DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Supplement 2009

Website: www.aimSciences.org

pp. **240–249**

TIME-DEPENDENT OBSTACLE PROBLEM IN THERMOHYDRAULICS

Takeshi Fukao[†]

Department of Mathematics
Kyoto University of Education
Fujinomori 1, Fukakusa Fushimi-ku
Kyoto 612-8522, Japan

Masahiro Kubo[‡]

Department of Mathematics
Nagoya Institute of Technology
Gokiso-cho, Showa-ku
Nagoya 466-8555, Japan

Abstract. Obstacle problems, mathematical models of some nonlinear phenomena accompanying a free boundary, have been well studied. In this paper, the existence and uniqueness of a system between the obstacle problem and the Navier-Stokes equations is considered. The abstract theory for evolution equations governed by a subdifferential of the indicator functional on a time-dependent, closed, and convex set is applied to show the main theorem. L^∞-estimate is an important lemma to prove the existence theorem.

1. **Introduction.** Let $0 < T < +\infty$, $\Omega \subset \mathbb{R}^2$ be a bounded domain with a smooth boundary: $\Gamma := \partial\Omega$. We consider the following time-dependent single obstacle problem (P):$= \{(1)$–$(7)\}$, for a prescribed obstacle function $\psi := \psi(t,x)$.

$$\theta \ge \psi \quad \text{in } Q := (0,T)\times\Omega, \tag{1}$$

$$\frac{\partial\theta}{\partial t} + \boldsymbol{v}\cdot\nabla\theta - \Delta\theta = f \quad \text{in } Q(\theta) := \{(t,x)\in Q; \theta > \psi\}, \tag{2}$$

$$\frac{\partial\theta}{\partial t} + \boldsymbol{v}\cdot\nabla\theta - \Delta\theta \ge f \quad \text{in } Q_1(\theta) := \{(t,x)\in Q; \theta = \psi\}, \tag{3}$$

$$\frac{\partial\boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} - \Delta\boldsymbol{v} = \boldsymbol{g}(\theta) - \nabla p \quad \text{in } Q, \tag{4}$$

$$\mathrm{div}\boldsymbol{v} = 0 \quad \text{in } Q, \tag{5}$$

$$\theta = h, \quad \boldsymbol{v} = 0 \quad \text{on } \Sigma := (0,T)\times\Gamma, \tag{6}$$

$$\theta(0) = \theta_0, \quad \boldsymbol{v}(0) = \boldsymbol{v}_0 \quad \text{in } \Omega, \tag{7}$$

where $\theta = \theta(t,x)$ is the temperature, $\boldsymbol{v} := (v_1(t,x), v_2(t,x))$ is the velocity, and $p := p(t,x)$ is the pressure; $f : Q \to \mathbb{R}, \boldsymbol{g} := (g_1, g_2) : \mathbb{R} \to \mathbb{R}^2, h : \Sigma \to \mathbb{R}, \theta_0 : \Omega \to \mathbb{R}, \boldsymbol{v}_0 : \Omega \to \mathbb{R}^2$ are given functions. From a physical viewpoint, $\boldsymbol{g}(\theta)$ represents the Boussinesq approximation of the buoyancy force in Navier-Stokes equations. The solvability of the Boussinesq system between the linear heat equation and the Navier-Stokes equations has been treated in many papers. For example, Morimoto

2000 *Mathematics Subject Classification.* Primary: 35K65, 76D05; Secondary: 35G30.
Key words and phrases. Variational inequality, Obstacle problem, Navier-Stokes equations.
[†]Supported by a Grant-in-Aid for Encouragement of Young Scientists (B) (No.18740095), JSPS.
[‡]Supported by a Grant-in-Aid for Scientific Research (C) (No.17540166), JSPS.

