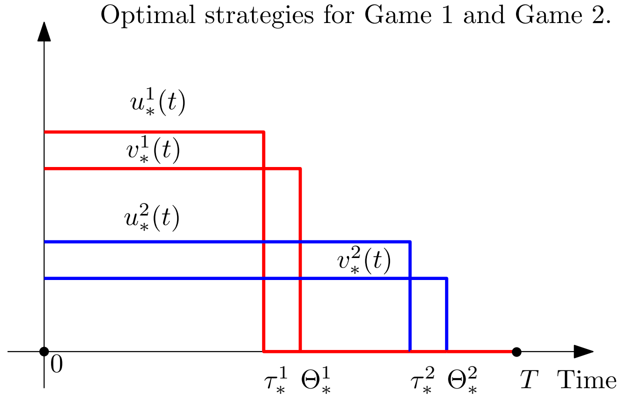
The Leader does not always achieve a higher profit than the Follower. This fact depends on initial parameters of the model, i.e. prices. The computer simulations confirmed the following interesting theoretical result that a player (for example, a manufacturer) always achieves a better profit in the games where he is not the Leader and vice versa ([21]). This can be explained by the different information available to the Leader with respect to the Follower's action. Thus, Game 1 corresponds to minimal Leader's information and maximal Follower's information. In Game 2 we have the opposite situation. Recall that for the manufacturer the maximum profit in Game 2 ($3341.19) is greater than the maximum profit in Game 1 ($2662.38). Likewise, for the retailer the maximum profit in Game 2 ($4440.98) is less than the maximum profit in Game 1 ($4742.68).

2. All else being equal, what is the best optimal policy for the player in Game 1 and Game 2?

2.1. If a manufacturer is the Leader, then the optimal game scenario is given by formulas (5)-(9). Over the planning period $[0, T]$ the manufacturer produces the consumer good until time τ_*^1 at the maximum production rate, then ceases its production. The retailer buys the good until time θ_*^1, $\theta_*^1 > \tau_*^1$ at the maximum purchasing rate then stops buying from manufacturer. Ultimately, the retailer resells his goods to the third party (or his customer) at the maximum rate during the entire planning period $[0, T]$.

2.2. If a retailer (e.g. Wal-Mart) is the Leader, then Game 2 becomes more difficult from mathematical point of view. The optimal game strategy is given by formulas (15),(31) and (37). In order to maximize profit, retailer buys from the manufacturer with the highest rate of purchasing $v_*(t) = v_0$ until time θ_*^2, $\theta_*^2 > \tau_*^2$, then stops buying and constantly resells it. The manufacturer, in turn, produces the good until time τ_*^2, then ceases its production.

Optimal strategies for Game 1 and Game 2.



Next, on the time axis for Game 2 the moments of switching of both players lie to the right of the moments of switching for Game 1. This can be easily seen from the definitions of the switching function $G_v(t)$, the value θ_*^2 and the second inequality from (42). Furthermore, from relationships (7),(9),(38) it follows that in these games the delay between switchings of both players is the same and can be written as

$$\theta_*^i = \tau_*^i + \frac{1}{v_0} \ln \frac{p_1}{p_1 - p_0}, \; i = 1, 2.$$

Optimal policies of the manufacturer and the retailer in Game 1 and Game 2 are shown on the Diagram above.

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