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APPROXIMATING THE BASIN OF ATTRACTION OF TIME-PERIODIC ODES BY MESHLESS COLLOCATION OF A CAUCHY PROBLEM

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Abstract. The basin of attraction of an equilibrium or periodic orbit can be determined by sublevel sets of a Lyapunov function. A Lyapunov function is a function with negative orbital derivative, which is defined by $LV(t,x) = \langle \nabla_x V(t,x), f(t,x)\rangle + \partial_t V(t,x)$. We construct a Lyapunov function by approximately solving a Cauchy problem with a linear PDE for its orbital derivative and boundary conditions on a non-characteristic hypersurface. For the approximation we use meshless collocation. We describe the general approximate reconstruction of multivariate functions, which are periodic in one variable, from discrete data sets and derive error estimates. This method has already been applied to autonomous dynamical systems. In this paper, however, we consider a time-periodic ODE $\dot{x} = f(t,x)$, $x \in \mathbb{R}^n$, and study the basin of attraction of an exponentially asymptotically stable periodic orbit.

1. **Introduction.** In this article we consider the time-periodic ODE $\dot{x} = f(t,x)$, $f \in C^\sigma(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$, $\sigma \geq 1$, and assume that Γ is an exponentially asymptotically stable periodic orbit with basin of attraction $A(\Gamma)$. A Lyapunov function is a function with negative orbital derivative, given by $LV(t,x) = \langle \nabla_x V(t,x), f(t,x)\rangle + \partial_t V(t,x)$. If V is such a Lyapunov function and if s_V is a good approximation to V, which means that it satisfies $|LV(t,x) - Ls_V(t,x)| \leq \epsilon$, then we can conclude

$$Ls_V(t,x) \leq \epsilon + LV(t,x) < 0$$

for sufficiently small $\epsilon > 0$. Hence, s_V is, in principle, a Lyapunov function itself and can be used to determine the basin of attraction.

We will show in Theorem 2.4 that there is a Lyapunov function $V \in C^\sigma(A(\Gamma) \setminus \Gamma, \mathbb{R})$ satisfying

$$\begin{aligned} LV(t,x) &= -c_1 \text{ for all } (t,x) \in A(\Gamma) \setminus \Gamma \text{ and} \qquad (1)\\ V(t,x) &= c_2 \text{ for all } x \in \Omega, \qquad (2) \end{aligned}$$

where $c_1 > 0$, $c_2 \in \mathbb{R}$ and Ω is a non-characteristic hypersurface, cf. Definition 2.2. Equation (1) is a linear first order partial differential equation with boundary conditions (2).

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Hence, it seems to be natural to approximate the Lyapunov function V by choosing *collocation* points $\widetilde{x}_j := (t_j, x_j) \in A(\Gamma) \setminus \Gamma$, $1 \leq j \leq N$ and $\widetilde{x}_j := (t_j, x_j) \in \Omega$, $N+1 \leq j \leq N+M$ to enforce the collocation conditions for the approximation s_V

$$Ls_V(t_j, x_j) = -c_1 \qquad 1 \leq j \leq N, \tag{3}$$
$$s_V(t_j, x_j) = c_2 \qquad N+1 \leq j \leq N+M. \tag{4}$$

Such a problem can be solved efficiently within the framework of generalized interpolation, as it has recently been done for Lyapunov functions of autonomous systems (see [6]). The theory was developed in a general framework based upon earlier results on generalized interpolation in [3, 4, 8, 13] and also applies to this more general situation. The meshless reconstruction of the function V and the error estimates face the problem that V and thus s_V are periodic in the t-variable. Therefore, a kernel is constructed which is periodic in the t-argument. The t-periodic case without boundary conditions was studied in [7].

In Section 2 the existence and smoothness of the Lyapunov function satisfying the Cauchy problem (1)-(2) is shown. In Section 3 the meshless reconstruction of a multivariate function which is periodic in one variable is discussed and applied to the construction of a Lyapunov function. We also give an example comparing this approach to [7], which considered a first-order PDE without boundary conditions.

2. **Dynamical Systems.**

We consider a time-periodic ODE of the form

$$\dot{x} = f(t, x) \tag{5}$$

where $f(t+T, x) = f(t, x)$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}^n$ and $f \in C^\sigma(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$, $\sigma \geq 1$. The ODE induces a dynamical system on the cylinder $S^1_T \times \mathbb{R}^n$ with semi-flow $S_t \colon S^1_T \times \mathbb{R}^n \to S^1_T \times \mathbb{R}^n$,

$$S_t(t_0, x) = ((t_0 + t) \bmod T, x(t; t_0, x_0)),$$

where $x(t; t_0, x_0)$ denotes the solution $x(t)$ of (5) at time t with initial condition $x(t_0) = x_0$.

