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POSITIVE SOLUTIONS OF A NONLINEAR HIGHER ORDER BOUNDARY-VALUE PROBLEM

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Abstract. The authors study a higher order three point boundary value problem. Estimates for positive solutions are given; these estimates improve some recent results in the literature. Using these estimates, new sufficient conditions for the existence and nonexistence of positive solutions of the problem are obtained. An example illustrating the results is included.

1. **Introduction.** In 1998, Ma [18] considered the second order boundary value problem

$$u''(t) + a(t)f(u(t)) = 0, \quad 0 \le t \le 1, \tag{1}$$

$$u(0) = 0, \quad \alpha u(\eta) = u(1), \tag{2}$$

and gave sufficient conditions under which this problem has at least one positive solution. In 2007, Guo, Sun, and Zhao [11] studied the same question for the third order boundary value problem

$$u'''(t) + h(t)f(u(t)) = 0, \quad 0 \le t \le 1, \tag{3}$$

$$u(0) = u'(0) = 0, \quad u'(1) = \alpha u'(\eta). \tag{4}$$

Motivated by these works, we consider the nonlinear n-th order ordinary differential equation

$$u^{(n)}(t) + g(t)f(u(t)) = 0, \quad 0 \le t \le 1, \tag{5}$$

subject to boundary conditions

$$u^{(i)}(0) = 0, \quad 0 \le i \le n-2, \quad \alpha u^{(n-2)}(\eta) = u^{(n-2)}(1), \tag{6}$$

where

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(H1) $n \geq 3$ is a fixed integer, α and η are constants such that $0 < \eta < 1$ and $1 < \alpha < 1/\eta$, and

(H2) $f : [0, \infty) \to [0, \infty)$ and $g : [0, 1] \to [0, \infty)$ are continuous, and $g(t) \not\equiv 0$ on $[0, 1]$.

Our interest here is in obtaining positive solutions to this boundary-value problem, that is, solutions $u(t)$ of (5)–(6) such that $u(t) > 0$ for $t \in (0, 1)$.

The importance of boundary-value problems in a wide variety of applications in the physical, biological and engineering sciences is now well documented in the literature, and in the last ten years this has become an extremely active area of research. The monographs of Agarwal [1] and Agarwal, O'Regan, and Wong [3] contain excellent surveys of known results. More recent contributions to the study of multipoint boundary-value problems can be found in the papers of Agarwal and Kiguradze [2], Anderson and Davis [4], Cao and Ma [5], Graef, Qian, and Yang [6, 7], Graef and Yang [8, 9], Hu and Wang [12], Infante [13], Infante and Webb [14], Kong and Kong [15], Ma [17, 18, 19], Maroun [20], Raffoul [21], Webb [22, 23], and Zhou and Xu [24].

We need the indicator function I to write the expression of the Green's function for the problem (5)–(6). Recall that if $[a, b] \subset R := (-\infty, +\infty)$ is a closed interval, then the indicator function I of $[a, b]$ is given by

$$I_{[a,b]}(t) = \begin{cases} 1, & \text{if } t \in [a, b], \\ 0, & \text{if } t \notin [a, b]. \end{cases}$$

Let $G_2 : [0, 1] \times [0, 1] \to [0, \infty)$ be defined by

$$G_2(t, s) = \frac{t(1-s)}{1-\alpha\eta} - \frac{\alpha t(\eta - s)}{1-\alpha\eta} I_{[0,\eta]}(s) - (t-s) I_{[0,t]}(s).$$

According to Ma [18], G_2 is the Green's function for the boundary value problem (1)–(2). For $n \geq 3$, we define

$$G_n(t, s) = \int_0^t G_{n-1}(v, s) dv, \quad (t, s) \in [0, 1] \times [0, 1].$$