[21] and the references quoted therein. The Boussinesq system with nonlinear thermal diffusion has been studied by Diaz and Galiano [8] and Lorca and Boldrini [20]. Under the Dirichlet boundary condition they proved the global existence of a weak solution, its uniqueness in the 2-dimensional case, and the local existence of a strong solution. Kubo [17] generalized the existence to a general nonlinear heat flux under the Dirichlet boundary condition, and the Neumann boundary condition in Fukao and Kubo [11], envisaging the application to the free boundary problem. On the other hand, the obstacle makes the engineering situation of the controlled problem, namely more interesting situation for the thermohydraulics. See the detail in Duvaut and Lions [9] and Rodrigues [22]. In the double obstacles case of (P), the existence and uniqueness problem has been studied in Fukao and Kubo [12] under a homogeneous Dirichlet boundary condition. See also Brezis [5], Biroli [3], Kenmochi [14], Kubo [16] and Yamazaki [26] for the abstract approach to the time-dependent obstacle problem.

Mathematically, our problem relates to the system between the nonlinear variational inequality and the Navier-Stokes equations. The key point of the existence is a $L^\infty(Q)$-boundedness of θ independent of $\boldsymbol{v}$. In [12], it was obtained automatically from the constraint of double obstacles. But for the single obstacle case we need to show this boundedness.

The strategy of the proof consists of three steps: the approximations, the uniform estimate, and the limiting. We use the Yosida approximation for maximal monotone operators and approximation for the convection. For a given velocity, namely the convection, we determine the solutions for the variational inequality with uniform estimates. On the other hand, for given temperatures, namely the buoyancy force, the velocity is obtained by a weak solution of the Navier-Stokes equation. On this iteration, by virtue of uniform estimates, we can apply the Schauder fixed point theorem to prove the existence of the system.

2. **Main theorem.** In this section, we define our solution and provide our main theorem. Hereafter, we use the following notations: $H := L^2(\Omega)$, $V := H^1(\Omega)$ with the usual norms. V^* is the dual space of V. These are Hilbert spaces with standard inner products. Then, the dense and compact imbedding $V \hookrightarrow H \hookrightarrow V^*$ holds. In terms of vector-valued function spaces, $\boldsymbol{\mathcal{D}}_\sigma(\Omega) := \{\boldsymbol{u} \in \boldsymbol{C}_0^\infty(\Omega) := (C_0^\infty(\Omega))^2; \mathrm{div}\boldsymbol{u} = 0 \text{ in } \Omega\}$, $\boldsymbol{H} := \boldsymbol{L}^2_\sigma(\Omega)$, $\boldsymbol{Y} := \boldsymbol{L}^4_\sigma(\Omega)$, $\boldsymbol{V} := \boldsymbol{H}^1_\sigma(\Omega)$ with the usual norms where $\boldsymbol{L}^2_\sigma(\Omega)$, $\boldsymbol{L}^4_\sigma(\Omega)$ and $\boldsymbol{H}^1_\sigma(\Omega)$ are the closures of $\boldsymbol{\mathcal{D}}_\sigma(\Omega)$ in spaces $\boldsymbol{L}^2(\Omega)$, $\boldsymbol{L}^4(\Omega)$, and $\boldsymbol{H}^1(\Omega)$, respectively. These are Hilbert spaces with standard inner products, and the relation $\boldsymbol{V} \hookrightarrow \boldsymbol{Y} \subset \boldsymbol{H} \subset \boldsymbol{Y}^* \hookrightarrow \boldsymbol{V}^*$ holds.