We assume that the system has a periodic solution $x(t)$ defining the periodic orbit Γ; without loss of generality we assume that $\Gamma = S^1_T \times \{0\}$ is a solution, i.e. $f(t, 0) = 0$. Moreover, we assume that Γ is exponentially asymptotically stable, i.e. all Floquet exponents are strictly negative. The set of all initial values such that the corresponding solutions approach Γ defines the basin of attraction $A(\Gamma) = \{(t_0, x_0) \in S^1_T \times \mathbb{R}^n \mid \lim_{t\to\infty} \operatorname{dist}(S_t(t_0, x_0), \Gamma) = 0\}$.

2.1. **Basin of attraction and Lyapunov functions.**

The basin of attraction can be determined using a Lyapunov function. The main characteristic of a Lyapunov function $V \colon S^1_T \times \mathbb{R}^n \to \mathbb{R}$ is that V is decreasing along solutions of (5). This is expressed by a negative orbital derivative, i.e. the derivative along solutions of (5). The orbital derivative of a function $V \in C^1(S^1_T \times \mathbb{R}^n, \mathbb{R})$ is given by

$$LV(t, x) = \langle \nabla_x V(t, x), f(t, x) \rangle + \partial_t V(t, x).$$

Note that the chain rule implies $LV(t, x) = \frac{d}{d\tau} V(S_\tau(t, x))\big|_{\tau=0}$.

For Theorem 2.1 we assume that the function V has negative orbital derivative in a sublevel set K with the exception of a certain set E. If we know that $E \subseteq A(\Gamma)$, then $K \subseteq A(\Gamma)$ by the following well-known theorem. The reason for this exceptional set is that the function s_V, which we will construct later, has no negative orbital derivative in a neighborhood of Γ in general. A set E with $E \subseteq A(\Gamma)$ will later be obtained by considering a local Lyapunov function.

Theorem 2.1 ([7, Theorem 2.4]). *Consider (5) with $f \in C^1(S^1_T \times \mathbb{R}^n, \mathbb{R}^n)$ and let $\Gamma = S^1_T \times \{0\}$ be a periodic orbit of (5). Let $B \subseteq \mathbb{R}^n$ be an open neighborhood of 0.*

Let $V \in C^1(S^1_T \times \mathbb{R}^n, \mathbb{R})$, $K \subseteq S^1_T \times B$ be a compact set and $\Gamma \subseteq E \subseteq S^1_T \times B$ be an open set. Let

1. $\Gamma \subseteq \overset{\circ}{K}$,
2. $LV(t,x) < 0$ *for all* $(t,x) \in K \setminus E$,
3. $K = \{(t,x) \in S^1_T \times B \mid V(t,x) \le R\}$ *for* $R \in \mathbb{R}$, *i.e. K is a sublevel set of V*,
4. $E \subseteq A(\Gamma)$.

Then $K \subseteq A(\Gamma)$.

2.2. **Existence of Lyapunov functions.** We can easily calculate a Lyapunov function for a linear autonomous differential equation of the form $\dot{x} = Ax$, where $A \in \mathbb{R}^{n\times n}$. This enables us to find a local Lyapunov function for special systems which are adjacent to an autonomous one, i.e. they are of the form

$$\dot{x} = f(t,x,\lambda) = g(x) + h(t,x,\lambda) \tag{6}$$

with λ small and some conditions on g and h, cf. [7, Proposition 2.6]. This function is called a local Lyapunov function, since the orbital derivative is negative only in a small neighborhood $\|x\| \le \delta$ of the zero solution.

On the other hand we have the existence Theorem 2.4 for a global Lyapunov function which has negative orbital derivative in the whole basin of attraction, satisfying an equation for its orbital derivative. However, this Lyapunov function cannot be calculated in general, but it will be approximated by meshless collocation.

In contrast to [7, Theorem 2.7] where the Lyapunov function with $LV(t,x) = -\|x\|^2$ was considered, we study a Lyapunov function satisfying $LV(t,x) = -c_1$. This Lyapunov function is only defined on $A(\Gamma) \setminus \Gamma$ and tends to $-\infty$ for $x \to \Gamma$. By fixing V on a non-characteristic hypersurface, V is uniquely determined as the solution of a non-characteristic Cauchy problem with a linear first-order PDE.

Let us first define a *non-characteristic hypersurface* which is a non-characteristic datum manifold; the definition is similar to the autonomous case, cf. [5, Definition 2.36]. We show in Lemma 2.3 how to obtain a non-characteristic hypersurface through a Lyapunov function.

Definition 2.2 (Non-characteristic hypersurface). *Consider $\dot{x} = f(t,x)$, where $f \in C^\sigma(S^1_T \times \mathbb{R}^n, \mathbb{R}^n)$, $\sigma \ge 1$. Let $h \in C^\sigma(S^1_T \times \mathbb{R}^n, \mathbb{R})$. The set $\Omega \subseteq S^1_T \times \mathbb{R}^n$ is called a* non-characteristic hypersurface *if*

1. Ω *is compact in* $S^1_T \times \mathbb{R}^n$,
2. $h(t,x) = 0$ *if and only if* $(t,x) \in \Omega$,
3. $Lh(t,x) < 0$ *holds for all* $(t,x) \in \Omega$, *and*
4. *for each* $(t,x) \in A(\Gamma)\setminus\Gamma$ *there is a time* $\theta(t,x) \in \mathbb{R}$ *such that* $S_{\theta(t,x)}(t,x) \in \Omega$.