Then, for $n \geq 3$, $G_n(t, s)$ is the Green's function for the equation

$$u^{(n)}(t) = 0$$

subject to the boundary conditions (6). Moreover, solving the problem (5)–(6) is equivalent to finding a solution to the integral equation

$$u(t) = \int_0^1 G_n(t, s) g(s) f(u(s))\, ds, \quad 0 \leq t \leq 1.$$

It is obvious that

$$G_n(t, s) > 0, \quad \text{for} \quad t, s \in (0, 1) \text{ and } n \geq 3.$$

Throughout this paper, we let

$$F_0 = \limsup_{x \to 0^+} (f(x)/x), \quad f_0 = \liminf_{x \to 0^+} (f(x)/x),$$
$$F_\infty = \limsup_{x \to +\infty} (f(x)/x), \quad f_\infty = \liminf_{x \to +\infty} (f(x)/x).$$

To prove our results, we will use the following fixed point theorem known as the Guo-Krasnosel'skii fixed point theorem [10, 16].

Theorem 1.1. *Let X be a Banach space over the reals, and let $P \subset X$ be a cone in X. Assume that Ω_1 and Ω_2 are bounded open subsets of X with $0 \in \Omega_1 \subset \overline{\Omega_1} \subset \Omega_2$, and let*

$$L : P \cap (\overline{\Omega_2} - \Omega_1) \to P$$

be a completely continuous operator such that either

(K1) $\|Lu\| \le \|u\|$ for $u \in P \cap \partial\Omega_1$ and $\|Lu\| \ge \|u\|$ for $u \in P \cap \partial\Omega_2$, or

(K2) $\|Lu\| \ge \|u\|$ for $u \in P \cap \partial\Omega_1$ and $\|Lu\| \le \|u\|$ for $u \in P \cap \partial\Omega_2$.

Then L has a fixed point in $P \cap (\overline{\Omega_2} - \Omega_1)$.

The next section contains some preliminary lemmas; our main results appear in Sections 3 and 4.

2. Preliminary lemmas.

The following lemmas will be used in the proofs of our main results.

Lemma 2.1. *If $u \in C^n[0,1]$ satisfies the boundary conditions (6) and*

$$u^{(n)}(t) \le 0 \quad \text{for} \quad 0 \le t \le 1, \tag{7}$$

then for each $i = 0, 1, 2, \ldots, n-2$, we have

$$u^{(i)}(t) \ge 0 \quad \text{for} \quad 0 \le t \le 1. \tag{8}$$

Proof. If we define $w(t) = u^{(n-2)}(t)$ for $0 \le t \le 1$, then we have

$$w''(t) \le 0 \quad \text{for} \quad 0 \le t \le 1,$$
$$w(0) = 0, \quad \alpha w(\eta) = w(1).$$

Therefore,

$$u^{(n-2)}(t) = w(t) = \int_0^1 G_2(t,s)(-w''(t))\,dt \ge 0, \quad 0 \le t \le 1.$$

Since $u(0) = u'(0) = \cdots = u^{(n-3)}(0) = 0$, we have

$$u^{(i)}(t) \ge 0 \quad \text{for} \quad 0 \le t \le 1 \text{ and } i = 0, 1, \ldots, n-3,$$

which completes the proof of the lemma. □

The next two lemmas give estimates on the growth of $u(t)$.

Lemma 2.2. *If $u \in C^n[0,1]$ satisfies (6) and (7), then*

$$u(t) \ge t^{n-1}u(1) \quad \text{for} \quad 0 \le t \le 1.$$

Proof. If we define

$$h(t) = u(t) - t^{n-1}u(1), \quad 0 \le t \le 1, \tag{9}$$

then

$$h^{(n)}(t) = u^{(n)}(t) \le 0, \quad 0 \le t \le 1. \tag{10}$$

To prove the lemma, it suffices to show that $h(t) \ge 0$ for $0 \le t \le 1$. It is easy to see from (9) that

$$h(0) = h'(0) = \cdots = h^{(n-2)}(0) = h(1) = 0.$$

Since $h(0) = h(1) = 0$, by the Mean Value Theorem, there exists $r_1 \in (0,1)$ such that $h'(r_1) = 0$. Similarly, $h'(0) = h'(r_1) = 0$ implies that there exists $r_2 \in (0, r_1)$ such that $h''(r_2) = 0$. Continuing this procedure, we can find a sequence of numbers

