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POSITIVE SOLUTIONS TO A FOURTH ORDER THREE POINT BOUNDARY VALUE PROBLEM

John R. Graef

Department of Mathematics
University of Tennessee at Chattanooga, Chattanooga, TN 37403, USA

Johnny Henderson

Department of Mathematics
Baylor University, Waco, TX 76798, USA

Bo Yang

Department of Mathematics and Statistics
Kennesaw State University, Kennesaw, GA 30144, USA

Abstract. We consider a three-point boundary value problem for the beam equation. Some *a priori* estimates to the positive solutions for the boundary value problem are obtained. Sufficient conditions for the existence and nonexistence of positive solutions for the boundary value problem are established. The results are illustrated with an example.

1. **Introduction.**

Fourth order ordinary differential equations are models for bending or deformation of elastic beams, and therefore have important applications in engineering and physical sciences. Two-point and multi-point boundary value problems for fourth order ordinary differential equations have attracted a lot of attention recently. Many authors have studied the beam equation under various boundary conditions and by different approaches. In 2003, Ma [10] considered the fourth order right focal two point boundary value problem

$$u''''(t) = \lambda f(t, u(t), u'(t)), \quad 0 < t < 1, \tag{1}$$

$$u(0) = u'(0) = u''(1) = u'''(1) = 0. \tag{2}$$

In 2006, Anderson and Avery [1] considered the fourth order right focal four-point boundary value problem

$$u''''(t) + f(u(t)) = 0, \quad 0 < t < 1,$$

$$u(0) = u'(q) = u''(r) = u'''(1) = 0,$$

and in 2004, Kosmotov [8] studied the existence of countably many solutions for the fourth order two-point conjugate boundary value problem

$$u''''(t) = g(t)f(u(t)), \quad 0 < t < 1, \tag{3}$$

$$u(0) = u'(0) = u'(1) = u(1) = 0. \tag{4}$$

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In addition, in 2003, Graef *et al.* [4] considered positive solutions for the fourth order nonlocal boundary value problem

$$u''''(t) = g(t)f(u(t)), \quad 0 < t < 1,$$

$$u(0) = u'(1) = u''(0) = u''(p) - u''(1) = 0,$$

and in 2004, Henderson and Ma [7] studied some uniqueness questions for fourth order nonlocal boundary value problems. For some other results on boundary-value problems for the beam equation, we refer the reader to the papers of Davis and Henderson [2], Eloe, Henderson, and Kosmatov [3], Graef and Yang [5], Gupta [6], Ma and Wang [11], and Yang [12].

Motivated by these works, in this paper we consider the fourth order beam equation

$$u''''(t) = g(t)f(u(t)), \quad 0 \le t \le 1, \tag{5}$$

together with the boundary conditions

$$u(0) = u'(0) = u''(\beta) = u''(1) = 0. \tag{6}$$

Throughout this paper, we assume that

(H) $\beta \in [2/3, 1)$ is a constant, $f : [0,\infty) \to [0,\infty)$ and $g : [0,1] \to [0,\infty)$ are continuous functions, and $g(t) \not\equiv 0$ on $[0,1]$.

By a positive solution of (5)–(6), we mean a solution $u(t)$ such that $u(t) > 0$ for $0 < t < 1$. Our motivation is actually two fold. First, the boundary conditions (6) are related to (2). To see this, note that if $u \in C^4[0,1]$ satisfies (6), then by the Mean Value Theorem there exists $\alpha \in (\beta, 1)$ such that $u'''(\alpha) = 0$. If we let $\beta \to 1^-$, then $\alpha \to 1^-$, and the boundary conditions (6) "tend to" (2). In this sense, (2) is the limiting case of (6).