An example for a non-characteristic hypersurface is the level set of a Lyapunov function within its basin of attraction. The proof of the following lemma is similar to [5, Lemma 2.37].

Lemma 2.3 (Level sets define a non-characteristic hypersurface). *Let $W \in C^1(S^1_T \times \mathbb{R}^n, \mathbb{R}^n)$ be a Lyapunov function. In particular, let $K \subseteq S^1_T \times \mathbb{R}^n$ be a compact set and $K \subseteq B \subseteq S^1_T \times \mathbb{R}^n$ be an open set such that*

1. $\Gamma \subseteq \overset{\circ}{K}$,
2. $W(t,x) = 0$ *for all* $(t,x) \in \Gamma$,
3. $LW(t,x) < 0$ *for all* $x \in K \setminus \Gamma$, *where L denotes the orbital derivative*,
4. $K = \{(t,x) \in B \mid W(t,x) \le R\}$ *for an* $R \in \mathbb{R}^+$.

Then each set $\Omega_r := \{(t,x) \in B \mid W(t,x) = r\}$ *with* $0 < r \leq R$ *is a non-characteristic hypersurface.*

Now we prove the existence theorem of the function V. The proof is similar to the autonomous case, cf. [5, Theorem 2.38], and follows the ideas of [2, Theorem V 2.9].

Theorem 2.4 (Existence of V)**.** *Let* $\dot{x} = f(t,x)$*,* $f \in C^\sigma(S^1_T \times \mathbb{R}^n, \mathbb{R}^n)$*,* $\sigma \geq 1$*. Let* $\Gamma = S^1_T \times \{0\}$ *be an exponentially asymptotically stable solution and let* Ω *be a non-characteristic hypersurface.*

Then for all $c_1 \in \mathbb{R}^+$ *and* $c_2 \in \mathbb{R}$ *there is a function* $V \in C^\sigma(A(\Gamma) \setminus \Gamma, \mathbb{R})$ *satisfying*

$$\begin{aligned} LV(t,x) &= -c_1 \text{ for all } (t,x) \in A(\Gamma) \setminus \Gamma \text{ and} \\ V(t,x) &= c_2 \text{ for all } (t,x) \in \Omega. \end{aligned}$$

Proof. We first show that a function $\theta \in C^\sigma(A(\Gamma) \setminus \Gamma, \mathbb{R})$ satisfying

$$S_\tau(t,x) \in \Omega \Leftrightarrow \tau = \theta(t,x) \tag{7}$$

exists and satisfies $\theta'(t,x) = -1$. We define θ implicitly by $h(S_{\theta(t,x)}(t,x)) = 0$, where h is the function defining the non-characteristic hypersurface, cf. Definition 2.2. By definition of a non-characteristic hypersurface, for a given $(t,x) \in A(\Gamma) \setminus \Gamma$, there exists a $\theta \in \mathbb{R}$ such that $S_\theta(t,x) \in \Omega$ and thus $h(S_\theta(t,x)) = 0$ holds. We show that θ is unique and, hence, the function $\theta(t,x)$ is well-defined: Indeed, assume that $h(S_\theta(t,x)) = h(S_{\theta+\tau^*}(t,x)) = 0$ for $\theta \in \mathbb{R}$ and $\tau^* > 0$ which is assumed to be minimal with this property. Since $Lh(S_\theta(t,x)) < 0$, we have $h(S_{\theta+\tau}(t,x)) < 0$ for $\tau \in (0,\tau^*)$, which contradicts $Lh(S_{\theta+\tau^*}(t,x)) < 0$. Hence, the function $\theta(t,x)$ is well-defined.

We show that $\theta \in C^\sigma(A(\Gamma) \setminus \Gamma, \mathbb{R})$ using the implicit function theorem. The function θ is the solution τ of

$$F((t,x),\tau) := h(S_\tau(t,x)) = 0.$$

Let $((t,x),\tau)$ be a point satisfying $S_\tau(t,x) \in \Omega$, i.e. $h(S_\tau(t,x)) = 0$. Then $\frac{\partial F}{\partial \tau}((t,x),\tau) = Lh(S_\tau(t,x)) < 0$. Since $S_\tau(t,x)$ is a C^σ function with respect to (t,x) and τ, $\theta \in C^\sigma$ follows with the implicit function theorem.

By definition $\theta(S_\tau(t,x)) = \theta(t,x) - \tau$. Thus, $L\theta(t,x) = \frac{d}{d\tau}\theta(S_\tau(t,x))\big|_{\tau=0} = -1$.

Define $V(t,x) := c_1\,\theta(t,x) + c_2$. The function V is C^σ, satisfies $LV(t,x) = -c_1$ and $V(t,x) = c_2$ for $(t,x) \in \Omega$. ☐

We have also some information about the smoothness of level sets of Lyapunov functions in general, cf. [5, Corollary 2.43]. This gives the following proposition for the Lyapunov function V in particular.