We employ the standard framework for the Navier-Stokes equations. Accordingly, we define the bilinear functional $a(\cdot,\cdot) : \boldsymbol{V} \times \boldsymbol{V} \to \mathbb{R}$ and trilinear functional $b(\cdot,\cdot,\cdot) : \boldsymbol{V} \times \boldsymbol{V} \times \boldsymbol{V} \to \mathbb{R}$ by

$$a(\boldsymbol{u},\boldsymbol{w}) := \sum_{i,j=1}^{2} \int_\Omega \frac{\partial u_j}{\partial x_i}\frac{\partial w_j}{\partial x_i} dx, \quad b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) := \sum_{i,j=1}^{2} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j dx,$$

for all $\boldsymbol{u}, \boldsymbol{v}$, and $\boldsymbol{w} \in \boldsymbol{V}$. Moreover we define the time-dependent, closed, and convex subset of H for all $t \in [0,T]$ as follows:

$$K(t) := \{z \in H; z \geq \psi(t) \text{ a.e. on } \Omega\}.$$

Under these settings, we define our solution.

Definition 2.1. The pair $\{\theta, \boldsymbol{v}\} \in L^\infty(Q) \times \boldsymbol{L}^2(Q)$ is called a solution of our problem (P) if (D1)–(D3) are satisfied:

(D1) $\theta \in W^{1,2}(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega))$ and $\theta(t) \in K(t)$ a.e. $t \in [0,T]$, $\boldsymbol{v} \in W^{1,2}(0,T;\boldsymbol{V}^*) \cap L^\infty(0,T;\boldsymbol{H}) \cap L^2(0,T;\boldsymbol{V})$;

(D2) For any $z \in K(t)$ and $\boldsymbol{z} \in \boldsymbol{V}$, functions θ and $\boldsymbol{v}$ satisfy the following relations

$$(\theta' - \Delta\theta + \boldsymbol{v} \cdot \nabla\theta - f, \theta - z)_H \le 0 \quad \text{a.e. on } (0,T],$$

$$\langle \boldsymbol{v}', \boldsymbol{z} \rangle_{\boldsymbol{V}^*,\boldsymbol{V}} + a(\boldsymbol{v},\boldsymbol{z}) + b(\boldsymbol{v},\boldsymbol{v},\boldsymbol{z}) = (\boldsymbol{g}(\theta), \boldsymbol{z})_{\boldsymbol{H}} \quad \text{a.e. on } (0,T];$$

(D3) $\theta(0) = \theta_0$, $\boldsymbol{v}(0) = \boldsymbol{v}_0$ a.e. in Ω and $\theta = h$ a.e. on Σ.

In this paper, we assume the following conditions (A1)–(A3):

(A1) $\psi \in W^{1,2}(0,T;V) \cap L^2(0,T;H^2(\Omega)) \subset L^\infty(Q)$;

(A2) $h \in W^{1,2}(0,T;H^{3/2}(\Gamma)) \subset L^\infty(\Sigma)$ with $h(t) \ge \psi(t)$ a.e. on Γ for all $t \in (0,T]$;

(A3) $f \in L^\infty(Q)$, g_i are Lipschitz continuous for $i = 1, 2$, $\theta_0 \in V \cap L^\infty(\Omega) \cap K(0)$, and $\boldsymbol{v}_0 \in \boldsymbol{H}$.

Main Theorem. *Under the assumptions (A1), (A2), and (A3), at least one solution exists, $\{\theta, \boldsymbol{v}\}$ of the problem (P).*

It is well-known that our problem is formulated as evolution equations governed by a subdifferential operator. We prepare related notations and definitions. For a proper, lower semi-continuous, and convex function $\phi : H \to \mathbb{R} \cup \{+\infty\}$, the subdifferential of ϕ is a possibly multivalued operator in H, and is defined by $u^* \in \partial\phi(u)$ if and only if $u \in D(\phi) = \{u \in H; \phi(u) < +\infty\}$ and

$$(u^*, z - u)_H \le \phi(z) - \phi(u) \quad \text{for all } z \in H.$$

Let $I_{K(t)}$ be the indicator function of $K(t)$, namely

$$I_{K(t)}(z) := \begin{cases} 0 & \text{if } z \in K(t), \\ +\infty & \text{if } z \in H \setminus K(t). \end{cases}$$