Secondly, the boundary conditions (6) are also related to (4). If $u \in C^4[0,1]$ satisfies (4), then $u(0) = u(1) = 0$ and $u'(0) = u'(1) = 0$. Since $u(0) = u(1) = 0$, there exists $\gamma \in (0,1)$ such that $u'(\gamma) = 0$. Since $u'(0) = u'(\gamma) = u'(1) = 0$, there exist $\beta \in (0,\gamma)$ and $\alpha \in (\gamma, 1)$ such that $u''(\beta) = u''(\alpha) = 0$. Therefore, we have $u(0) = u'(0) = u''(\beta) = u''(\alpha) = 0$, where $\beta < \alpha$. Hence, to fully understand boundary conditions (4), we need to study (6).

This paper is organized as follows. In Section 2, we give the Green's function for the problem (5)–(6), state the Krasnosel'skii fixed point theorem, and fix some notations. In Section 3, we present some *a priori* estimates to positive solutions to the problem (5)–(6). In Section 4, we establish some existence and nonexistence results for positive solutions to the problem (5)–(6), and we give an example to illustrate our results.

2. **Preliminaries.**

The Green's function $G : [0,1] \times [0,1] \to [0,\infty)$ for the problem consisting of the equation

$$u''''(t) = 0$$

and the boundary conditions (6) is

$$G(t,s)=\begin{cases}\dfrac{(1-s)(3\beta-t)t^2}{6(1-\beta)}, & \text{if } s\geq\beta \text{ and } s\geq t,\\ \dfrac{(3t-s)s^2}{6}, & \text{if } s\leq\beta \text{ and } s\leq t,\\ \dfrac{(3s-t)t^2}{6}, & \text{if } t\leq s\leq\beta,\\ \dfrac{(1-s)t^2(3\beta-t)}{6(1-\beta)}+\dfrac{(t-s)^3}{6}, & \text{if } \beta\leq s\leq t.\end{cases}$$

Then problem (5)–(6) is equivalent to the integral equation

$$u(t)=\int_0^1 G(t,s)g(s)f(u(s))\,ds,\quad 0\leq t\leq 1.$$

It is easy to verify that G is a continuous function and that $G(t,s)>0$ if $t,s\in(0,1)$. We will need the following fixed point theorem, which is due to Krasnosel'skii [9], to prove some of our results.

Theorem 2.1. *Let $(X,\|\cdot\|)$ be Banach space over the reals, and let $P\subset X$ be a cone in X. Let H_1 and H_2 be real numbers such that $H_2>H_1>0$, and let*

$$\Omega_i=\{v\in X\mid \|v\|<H_i\},\quad i=1,2.$$

Let $L:P\cap(\overline{\Omega_2}-\Omega_1)\to P$ be a completely continuous operator such that, either
(K1) $\|Lv\|\leq\|v\|$ if $v\in P\cap\partial\Omega_1$, and $\|Lv\|\geq\|v\|$ if $v\in P\cap\partial\Omega_2$, or
(K2) $\|Lv\|\geq\|v\|$ if $v\in P\cap\partial\Omega_1$, and $\|Lv\|\leq\|v\|$ if $v\in P\cap\partial\Omega_2$.
Then L has a fixed point in $P\cap(\overline{\Omega_2}-\Omega_1)$.

For the rest of this paper, we let $X=C[0,1]$ with the norm

$$\|v\|=\max_{t\in[0,1]}|v(t)|\quad\text{for all}\quad v\in X.$$

Clearly, X is a Banach space. We define $Y=\{v\in X\mid v(t)\geq 0 \text{ for } 0\leq t\leq 1\}$, and define the operator $T:Y\to X$ by

$$(Tu)(t)=\int_0^1 G(t,s)g(s)f(u(s))\,ds,\quad 0\leq t\leq 1.$$

It is clear that if (H) holds, then $T:Y\to Y$ is a completely continuous operator. We also define the constants

$$F_0=\limsup_{x\to 0^+}\frac{f(x)}{x},\quad f_0=\liminf_{x\to 0^+}\frac{f(x)}{x},$$
$$F_\infty=\limsup_{x\to +\infty}\frac{f(x)}{x},\quad f_\infty=\liminf_{x\to +\infty}\frac{f(x)}{x}.$$

These constants associated with the function f will be used in Section 4.