Proposition 2.5. *Let* $\dot{x} = f(t,x)$*,* $f \in C^\sigma(S^1_T \times \mathbb{R}^n, \mathbb{R}^n)$*,* $\sigma \geq 1$*. Let* $\Gamma = S^1_T \times \{0\}$ *be an exponentially asymptotically stable solution. Let* V *be the Lyapunov function of Theorem 2.4, and, moreover, let* f *be bounded in* $A(\Gamma)$*. Then for all* $r \in \mathbb{R}$ *the set* $\{(t,x) \in A(\Gamma) \setminus \Gamma \mid V(t,x) = r\}$ *is compact. Moreover, there is a* C^σ*-diffeomorphism* $\phi \in C^\sigma(\widetilde{T}, N_r)$*, where* $\widetilde{T} = \{(t,x) \in S^1_T \times \mathbb{R}^n \mid \|x\| = 1\}$ *and* $N_r = \{(t,x) \in A(\Gamma) \setminus \Gamma \mid V(t,x) = r\}$*.*

3. **Meshless Reconstruction.** In this section we will describe the general approximate reconstruction of multivariate functions, which are periodic in one variable, i.e. which are of the form $g : O \to \mathbb{R}$ with $O \subseteq S^1_T \times \mathbb{R}^n$, from discrete data sets. The main application will later be the reconstruction of the Lyapunov function V,

cf. Theorem 2.4. We restrict ourselves to the case $T = 2\pi$ and denote by $S^1 = S^1_{2\pi}$ the circle of circumference 2π or radius 1.

3.1. **Positive Definite Functions and Fourier Transforms.** In this section we will define a reproducing kernel Hilbert space with a positive definite kernel. This will ensure that the interpolation problem leads to a system of linear equations with a positive definite matrix and thus has a unique solution. We take into account that the functions are periodic with respect to t.

Definition 3.1. *A function $\Phi : S^1 \times \mathbb{R}^n \to \mathbb{R}$, periodic in t, is called positive definite if for all choices of pairwise distinct points $(t_j, x_j) \in S^1 \times \mathbb{R}^n$, $1 \le j \le N$, and all $\alpha = (\alpha_1, \ldots, \alpha_N)^T \in \mathbb{R}^N \setminus \{0\}$, we have*

$$\sum_{j,k=1}^{N} \alpha_j \alpha_k \Phi(t_j - t_k, x_j - x_k) > 0. \tag{8}$$

Positive definite functions are often characterised using Fourier transform. Since we are here dealing with functions, which are periodic in their t argument, the appropriate form of the Fourier transform of a function $f : S^1 \times \mathbb{R}^n \to \mathbb{R}$ is defined by

$$\widehat{f}_\ell(\omega) := (2\pi)^{-(n+1)} \int_0^{2\pi} \int_{\mathbb{R}^n} \Phi(t, x) e^{-ix^T\omega} e^{-i\ell t}\, dx\, dt. \tag{9}$$

The inverse Fourier transform is then given by

$$f(t, x) = \sum_{\ell \in \mathbb{Z}} \int_{\mathbb{R}^n} \widehat{f}_\ell(\omega) e^{i(\ell t + x^T\omega)}\, d\omega. \tag{10}$$

The following characterisation has been proven in [7, Lemma 3.7].

Lemma 3.2. *Let the kernel $\Phi : S^1 \times \mathbb{R}^n \to \mathbb{R}$ have a pointwise representation of the form (10) with positive Fourier coefficients $\widehat{\Phi}_\ell(\omega)$ for all $\ell \in \mathbb{Z}$ and all $\omega \in \mathbb{R}^n$. Then, $\Phi : S^1 \times \mathbb{R}^n \to \mathbb{R}$ is positive definite.*

As an example, in [7] it was shown that a t-periodic positive definite kernel can be constructed from a positive definite function $\Psi \colon \mathbb{R}^{n+1} \to \mathbb{R}$ by making it periodic in the first argument:

$$\Phi(t, x) = \sum_{k \in \mathbb{Z}} \Psi(t + 2\pi k, x). \tag{11}$$

Note that this sum is finite if Ψ has compact support.

The associated reproducing kernel Hilbert space for a kernel $\Phi(t, x)$ with positive Fourier coefficients $\widehat{\Phi}_\ell(\omega)$ can be defined by

$$\mathcal{N}_\Phi(S^1 \times \mathbb{R}^n) := \left\{ g : \sum_{\ell \in \mathbb{Z}} \int_{\mathbb{R}^n} \frac{|\widehat{g}_\ell(\omega)|^2}{\widehat{\Phi}_\ell(\omega)} d\omega < \infty \right\}.$$

The space is a Hilbert space with the inner product

$$(g, h)_{\mathcal{N}_\Phi} := \sum_{\ell \in \mathbb{Z}} \int_{\mathbb{R}^n} \frac{\widehat{g}_\ell(\omega)\overline{\widehat{h}_\ell(\omega)}}{\widehat{\Phi}_\ell(\omega)} d\omega.$$

Now, suppose we are given a kernel possessing Fourier coefficients $\widehat{\Phi}_\ell(\omega)$ behaving like

$$c_1(1 + \ell^2 + \|\omega\|^2)^{-\tau} \le \widehat{\Phi}_\ell(\omega) \le c_2(1 + \ell^2 + \|\omega\|^2)^{-\tau} \tag{12}$$

with $0 < c_1 \le c_2$. Then, according to [7, Lemma 3.5], see also [7, Section 3.3], the associated function space $\mathcal{N}_\Phi(S^1 \times \mathbb{R}^n)$ is norm equivalent to the Sobolev space

$\widetilde{W}_2^\tau(S^1 \times \mathbb{R}^n)$ of functions which are periodic in t; for the definition of this space see [7, Section 3.1].