Moreover

$$\varphi^t(z) := \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega |\nabla z|^2 dx & \text{if } z \in V \text{ and } z = h(t) \text{ a.e. on } \Gamma, \\ +\infty & \text{otherwise.} \end{cases}$$

Lemma 2.2. *Both $I_{K(t)}$ and φ^t are proper, lower semi-continuous, and convex functions on H. Moreover $z^* = \partial\varphi^t(z)$ in H if and only if $z^* = -\Delta z$ a.e. on Ω and $z = h(t)$ a.e. on Γ, namely*

$$D(\partial\varphi^t) = \left\{ z \in H^2(\Omega); z = h(t) \text{ a.e. on } \Gamma \right\} \quad \text{for all } t \in [0,T].$$

We omit the proof. See the reference work of Barbu [2].

Let $G(\boldsymbol{v}(t), \cdot) : H \to H$ be the operator defined by $G(\boldsymbol{v}(t), z) := \boldsymbol{v}(t) \cdot \nabla z$, which is monotone if $\boldsymbol{v}(t) \in \boldsymbol{V}$; and $A : \boldsymbol{H} \to \boldsymbol{H}$ be the Stokes operator $-P\Delta$ where P is the Helmholtz projection $P : \boldsymbol{L}^2(\Omega) \to \boldsymbol{H}$. Moreover, we define $\langle B(\boldsymbol{u},\boldsymbol{v}), \boldsymbol{z} \rangle_{\boldsymbol{V}^*,\boldsymbol{V}} := b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{z})$ for all $\boldsymbol{u}, \boldsymbol{v}$, and $\boldsymbol{z} \in \boldsymbol{V}$. Then the relations (D2) with the initial-boundary condition (D3) is equivalent to the following evolution equations:

$$\theta'(t) + \partial\varphi^t(\theta(t)) + \partial I_{K(t)}(\theta(t)) + G(\boldsymbol{v}(t), \theta(t)) \ni f(t) \quad \text{in } H \quad \text{for a.e. } t \in (0,T], \tag{8}$$

$$\theta(0) = \theta_0 \quad \text{in } H, \tag{9}$$

$$\boldsymbol{v}'(t) + A\boldsymbol{v}(t) + B(\boldsymbol{v}(t), \boldsymbol{v}(t)) = P\boldsymbol{g}(\theta(t)) \quad \text{in } \boldsymbol{V}^* \quad \text{for a.e. } t \in (0,T], \tag{10}$$

$$\boldsymbol{v}(0) = \boldsymbol{v}_0 \quad \text{in } \boldsymbol{H}. \tag{11}$$

Proposition 1 ([12], Theorem 2)**.** *Let $\{\theta_i, \boldsymbol{v}_i\}$ be the solutions of the problem (P) corresponding to the initial values $\theta_{0,i}$ and $\boldsymbol{v}_{0,i}$ for $i = 1, 2$. Then the following continuous dependence for the data holds:*

$$|\theta_1(t) - \theta_2(t)|_H^2 + |\boldsymbol{v}_1(t) - \boldsymbol{v}_2(t)|_{\boldsymbol{H}}^2 \le \Big(|\theta_{0,1} - \theta_{0,2}|_H^2 + |\boldsymbol{v}_{0,1} - \boldsymbol{v}_{0,2}|_{\boldsymbol{H}}^2\Big)e^{M_0 t},$$

for a.e. $t \in [0, T]$, where M_0 is a positive constant. Namely, the solution of the problem (P) is unique.