3. **Estimates for Positive Solutions.** In this section, we shall give some estimates for positive solutions of the problem (5)–(6). To this purpose, we define the function $a:[0,1]\to[0,1]$ by

$$a(t)=(3t^2-t^3)/2.$$

Lemma 3.1. *If $u \in C^4[0,1]$ satisfies the boundary conditions (6), and*

$$u''''(t) \geq 0 \quad \textit{for} \quad 0 \leq t \leq 1, \tag{7}$$

then $u(t) \geq 0$ for $0 \leq t \leq 1$.

Proof. The lemma follows easily from the fact that $G(t,s) \geq 0$ for t, $s \in [0,1]$. □

Lemma 3.2. *$tG(1,s) \geq G(t,s)$ for $t, s \in [0,1]$.*

Proof. If $s \geq \beta$ and $s \geq t$, then

$$tG(1,s) - G(t,s) = \frac{t(1-s)[(1-s)^2(1-\beta) + (1-t)(3\beta - 1 - t)]}{6(1-\beta)} \geq 0.$$

If $s \leq t$ and $s \leq \beta$, then

$$tG(1,s) - G(t,s) = \frac{(1-t)s^3}{6} \geq 0.$$

If $t \leq s \leq \beta$, then

$$tG(1,s) - G(t,s) = \frac{t((2s-t)(s-t) + s^2(1-s))}{6} \geq 0.$$

If $\beta \leq s \leq t$, then

$$\begin{aligned} tG(1,s) - G(t,s) &= \frac{(1-t)(-\beta t^2 + t^2 s + 2\beta t - 2st + s^3 - s^3\beta)}{6(1-\beta)} \\ &= \frac{(1-t)[(s-\beta)(1-t)^2 + \beta(1-s^3) - s + s^3]}{6(1-\beta)} \\ &\geq \frac{(1-t)[(s-\beta)(1-t)^2 + (2/3)(1-s^3) - s + s^3]}{6(1-\beta)} \\ &\geq 0. \end{aligned}$$

In the last inequality, we used the fact that $(2/3)(1-s^3) - s + s^3 \geq 0$, for $2/3 \leq s \leq 1$. The proof is complete. □

Lemma 3.3. *If $u''''(t) \geq 0$ on $[0,1]$ and $u(t)$ satisfies the boundary conditions (6), then*

$$u(t) \leq tu(1) \quad \textit{for} \quad 0 \leq t \leq 1. \tag{8}$$

Proof. If $u''''(t) \geq 0$ on $[0,1]$ and $u(t)$ satisfies the boundary conditions (6), then

$$u(t) = \int_0^1 G(t,s)u''''(s)\,ds \leq t\int_0^1 G(1,s)u''''(s)\,ds = tu(1), \quad 0 \leq t \leq 1,$$

by Lemma 3.2. This proves the lemma. □

Lemma 3.4. *If $u''''(t) \geq 0$ on $[0,1]$ and $u(t)$ satisfies the boundary conditions (6), then*

$$u(t) \geq a(t)u(1) \quad \textit{for} \quad 0 \leq t \leq 1. \tag{9}$$

Proof. Let $h(t) = u(t) - a(t)u(1)$, $0 \leq t \leq 1$. Then,

$$h'(t) = u'(t) - u(1)(6t - 3t^2)/2,$$
$$h''(t) = u''(t) - 3(1-t)u(1),$$
$$h''''(t) = u''''(t) \geq 0.$$

From the above equations we see that $h(0) = h'(0) = h''(1) = h(1) = 0$. By the Mean Value Theorem, $h(0) = h(1) = 0$ implies there exists $r \in (0,1)$ such that