$$1 > r_1 > r_2 > \cdots > r_{n-2} > 0$$

such that

$$h^{(i)}(r_i) = 0, \quad 1 \le i \le n-2.$$

Since

$$h^{(n)}(t) \le 0 \quad \text{for} \quad 0 \le t \le 1, \tag{11}$$

$h^{(n-2)}$ is concave downward on $(0,1)$. Since $h^{(n-2)}(0) = h^{(n-2)}(r_{n-2}) = 0$, we have

$$h^{(n-2)}(t) \ge 0 \quad \text{on} \quad [0, r_{n-2}] \quad \text{and} \quad h^{(n-2)}(t) \le 0 \quad \text{on} \quad [r_{n-2}, 1].$$

Since $h^{(n-3)}(0) = h^{(n-3)}(r_{n-3}) = 0$, we have

$$h^{(n-3)}(t) \ge 0 \quad \text{on} \quad [0, r_{n-3}] \quad \text{and} \quad h^{(n-3)}(t) \le 0 \quad \text{on} \quad [r_{n-3}, 1].$$

If we continue this procedure, we finally obtain

$$h'(t) \ge 0 \quad \text{on} \quad [0, r_1] \quad \text{and} \quad h'(t) \le 0 \quad \text{on} \quad [r_1, 1]. \tag{12}$$

Combining (12) with the fact that $h(0) = h(1) = 0$ yields

$$h(t) \ge 0 \quad \text{for} \quad 0 \le t \le 1,$$

which completes the proof of the lemma. ☐

Lemma 2.3. *If $u \in C^n[0,1]$ satisfies (6) and (7), then*

$$u(t) \le t^{n-3} u(1) \quad \textit{for} \quad t \in [0,1].$$

Proof. If we define

$$h(t) = t^{n-3} u(1) - u(t), \quad t \in [0,1], \tag{13}$$

then

$$h^{(n)}(t) = -u^{(n)}(t) \ge 0, \quad 0 \le t \le 1. \tag{14}$$

To prove the lemma, it suffices to show that $h(t) \ge 0$ for $0 \le t \le 1$. It is easy to see from (13) that

$$h(0) = h'(0) = \cdots = h^{(n-4)}(0) = h(1) = 0.$$

By the Mean Value Theorem, in view of the fact that $h(0) = h(1) = 0$, there exists $r_1 \in (0,1)$ such that $h'(r_1) = 0$. Because $h'(0) = h'(r_1) = 0$, there exists $r_2 \in (0, r_1)$ such that $h''(r_2) = 0$. If we continue his procedure, then we can find a sequence of numbers

$$1 > r_1 > r_2 > \cdots > r_{n-3} > 0$$

such that

$$h^{(i)}(r_i) = 0, \quad 1 \le i \le n-3.$$

We can also see from (13) that

$$h^{(n-2)}(0) = 0, \quad \alpha h^{(n-2)}(\eta) = h^{(n-2)}(1).$$

Therefore, we have

$$h^{(n-2)}(t) = \int_0^1 G_2(t,s)(-h^{(n)}(s))\, ds \le 0, \quad 0 \le t \le 1.$$

This means that $h^{(n-3)}(t)$ is nonincreasing. Since $h^{(n-3)}(r_{n-3}) = 0$, we have

$$h^{(n-3)}(t) \ge 0 \quad \text{on} \quad [0, r_{n-3}] \quad \text{and} \quad h^{(n-3)}(t) \le 0 \quad \text{on} \quad [r_{n-3}, 1].$$

Since $h^{(n-4)}(0) = h^{(n-4)}(r_{n-4}) = 0$, we have

$$h^{(n-4)}(t) \ge 0 \quad \text{on } [0, r_{n-4}] \quad \text{and} \quad h^{(n-4)}(t) \le 0 \quad \text{on } [r_{n-4}, 1].$$

Continuing in this way, we finally obtain

$$h'(t) \ge 0 \quad \text{on } [0, r_1] \quad \text{and} \quad h'(t) \le 0 \quad \text{on } [r_1, 1]. \tag{15}$$

Combining (15) with the fact that $h(0) = h(1) = 0$ yields

$$h(t) \geq 0 \quad \text{for} \quad 0 \leq t \leq 1,$$

which completes the proof of the lemma. □

The next theorem is an immediate consequence of Lemmas 2.1, 2.2, and 2.3.