Typical kernels satisfying (12) are Wendland's compactly supported radial basis functions $\Psi(\widetilde{x}) = \psi_{l,k}(\|\widetilde{x}\|)$, where $k \in \mathbb{N}$ is the smoothness index of the compactly supported Wendland function and $l = \lfloor \frac{n+1}{2} \rfloor + k + 1$, $\widetilde{x} = (t,x) \in \mathbb{R} \times \mathbb{R}^n$. Then (12) holds for the kernel (11) with $\tau = k + n/2 + 1$.

Finally, we will often assume that our target function V is not defined on all of $S^1 \times \mathbb{R}^n$ but only on a subset O. We assume that the open set $O \subseteq S^1 \times \mathbb{R}^n$ has a bounded C^K boundary, where $K \in \mathbb{N}$ and $0 \le \tau \le K$. By [7, Theorem 3.12] there exists a bounded extension operator from $\widetilde{W}_2^\tau(O)$ to $\widetilde{W}_2^\tau(S^1 \times \mathbb{R}^n)$.

3.2. **Generalized Interpolation.** To approximate the Lyapunov function V of Theorem 2.4 we choose *collocation* points $\widetilde{x}_j := (t_j, x_j) \in O \subseteq A(\Gamma) \setminus \Gamma$, $1 \le j \le N$ and $\widetilde{x}_j := (t_j, x_j) \in \Omega$, $N+1 \le j \le N+M$ to enforce the collocation conditions

$$\begin{aligned} Ls_V(t_j, x_j) &= -c_1 \qquad 1 \le j \le N, \\ s_V(t_j, x_j) &= c_2 \qquad N+1 \le j \le N+M. \end{aligned}$$

The approach to solve this problem is as follows and works for arbitrary linearly independent functionals over reproducing kernel Hilbert spaces.

Theorem 3.3 ([11, Theorem 16.1]). *Suppose $\mathcal{N}_\Phi$ is a reproducing kernel Hilbert space with reproducing kernel Φ. Suppose further that there are linearly independent linear functionals $\lambda_1, \ldots, \lambda_N \in \mathcal{N}_\Phi^*$. Then, to every $V \in \mathcal{N}_\Phi$, there exists one and only one norm-minimal generalized interpolant s_V, i.e. s_V is the unique solution to*

$$\min\{\|s\|_{\mathcal{N}_\Phi} : s \in \mathcal{N}_\Phi \text{ with } \lambda_j(s) = \lambda_j(V)\}.$$

Moreover, s_V has the representation

$$s_V(\widetilde{x}) = \sum_{j=1}^N \alpha_j \lambda_j^{\widetilde{y}} \Phi(\widetilde{x} - \widetilde{y}), \tag{13}$$

where the coefficients are determined by solving the linear system $\lambda_i(s_V) = \lambda_i(V)$, $1 \le i \le N$.

In order to solve our boundary value problem we have two sets of functionals. Using the notation $\widetilde{x} = (t,x) \in S^1 \times \mathbb{R}^n$, we choose two sets of points, $X_1 := \{\widetilde{x}_1, \ldots, \widetilde{x}_N\} \subseteq O$ and $X_2 := \{\widetilde{x}_{N+1}, \ldots, \widetilde{x}_{N+M}\} \subseteq \Omega \subseteq \partial O$ and define the functionals by

$$\lambda_j = \begin{cases} \delta_{\widetilde{x}_j} \circ L, & \text{for } 1 \le j \le N, \\ \delta_{\widetilde{x}_j} & \text{for } N+1 \le j \le N+M. \end{cases} \tag{14}$$

This leads to a generalized interpolant of the form

$$s_V(\widetilde{x}) = \sum_{k=1}^N \alpha_k (\delta_{\widetilde{x}_k} \circ L)^{\widetilde{y}} \Phi(\widetilde{x} - \widetilde{y}) + \sum_{k=N+1}^{N+M} \alpha_k \Phi(\widetilde{x} - \widetilde{x}_k).$$

The coefficient vector $\alpha \in \mathbb{R}^{N+M}$ is determined by the interpolation conditions

$$(\delta_{\widetilde{x}_j} \circ L)(s_V) = (\delta_{\widetilde{x}_j} \circ L)(V) = -c_1, \quad 1 \le j \le N \tag{15}$$

$$s_V(\widetilde{x}_j) = V(\widetilde{x}_j) = c_2, \quad N+1 \le j \le N+M. \tag{16}$$

As in the case of one operator, it is easy to show that the functionals λ_j, this time defined by (14), are linearly independent, for a similar proof cf. [6, Proposition 3.3]; note that L, being the orbital derivative of a time-periodic ODE, does not have any singular points.