3. **Auxiliary problem.** In this section we consider the auxiliary problem. First, we prepare the following lemmas. See Attouch and Damlamian [1], Kenmochi [13], Kubo [16, 18], Shirakawa, et al. [23], Yamazaki [26] and their references. Hereafter, for any $\lambda, \mu \in (0, 1)$, we use the following standard notations for the resolvents and Yosida approximations:

$$J_\lambda^{\partial\varphi^t} := (I + \lambda\partial\varphi^t)^{-1}, \quad J_\mu^{\partial I_{K(t)}} = (I + \mu\partial I_{K(t)})^{-1} = P_{K(t)},$$

$$(\partial\varphi^t)_\lambda(z) := \frac{1}{\lambda}(z - J_\lambda^{\partial\varphi^t} z) = \partial\varphi_\lambda^t(z),$$

$$(\partial I_{K(t)})_\mu(z) = \frac{[z - \psi(t)]^-}{\mu} = \partial I_{K(t)}^\mu(z),$$

$$\varphi_\lambda^t(z) := \inf_{y \in H}\left\{\frac{1}{2\lambda}|z - y|_H^2 + \varphi^t(y)\right\}, \quad I_{K(t)}^\mu(z) = \frac{1}{2\mu}|P_{K(t)}z - z|_H^2,$$

where $P_{K(t)} : H \to K(t)$ is the projection on $K(t)$. We refer to the reference works by Barbu [2], Kinderlehrer and Stampacchia [15] for the basic concepts.

Lemma 3.1 (see Kenmochi [13], Section 1.5)**.** *There are a number $a \in [0, 1)$ and non-negative functions $b, c \in L^1(0, T)$ such that*

$$\begin{aligned}&\frac{d}{dt}\varphi_\lambda^t(v(t)) - \Big(v'(t), \partial\varphi_\lambda^t(v(t))\Big)_H \\ &\quad \le \quad a|\partial\varphi_\lambda^t(v(t))|_H^2 + b(t)|\varphi_\lambda^t(v(t))| + c(t)(1 + |v(t)|_H^2) \quad \text{for a.e. } t \in (0, T],\end{aligned}$$

for all $v \in W^{1,1}(0, T; H)$ and $\lambda \in (0, 1)$.

Lemma 3.2 (see Brezis-Crandall-Pazy [7], Section 3 or Brezis [4], Chapter 1)**.**

$$\frac{d}{dt}I_{K(t)}^\mu(v(t)) - \Big(v'(t), \partial I_{K(t)}^\mu(v(t))\Big)_H = \Big(\psi'(t), \partial I_{K(t)}^\mu(v(t))\Big)_H \quad \text{for a.e. } t \in (0, T],$$

for all $v \in W^{1,1}(0, T; H)$ and $\mu \in (0, 1)$.

Lemma 3.3 (see Fukao-Kubo [12], Lemma 2)**.** *There exists a non-negative function $d \in L^2(0, T)$ such that*

$$(\partial I_{K(t)}^\mu(z), \partial\varphi_\lambda^t(z))_H \ge -d(t)|\partial I_{K(t)}^\mu(z)|_H \quad \text{for a.e. } t \in (0, T],$$

for all $z \in H$ and $\lambda, \mu \in (0, 1)$.

Lemma 3.4 (see Brezis [5, 6] or Barbu [2], Chapter 2)**.** *Let $\bar{\varphi}^t := \varphi^t + I_{K(t)}$ with $D(\bar{\varphi}^t) = D(\varphi^t) \cap K(t)$. Then*

$$\partial\bar{\varphi}^t = \partial\varphi^t + \partial I_{K(t)},$$

and $D(\partial\bar{\varphi}^t) = \{z \in H^2(\Omega) \cap K(t); z = h(t) \text{ a.e. on } \Gamma\}$ for all $t \in [0, T]$.

We omit the proof.

Lemma 3.5. *There exists a positive constant* c_1 *such that*

$$|G(\boldsymbol{v},\theta)|^2_{L^2(0,T;H)} \le c_1 |\boldsymbol{v}|^2_{L^4(0,T;\boldsymbol{Y})} |\theta|_{L^\infty(Q)} \Big(|\Delta\theta|_{L^2(0,T;H)} + |h|_{L^2(0,T;H^{3/2}(\Gamma))} \Big),$$

for all $\boldsymbol{v} \in L^4(0,T;\boldsymbol{Y})$ *and* $\theta \in L^2(0,T;D(\partial\varphi^t)) \cap L^\infty(Q)$.