Theorem 2.4. *If $u \in C^n[0,1]$ satisfies (6) and (7), then $0 \leq u(t) \leq u(1)$ for $0 \leq t \leq 1$, and*

$$t^{n-3}u(1) \geq u(t) \geq t^{n-1}u(1) \quad \textit{for} \quad 0 \leq t \leq 1. \tag{16}$$

In particular, if $u(t)$ is a nonnegative solution to the problem (5)–(6), then $u(t)$ satisfies (16).

Note that Theorem 2.4 provides both an upper and a lower estimate to each positive solution to the problem (5)–(6).

Guo, Sun, and Zhao [11] obtained the following result.

Lemma 2.5. *Let $1 < \alpha < 1/\eta$. Then*

$$0 \leq G_3(t,s) \leq q(s) \quad \textit{for all} \quad (t,s) \in [0,1] \times [0,1]$$

and

$$\gamma q(s) \leq G_3(t,s) \quad \textit{for all} \quad (t,s) \in [\eta/\alpha, \eta] \times [0,1],$$

where $0 < \gamma = \frac{\eta^2}{2\alpha^2(1+\alpha)} \min\{\alpha - 1, 1\} < 1$, and

$$q(s) = \frac{1+\alpha}{1-\alpha\eta} s(1-s), \quad s \in [0,1]. \tag{17}$$

Remark 2.6. Each time we obtain a pair of upper and lower estimates, we face the question of the sharpness of these estimates. We propose to use a ratio − the ratio of the L^1 norm of the upper estimate on solutions to the L^1 norm of the lower estimate − as a measure of the sharpness. If the ratio is very large, then the gap between the lower and upper estimates is large, and so this indicates that there is room for improvement.

Based on our results in Theorem 2.4, we see that

$$\frac{u(1)}{n} = \int_0^1 t^{n-1}u(1)\,dt \leq \int_0^1 u(t)\,dt = \int_0^1 t^{n-3}u(1)\,dt = \frac{u(1)}{n-2},$$

and so the ratio in our case is $n/(n-2)$.

In order to make a comparison to the results of Guo, Sun, and Zhao above, first note that we must compare with the case of $n = 3$ in our problem. Moreover, their estimates are on the Green's function not the solutions. The ratio of their upper estimate $q(s)$ to their lower estimate $\gamma q(s)$ on $G(t,s)$ gives

$$\frac{1}{\gamma} = \frac{2\alpha^2(1+\alpha)}{\eta^2 \min\{\alpha - 1, 1\}},$$

and that this ratio $1/\gamma$ depends on the parameters α and η. In order to interpret this in terms of the solutions, observe that

$$u(t) = \int_0^1 G_3(t,s)g(s)f(u(s))\,ds \leq \int_0^1 q(s)g(s)f(u(s))\,ds$$

for $t \in [0,1]$. Also,

$$u(t) \geq \gamma \int_0^1 q(s)g(s)f(u(s))\,ds$$

for $t \in [\eta/\alpha, \eta]$. Therefore,

$$\int_0^1 u(\tau)\,d\tau \le \int_0^1 q(s)g(s)f(u(s))\,ds, \tag{18}$$

and

$$\int_0^1 u(\tau)\,d\tau \ge \int_{\eta/\alpha}^{\eta} u(\tau)\,d\tau \ge (\eta - \eta/\alpha)\gamma \int_0^1 q(s)g(s)f(u(s))\,ds. \tag{19}$$

Thus, from (18) and (19), we have that the ratio of the L^1 norms of the upper and lower bounds on the solutions is

$$R = \frac{1}{\gamma(\eta - \eta/\alpha)}.$$

For example, if we let $\alpha = 11/10$ and $\eta = 9/10$, then this ratio becomes $R \approx 766.83$. That is, in this case the ratio of the L_1 norm of the upper estimate to the solution is more than 760 times that of the lower estimate. But with $n = 3$, our result gives this ratio to be 3.