Proposition 3.4. *Suppose* $\Phi : S^1 \times \mathbb{R}^n \to \mathbb{R}$ *is a reproducing kernel of* $\widetilde{W}_2^\tau(S^1 \times \mathbb{R}^n)$ *with* $\tau > m + (n+1)/2$. *Let* L *be a linear differential operator of degree* m. *Let* $X_1 = \{\widetilde{x}_1, \ldots, \widetilde{x}_N\} \subseteq O$ *and* $X_2 = \{\widetilde{x}_{N+1}, \ldots, \widetilde{x}_{N+M}\} \subseteq \partial O$ *be two sets of pairwise distinct points. Then, the functionals* $\Lambda = \{\lambda_1, \ldots, \lambda_{N+M}\}$ *with* $\lambda_j = \delta_{\widetilde{x}_j} \circ L$, $1 \le j \le N$ *and* $\lambda_j = \delta_{\widetilde{x}_j}$ *for* $N+1 \le j \le N+M$ *are linearly independent over* $\widetilde{W}_2^\tau(S^1 \times \mathbb{R}^n)$.

3.3. **Error Estimates.** To derive error estimates, we have to make certain further assumptions on the boundary. We will assume that the bounded region $O \subseteq S^1 \times \mathbb{R}^n$ has a $C^{k,s}$-boundary ∂O, where $\tau = k + s$ with $k \in \mathbb{N}_0$ and $s \in [0, 1)$. This means in particular, that ∂O is a n dimensional $C^{k,s}$-sub-manifold of $S^1 \times \mathbb{R}^n$. It also means that O is Lipschitz continuous and satisfies the cone condition. For details, we refer the reader to [12].

In the following it is sufficient to identify S^1 with the interval $(0, 2\pi)$ and to consider spaces of functions which are not necessarily periodic. We will represent the boundary ∂O by a finite atlas consisting of $C^{k,s}$-diffeomorphisms. To be more precise, we assume that $\partial O \subseteq \cup_{j=1}^K V_j$, where $V_j \subseteq S^1 \times \mathbb{R}^n$ are open sets. Moreover, the sets V_j are images of $C^{k,s}$-diffeomorphism

$$\varphi_j : B \to V_j \cap \partial O,$$

where $B = B(0, 1)$ denotes the unit ball in $\mathbb{R}^n$.

Finally, suppose $\{w_j\}$ is a partition of unity with respect to $\{V_j\}$. Then, the Sobolev norms on ∂O can be defined via

$$\|u\|^p_{W_p^\mu(\partial O)} = \sum_{j=1}^K \|(uw_j) \circ \varphi_j\|^p_{W_p^\mu(B)}.$$

It is well known that this norm is independent of the chosen atlas $\{V_j, \varphi_j\}$. Furthermore, the trace theorem (see [12, Theorem 8.7]) guarantees under these conditions that the restriction of $u \in W_2^\tau(O)$ with $\tau = k + s$ to ∂O is well defined, belongs to $W_2^{\tau-1/2}(\partial O)$, and satisfies $\|u\|_{W_2^{\tau-1/2}(\partial O)} \le \|u\|_{W_2^\tau(O)}$.

To measure the quality of our approximants we will use *mesh norms*. The quantity $h_{X_1,O} = \sup_{\widetilde{x} \in O} \min_{\widetilde{x}_j \in X_1} \|\widetilde{x} - \widetilde{x}_j\|$ measures how well X_1 is distributed over O. However, since O is periodic in the t variable, it is more natural to use the measure

$$\widetilde{h}_{X_1,O} := \sup_{\widetilde{x} \in O} \min_{\widetilde{x}_j \in X_1} \|\widetilde{x} - \widetilde{x}_j\|^c$$

where the "cylinder"-norm is defined by $\|\widetilde{x}\|^c = ((t \bmod 2\pi)^2 + \|x\|^2)^{1/2}$ and $t \bmod 2\pi \in [-\pi, \pi)$.

For the boundary part we are using the fixed atlas $\{V_j, \varphi_j\}$ and define the mesh norm now as

$$h_{X_2,\partial O} := \max_{1 \le j \le K} h_{T_j,B}$$

with $T_j = \varphi_j^{-1}(X_2 \cap V_j) \subseteq B$. Then, we have the following result, which holds for arbitrary operators L of order m.

Theorem 3.5. *Suppose* Φ *is the reproducing kernel of* $\widetilde{W}_2^\tau(S^1 \times \mathbb{R}^n)$ *with* $\tau > m + (n+1)/2$.

Suppose further that $O \subseteq S^1 \times \mathbb{R}^n$ *has a* C^K *boundary with* $K \ge \tau$. *Let* L *be a linear differential operator of order* m, *i.e.* $Lu(t,x) = \sum_{|\alpha| \le m} c_\alpha(t,x) D^\alpha u(t,x)$ *with coefficients* c_α *in* $\widetilde{W}_\infty^{k-m+1}(O)$ *and* $0 \le m \le \lceil \tau - (n+1)(1/2 - 1/p)_+ \rceil - 1$.