$h'(r) = 0$. Since $h'(0) = h'(r) = 0$, there exists $s \in (0, r)$ such that $h''(s) = 0$. Note that h'' is convex. Therefore, we have $h''(t) \geq 0$ on $[0, s]$ and $h''(t) \leq 0$ on $[s, 1]$. Since $h'(0) = 0$, we have $h'(t) \geq 0$ on $[0, s]$. Since $h(0) = 0$, we have $h(t) \geq 0$ on $[0, s]$. Because h is concave on $[s, 1]$ and $h(s) \geq 0 = h(1)$, we have $h(t) \geq 0$ on $[s, 1]$. Thus, $h(t) \geq 0$ on $[0, 1]$, and this completes the proof of the lemma. □

Theorem 3.5. *Suppose that* (H) *holds. If $u(t)$ is a nonnegative solution to the problem* (5)–(6)*, then $u(t)$ satisfies* (8) *and* (9)*.*

Proof. If $u(t)$ is a nonnegative solution to the problem (5)–(6), then $u(t)$ satisfies the boundary conditions (6), and

$$u''''(t) = g(t)f(u(t)) \geq 0, \quad 0 \leq t \leq 1.$$

The conclusion follows directly from Lemmas 3.3 and 3.4. □

4. **Existence and Nonexistence Results.** First, we define some important constants:

$$A = \int_0^1 G(1, s)g(s)a(s)\, ds, \quad B = \int_0^1 G(1, s)g(s)s\, ds.$$

We also define

$$P = \{v \in X : v(1) \geq 0,\ a(t)v(1) \leq v(t) \leq tv(1) \text{ on } [0, 1]\}.$$

Clearly P is a positive cone in X. It is obvious that, if $u \in P$, then $u(1) = \|u\|$. We see from Theorem 3.5 that, if $u(t)$ is a nonnegative solution to the problem (5)–(6), then $u \in P$. In a similar fashion to Theorem 3.5, we can show that $T(P) \subset P$. To find a positive solution to the problem (5)–(6), we need only to find a fixed point u of T such that $u \in P$ and $u(1) > 0$.

The next two theorems provide sufficient conditions for the existence of at least one positive solution for the problem (5)–(6).

Theorem 4.1. *Suppose that* (H) *holds. If $BF_0 < 1 < Af_\infty$, then the problem* (5)–(6) *has at least one positive solution.*

Proof. First, we choose $\varepsilon > 0$ such that $(F_0 + \varepsilon)B \leq 1$. By the definition of F_0, there exists $H_1 > 0$ such that $f(x) \leq (F_0 + \varepsilon)x$ for $0 < x \leq H_1$. Now for each $u \in P$ with $\|u\| = H_1$, we have

$$\begin{aligned}(Tu)(1) &= \int_0^1 G(1, s)g(s)f(u(s))\, ds \\ &\leq \int_0^1 G(1, s)g(s)(F_0 + \varepsilon)u(s)\, ds \\ &\leq (F_0 + \varepsilon)\|u\| \int_0^1 G(1, s)g(s)\, s\, ds \\ &= (F_0 + \varepsilon)\|u\|B \leq \|u\|,\end{aligned}$$

which means $\|Tu\| \leq \|u\|$. Thus, if we let $\Omega_1 = \{u \in X \mid \|u\| < H_1\}$, then

$$\|Tu\| \leq \|u\| \text{ for } u \in P \cap \partial\Omega_1.$$

To construct Ω_2, we choose $\delta > 0$ and $\tau \in (0, 1/4)$ such that

$$(f_\infty - \delta) \int_\tau^1 G(1, s)g(s)a(s)\, ds \geq 1.$$

There exists $H_3 > 2H_1$ such that $f(x) \ge (f_\infty - \delta)x$ for $x \ge H_3$. Let $H_2 = H_3/\tau^2$. If $u \in P$ such that $\|u\| = H_2$, then for each $t \in [\tau, 1]$, we have