In addition to the fact that our ratio does not depend on the parameters α and η, another positive feature is that our estimates become sharper if the order of the boundary value problem increases. For example, if $n = 12$, then our upper estimate on the solution is $t^9 u(1)$ and our lower estimate is $t^{11}u(1)$, and the ratio of the L_1 norms is $6/5$. This means the upper estimate is just 20% larger than the lower estimate.

3. **Existence of positive solutions.** We begin by introducing some notation. Define

$$A = \int_0^1 G_n(1,s)g(s)s^{n-1}\,ds \quad \text{and} \quad B = \int_0^1 G_n(1,s)g(s)s^{n-3}\,ds.$$

Let $X = C[0,1]$ with the supremum norm

$$\|v\| = \max_{t\in[0,1]} |v(t)|, \quad v \in X,$$

and let

$$P = \big\{v \in X : v(1) \ge 0,\, t^{n-1}v(1) \le v(t) \le v(1)t^{n-3} \text{ on } [0,1]\big\}.$$

Clearly, X is a Banach space and P is a positive cone of X. Define the operator $T : P \to X$ by

$$Tu(t) = \int_0^1 G_n(t,s)g(s)f(u(s))ds, \quad 0 \le t \le 1,\ u \in P.$$

By a standard argument, we can show that $T : P \to X$ is a completely continuous operator. It is obvious that if $u \in P$, then $u(1) = \|u\|$. We see from Theorem 2.4 that if $u(t)$ is a nonnegative solution to the problem (5)–(6), then $u \in P$. In a similar fashion to the proof of Theorem 2.4, we can show that $T(P) \subset P$. To find a positive solution to the problem (5)–(6), we only need to find a fixed point u of T such that $u \in P$ and $u(1) = \|u\| > 0$.

We now give our first existence result.

Theorem 3.1. *If $BF_0 < 1 < Af_\infty$, then the problem (5)–(6) has at least one positive solution.*

Proof. Choose $\varepsilon > 0$ such that $(F_0 + \varepsilon)B \le 1$. There exists $H_1 > 0$ such that

$$f(x) \le (F_0 + \varepsilon)x \quad \text{for } 0 < x \le H_1.$$

For each $u \in P$ with $\|u\| = H_1$, we have

$$\begin{aligned}(Tu)(1) &= \int_0^1 G_n(1,s)g(s)f(u(s))\,ds \\ &\le (F_0 + \varepsilon)\int_0^1 G_n(1,s)g(s)u(s)\,ds \\ &\le (F_0 + \varepsilon)\|u\| \int_0^1 G_n(1,s)g(s)s^{n-3}ds \\ &\le (F_0 + \varepsilon)\|u\|B \le \|u\|,\end{aligned}$$

which means $\|Tu\| \le \|u\|$. If we let $\Omega_1 = \{u \in X : \|u\| < H_1\}$, then

$$\|Tu\| \le \|u\| \quad \text{for } u \in P \cap \partial\Omega_1.$$

Next we construct Ω_2. Since $1 < Af_\infty$, we can choose $c \in (0, 1/4)$ and $\delta > 0$ such that

$$(f_\infty - \delta)\int_c^1 G_n(1,s)g(s)s^{n-1}\,ds > 1.$$

There exists $H_3 > 0$ such that

$$f(x) \ge (f_\infty - \delta)x \quad \text{for } x \ge H_3.$$

Let $H_2 = \max\{H_3 c^{1-n}, 2H_1\}$. Now if $u \in P$ with $\|u\| = H_2$, then for $c \le t \le 1$, we have