Finally, let s_V be the generalized interpolant to $V \in \widetilde{W}_2^\tau(O)$. If the data sets have sufficiently small mesh norms then for $1 \le p \le \infty$, the error estimates

$$\|LV - Ls_V\|_{L_p(O)} \leq C\widetilde{h}_{X_1,O}^{\tau-m-(n+1)(1/2-1/p)_+}\|V\|_{\widetilde{W}_2^\tau(O)} \qquad (17)$$

$$\|V - s_V\|_{L_p(\partial O)} \leq Ch_{X_2,\partial O}^{\tau-1/2-n(1/2-1/p)_+}\|V\|_{\widetilde{W}_2^\tau(O)} \qquad (18)$$

are satisfied.

Proof. Estimate (17) has been proven in [7, Theorem 3.19] for $m = 1$, $p = \infty$ and no boundary points. Its proof generalizes to our situation and general p and m as it has been done for the non-periodic case in [6].

The second estimate is proven as in the non-periodic case in [6, Theorem 3.10]. Since the functions $v_j = ((V - s_V)w_j) \circ \varphi_j$ belong to $W_2^{\tau-1/2}(B)$ and vanish on T_j we can estimate

$$\begin{aligned}
\|V - s_V\|_{L_p(\partial O)}^p &= \sum_{j=1}^K \|v_j\|_{L_p(B)}^p \le C\sum_{j=1}^K h_{T_j,B}^{p(\tau-1/2-n(1/2-1/p)_+)}\|v_j\|_{W_2^{\tau-1/2}(B)}^p \\
&\leq Ch_{X_2,\partial O}^{p(\tau-1/2-n(1/2-1/p)_+)}\|V - s_V\|_{W_2^{\tau-1/2}(\partial O)}^p \\
&\leq Ch_{X_2,\partial O}^{p(\tau-1/2-n(1/2-1/p)_+)}\|V - s_V\|_{W_2^\tau(O)}^p \\
&\leq Ch_{X_2,\partial O}^{p(\tau-1/2-n(1/2-1/p)_+)}\|V - s_V\|_{\widetilde{W}_2^\tau(O)}^p
\end{aligned}$$

for $1 \le p < \infty$, where, in the first step, we used a result on Sobolev functions having lots of zeros (see [1, 9, 10]) in the formulation of [7, Proposition 3.15]. The case $p = \infty$ is treated in the same way. Finally, since s_V is a norm-minimal interpolant, the norm in the last expression can again be bounded by the norm of V. □

The proof of Theorem 3.5 shows, that the following alternative version of Theorem 3.5 is also true.

Corollary 3.6. *Suppose $\Omega \subseteq \partial O$ is a part of the boundary satisfying $\Omega = \bigcup_{j=1}^L (V_j \cap \partial O)$. This means, that the first L charts $\{V_j, \varphi_j\}_{j=1}^L$ are exclusive for Ω, or that, for $1 \le j \le L$, $V_j \cap (\partial O \setminus \Omega) = \emptyset$. Suppose further, that the boundary collocation points X_2 are chosen only on Ω, while the interior points are still chosen in O, then estimate (17) remains valid and (18) becomes*

$$\|V - s_V\|_{L_p(\Omega)} \le Ch_{X_2,\Omega}^{\tau-1/2-n(1/2-1/p)_+}\|V\|_{\widetilde{W}_2^\tau(O)}, \qquad (19)$$

where $h_{X_2,\Omega} = \max_{1\le j\le L} h_{T_j,B}$ with T_j defined as before.

For the orbital derivative we have $m = 1$ and hence the following result. Note that the set O in the corollary has a C^σ boundary by Proposition 2.5.

Corollary 3.7. *Assume that $\tau > (n + 1)/2 + 1$. Let $\sigma := \lceil \tau \rceil$. Consider the dynamical system defined by the periodic ordinary differential equation $\dot{x} = f(t, x)$, where $f \in C^\sigma(S^1 \times \mathbb{R}^n, \mathbb{R}^n)$. Let $x(t) = 0 \in \mathbb{R}^n$ be an exponentially asymptotically periodic solution defining the periodic orbit Γ. Denote by $V \in C^\sigma(A(\Gamma) \setminus \Gamma, \mathbb{R})$ the Lyapunov function of Theorem 2.4. Suppose further that $\Phi : S^1 \times \mathbb{R}^n \to \mathbb{R}$ is a positive definite kernel satisfying (12).*

Let $\Omega = \{(t, x) \in A(\Gamma) \setminus \Gamma\} \mid h(t, x) = 0\}$ be a non-characteristic hypersurface and set $O = \{(t, x) \in A(\Gamma) \setminus \Gamma\} \mid V(t, x) \le r$ and $h(t, x) \ge 0\}$, where $r > 0$ is large enough such that $\{(t, x) \in A(\Gamma) \setminus \Gamma\} \mid V(t, x) = r\} \cap \Omega = \emptyset$.