$$u(t) \ge H_2 a(t) \ge H_2 t^2 \ge H_2 \tau^2 \ge H_3.$$

Therefore, for each $u \in P$ with $\|u\| = H_2$, we have

$$\begin{aligned}(Tu)(1) &= \int_0^1 G(1,s)g(s)f(u(s))\,ds \\ &\ge \int_\tau^1 G(1,s)g(s)f(u(s))\,ds \\ &\ge \int_\tau^1 G(1,s)g(s)(f_\infty - \delta)u(s)\,ds \\ &\ge \int_\tau^1 G(1,s)g(s)a(s)\,ds \cdot (f_\infty - \delta)\|u\| \ge \|u\|,\end{aligned}$$

which means $\|Tu\| \ge \|u\|$. Thus, if we let $\Omega_2 = \{u \in X \mid \|u\| < H_2\}$, then $\overline{\Omega_1} \subset \Omega_2$, and

$$\|Tu\| \ge \|u\| \quad \text{for} \quad u \in P \cap \partial\Omega_2.$$

Now condition (K1) of Theorem 2.1 is satisfied, and so there exists a fixed point of T in $P \cap (\overline{\Omega_2} - \Omega_1)$. The proof is now complete. □

The following theorem is a companion result to Theorem 4.1. Its proof is very similar to the proof of Theorem 4.1 and is therefore omitted.

Theorem 4.2. *Suppose that* (H) *holds. If* $BF_\infty < 1 < Af_0$, *then the problem* (5)–(6) *has at least one positive solution.*

The next two theorems provide sufficient conditions for the nonexistence of positive solutions to the problem (5)–(6).

Theorem 4.3. *Suppose that* (H) *holds. If* $Bf(x) < x$ *for all* $x > 0$, *then the problem* (5)–(6) *has no positive solutions.*

Proof. Assume to the contrary that $u(t)$ is a positive solution of the problem (5)–(6). Then, $u \in P$, $u(t) > 0$ for $0 < t \le 1$, and

$$\begin{aligned}u(1) &= \int_0^1 G(1,s)g(s)f(u(s))\,ds \\ &< B^{-1}\int_0^1 G(1,s)g(s)u(s)\,ds \\ &\le B^{-1}u(1)\int_0^1 G(1,s)g(s)s\,ds \\ &= B^{-1}Bu(1) = u(1),\end{aligned}$$

which is a contradiction. The proof is complete. □

The proof of the following theorem is similar to the one above.

Theorem 4.4. *Suppose that* (H) *holds. If* $Af(x) > x$ *for all* $x > 0$, *then the problem* (5)–(6) *has no positive solutions.*

We conclude this paper with an example.

Example 4.5. Consider the boundary value problem

$$u''''(t) = \lambda(t + 2t^2)u(t)(1 + 3u(t))/(1 + u(t)), \quad 0 \le t \le 1, \tag{10}$$

$$u(0) = u'(0) = u''(3/4) = u''(1) = 0, \tag{11}$$

where $\lambda > 0$ is a parameter. In this example, $\beta = 3/4$, $g(t) = t + 2t^2$, and $f(u) = \lambda u(1 + 3u)/(1 + u)$. It is easy to see that $f_0 = F_0 = \lambda$, $f_\infty = F_\infty = 3\lambda$, and

$$\lambda x < f(x) < 3\lambda x \quad \text{for} \quad x > 0.$$

Calculations indicate that $A = 586631/7741440$ and $B = 14699/161280$. By Theorem 4.1, if

$$4.398813 \approx 1/(3A) < \lambda < 1/B \approx 10.972174,$$

then the problem (10)–(11) has at least one positive solution. From Theorems 4.3 and 4.4, we see that, if

$$\lambda < 1/(3B) \approx 3.657391 \quad \text{or} \quad \lambda > 1/A \approx 13.196439,$$

then the problem (10)–(11) has no positive solutions. This example shows that our existence and nonexistence results are quite sharp indeed.

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E-mail address: `John-Graef@utc.edu`
E-mail address: `Johnny_Henderson@baylor.edu`
E-mail address: `byang@kennesaw.edu`