We can show this lemma by applying Gagliardo-Nirenberg inequality $|\nabla\theta|^2_{\boldsymbol{L}^4(\Omega)} \le c_2 |\theta|_{H^2(\Omega)} |\theta|_{L^\infty(\Omega)}$ and a priori estimate for elliptic problems $|\theta|_{H^2(\Omega)} \le c_3(|\Delta\theta|_H + |h|_{H^{3/2}(\Gamma)})$ where c_2 and c_3 are positive constants.

For each fixed given function $\tilde{\theta} \in C([0,T];H) \cap L^\infty(Q)$, we recall the well-known result for the Navier-Stokes equations in a 2-dimensional case.

Proposition 2. *For each given* $\tilde{\theta} \in C([0,T];H) \cap L^\infty(Q)$, *there exists a unique solution of the following evolution equation:*

$$\tilde{\boldsymbol{v}}'(t) + A\tilde{\boldsymbol{v}}(t) + B(\tilde{\boldsymbol{v}}(t),\tilde{\boldsymbol{v}}(t)) = P\boldsymbol{g}(\tilde{\theta}(t)) \quad \text{in } \boldsymbol{V}^* \text{ a.e. } t \in (0,T], \tag{12}$$

$$\tilde{\boldsymbol{v}}(0) = \boldsymbol{v}_0 \quad \text{in } \boldsymbol{H}. \tag{13}$$

Moreover, there exist positive constants M_1 *and* M_2 *such that*

$$|\tilde{\boldsymbol{v}}|^2_{L^\infty(0,T;\boldsymbol{H})} + |\tilde{\boldsymbol{v}}|^2_{L^2(0,T;\boldsymbol{V})} \le M_1\Big(|\tilde{\theta}|^2_{L^\infty(Q)} + |\boldsymbol{v}_0|^2_{\boldsymbol{H}} \Big), \tag{14}$$

$$|\tilde{\boldsymbol{v}}'|^2_{L^2(0,T;\boldsymbol{V}^*)} \le M_2\Big(|\tilde{\boldsymbol{v}}|^2_{L^2(0,T;\boldsymbol{V})} + |\tilde{\boldsymbol{v}}|^2_{L^\infty(0,T;\boldsymbol{H})}|\tilde{\boldsymbol{v}}|^2_{L^2(0,T;\boldsymbol{V})} + |\tilde{\theta}|^2_{L^\infty(Q)} \Big). \tag{15}$$

This proposition is concluded by referring to the theory developed by Temam [25].

Here, from the representation of $\boldsymbol{Y} = \{\boldsymbol{z} \in \boldsymbol{L}^4(\Omega); \mathrm{div}\boldsymbol{v} = 0 \text{ a.e. on } \Omega,\ \boldsymbol{v}\cdot\boldsymbol{\nu} = 0 \text{ a.e. on } \Gamma\}$, (cf. Fujiwara-Morimoto [10]), if $\boldsymbol{v} \in L^\infty(0,T;\boldsymbol{H}) \cap L^2(0,T;\boldsymbol{V})$, then $\boldsymbol{v} \in L^4(0,T;\boldsymbol{Y})$ and there exists a positive constant c_4 such that

$$|\boldsymbol{v}|_{L^4(0,T;\boldsymbol{Y})} \le c_4 |\boldsymbol{v}|^{1/2}_{L^\infty(0,T;\boldsymbol{H})} |\boldsymbol{v}|^{1/2}_{L^2(0,T;\boldsymbol{V})}, \tag{16}$$