$$u(t) \ge t^{n-1}\|u\| \ge c^{n-1}H_2 \ge H_3,$$

and

$$\begin{aligned}(Tu)(1) &\ge \int_c^1 G_n(1,s)g(s)f(u(s))ds \\ &\ge (f_\infty - \delta)\int_c^1 G_n(1,s)g(s)u(s)ds \\ &\ge (f_\infty - \delta)\|u\| \int_c^1 G_n(1,s)g(s)s^{n-1}\,ds \ge \|u\|,\end{aligned}$$

which means $\|Tu\| \ge \|u\|$. So, if we let $\Omega_2 = \{u \in X \mid \|u\| < H_2\}$, then $\overline{\Omega_1} \subset \Omega_2$ and

$$\|Tu\| \ge \|u\| \quad \text{for } u \in P \cap \partial\Omega_2.$$

Since condition (K1) of Theorem 1.1 is satisfied, there exists a fixed point of T in P, and this completes the proof of the theorem. □

The proof of the following theorem is similar to that of Theorem 3.1 and is therefore omitted.

Theorem 3.2. *If $BF_\infty < 1 < Af_0$, then the problem (5)–(6) has at least one positive solution.*

4. **Nonexistence results and an example.** In this section, we establish some nonexistence results for the positive solutions of the problem (5)–(6). We also include an example to illustrate our existence and nonexistence criteria.

Theorem 4.1. *If* $Bf(x) < x$ *for all* $x > 0$*, then the problem* (5)–(6) *has no positive solutions.*

Proof. Assume to the contrary that $u(t)$ is a positive solution of the problem (5)–(6). Then $u \in P$, $u(t) > 0$ for $0 < t \leq 1$, and

$$\begin{aligned} u(1) &= \int_0^1 G_n(1,s)g(s)f(u(s))\,ds \\ &< B^{-1}\int_0^1 G_n(1,s)g(s)u(s)\,ds \\ &\leq B^{-1}u(1)\int_0^1 G_n(1,s)g(s)s^{n-3}\,ds = u(1), \end{aligned}$$

which is a contradiction. □

The proof of the next result is quite similar to that of Theorem 4.1, and so we omit the details.

Theorem 4.2. *If* $Af(x) > x$ *for all* $x > 0$*, then the problem* (5)–(6) *has no positive solutions.*

We conclude our paper with an example to show that our existence and nonexistence results are sharp.

Example 4.3. Consider the boundary-value problem

$$u^{(6)}(t) = 10t \cdot \frac{\lambda u(t)(1+3u(t))}{1+u(t)}, \quad 0 < t < 1, \tag{20}$$

$$u^{(i)}(0) = 0 \text{ for } 0 \leq i \leq 4, \quad u''''(1) = (11/10)\cdot u''''(9/10), \tag{21}$$

where $g(t) = 10t$,

$$f(u) = \frac{\lambda u(1+3u)}{1+u},$$

and $\lambda > 0$ is a parameter. This problem is a special case of the problem (5)–(6) in which $n = 6$, $\alpha = 11/10$, and $\eta = 9/10$. It is easy to see that $F_0 = f_0 = \lambda$, $F_\infty = f_\infty = 3\lambda$, and $\lambda u \leq f(u) \leq 3\lambda u$ for $u \geq 0$. For the problem (20)–(21), calculations show that

$$A = 52112120831/665280000000 \quad \text{and} \quad B = 87187129/756000000.$$

By Theorem 3.1, we have that if

$$4.2555 \approx \frac{1}{3A} < \lambda < \frac{1}{B} \approx 8.6710,$$

then problem (5)–(6) has at least one positive solution. By Theorems 4.1 and 4.2, we see that if either

$$\lambda < \frac{1}{3B} \approx 2.8903 \quad \text{or} \quad \lambda > \frac{1}{A} \approx 12.7664$$

then (20)–(21) has no positive solutions.

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