Then, the reconstruction s_V of the Lyapunov function V with respect to the operator $Lu(t,x) = \partial_t u(t,x) + \langle \nabla_x u(t,x), f(t,x)\rangle$ and sets $X_1 \subseteq O$ and $X_2 \subseteq \Omega \subseteq \partial O$ satisfies

$$\|LV - Ls_V\|_{L_\infty(O)} \leq C\widetilde{h}_{X_1,O}^{\tau-1-(n+1)/2}\|V\|_{\widetilde{W}_2^\tau(O)} \quad (20)$$

$$\|V - s_V\|_{L_\infty(\Omega)} \leq Ch_{X_2,\Omega}^{\tau-(n+1)/2}\|V\|_{\widetilde{W}_2^\tau(O)} \quad (21)$$

In order to apply Theorem 2.1 we need in addition to the negative orbital derivative of the Lyapunov function a sublevel set. The next theorem ensures that each compact set $\widetilde{K}$ in the basin of attraction can be covered by a sublevel set of s_V.

Theorem 3.8. *Let $\widetilde{K}$ be a compact set with $\Gamma \subseteq \overset{\circ}{\widetilde{K}} \subseteq \widetilde{K} \subseteq A(\Gamma)$.*

Then there are $r, h_1^*, h_2^* > 0$ such that for all approximations s_V of V as in Corollary 3.7 where $O = \{(t,x) \in A(\Gamma) \setminus \Gamma \mid V(t,x) \leq r \text{ and } h(t,x) \geq 0\}$, $\widetilde{h}_{X_1,O} \leq h_1^*$ and $h_{X_2,\Omega} \leq h_2^*$, there is a $\rho \in \mathbb{R}$ with $\widetilde{K} \subseteq \{(t,x) \in O \mid s_V(t,x) \leq \rho\}$.

The proof is similar to [5, Theorem 5.3]. We can use (21) near Γ and then the estimate (20) along solutions.

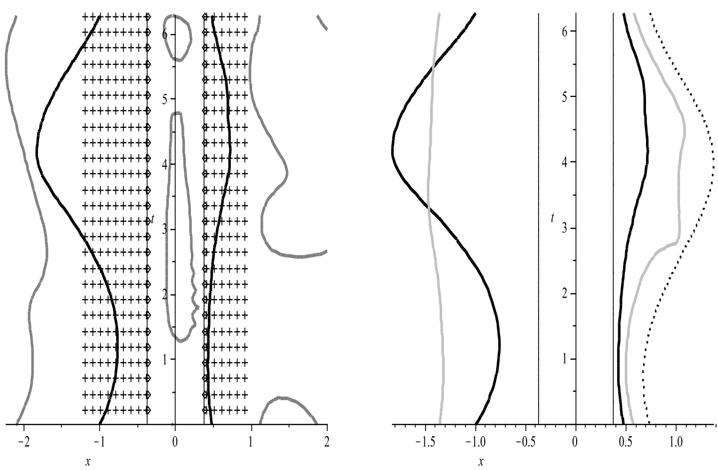

FIGURE 1. Left: Sublevel set E (thin black) of the local Lyapunov function $W(t,x) = \frac{1}{2}x^2$, the grids X_1 (+) and X_2 (∘) for the calculated Lyapunov function s_V, the level set $Ls_V(t,x) = 0$ (grey) determining the sign of Ls_V and a sublevel set of s_V (thick black) which is a subset of the basin of attraction. Right: different subsets of the basin of attraction (dotted): local Lyapunov function (thin), approximation of a PDE with boundary conditions (this paper, thick black) and without boundary conditions ([7], grey).

3.4. **Example.** As an example we consider the system [7, Section 4.1]

$$\dot{x} = x(\lambda \sin t - 1) + x^2 \quad (22)$$

with $\lambda = 1/2$. The function $x(t) = 0$ is a solution and $W(t,x) = \frac{1}{2}x^2$ is the Lyapunov function for the system with $\lambda = 0$, linearized at 0, i.e. $\dot{x} = -x$. Hence, the set $E = \{(t,x) \mid W(t,x) \leq 0.12\}$ is a subset of the basin of attraction, cf. [7, Section 4.1] and Figure 1, left.

Now we choose two grids of $N = 390$ and $M = 52$ points, where $X_2 \subseteq \partial E =: \Omega$, and approximate the solution $V(t,x)$ of $LV(t,x) = -1$ and $V(t,x) = 0$ on Ω by s_V using the kernel given by Wendland's function $\psi(r) = \psi_{5,3}(r/4.5)$. The sign of $Ls_V(t,x)$ and the sublevel set $K = \{(t,x) \mid s_V(t,x) \leq 0.68\}$ satisfying the conditions of Theorem 2.1 is shown in Figure 1, left (thick black). In Figure 1, right we compare this subset of the basin of attraction (thick black) with the subset obtained in [7] using a PDE without boundary conditions (grey) with 442 collocation points as well as the boundary of the basin of attraction, an unstable periodic orbit (dotted).

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