see p.197, Lemma 3.3 of Temam [25], where $c_4 = 2^{1/4}$. Thus, for each $\tilde{\boldsymbol{v}} \in L^\infty(0,T;\boldsymbol{H}) \cap L^2(0,T;\boldsymbol{V})$ there exists a sequence $\{\tilde{\boldsymbol{v}}_\varepsilon\} \subset L^4(0,T;\boldsymbol{\mathcal{D}}_\sigma(\Omega))$ such that $\tilde{\boldsymbol{v}}_\varepsilon \to \tilde{\boldsymbol{v}}$ in $L^4(0,T;\boldsymbol{Y})$ as $\varepsilon \to 0$. Then, we recall the known result for the evolution equation governed by a time dependent subdifferential operator. See the reference work of Brezis [6]. See also the paper of Kenmochi [13]. Thanks to Lemma 3.1–3.4 we get the following proposition.

Proposition 3. *Let* $\{\tilde{\boldsymbol{v}}\}_\varepsilon \subset L^4(0,T;\boldsymbol{\mathcal{D}}_\sigma(\Omega))$ *be the approximating sequence in the above, then there exists a unique solution* $\tilde{\theta}_\varepsilon$ *of the following evolution equation:*

$$\tilde{\theta}_\varepsilon'(t) + \partial\varphi^t(\tilde{\theta}_\varepsilon(t)) + \partial I_{K(t)}(\tilde{\theta}_\varepsilon(t)) + G(\tilde{\boldsymbol{v}}_\varepsilon(t),\tilde{\theta}_\varepsilon(t)) \ni f(t) \quad \text{in } H \text{ a.e. } t \in (0,T], \tag{17}$$

$$\tilde{\theta}_\varepsilon(0) = \theta_0 \quad \text{in } H. \tag{18}$$

To take the limit $\varepsilon \to 0$ for the solution $\tilde{\theta}_\varepsilon$ constructed by the above proposition, we need to obtain L^∞-estimates independent of ε.

Lemma 3.6. *Put a positive constant* M *by*

$$M := \max\{|f|_{L^\infty(Q)}, |h|_{L^\infty(\Sigma)}, |\theta_0|_{L^\infty(\Omega)}, |\psi|_{L^\infty(Q)}\}.$$

Then

$$-M \le \tilde{\theta}_\varepsilon(t,x) \le M(T+1) \quad \text{for all } (t,x) \in Q.$$

Proof. The equation (17) means that

$$(\tilde{\theta}_\varepsilon'(\tau) - \Delta\tilde{\theta}_\varepsilon(\tau) + \tilde{\boldsymbol{v}}_\varepsilon(\tau)\cdot\nabla\tilde{\theta}_\varepsilon(\tau) - f(\tau), \tilde{\theta}_\varepsilon(\tau) - z)_H \le 0 \quad \text{for all } z \in K(\tau),$$

and $\tilde{\theta}_\varepsilon(\tau) = h(\tau)$ a.e. on Γ for all $\tau \in (0,T]$. Now, put the test function $z = \tilde{\theta}_\varepsilon(\tau) - [\tilde{\theta}_\varepsilon(\tau) - M(\tau+1)]^+ \in K(\tau)$. Then

$$\begin{aligned}&(\tilde{\boldsymbol{v}}_\varepsilon(\tau)\cdot\nabla\tilde{\theta}_\varepsilon(\tau), [\tilde{\theta}_\varepsilon(\tau) - M(\tau+1)]^+)_H\\ &\quad = (\tilde{\boldsymbol{v}}_\varepsilon(\tau)\cdot\nabla(\tilde{\theta}_\varepsilon(\tau) - M(\tau+1)), [\tilde{\theta}_\varepsilon(\tau) - M(\tau+1)]^+)_H\\ &\quad = 0 \quad \text{for a.e. } \tau \in (0,T].\end{aligned}$$

So we have

$$\frac{1}{2}\frac{d}{d\tau}\Big|[\tilde{\theta}_\varepsilon(\tau) - M(\tau+1)]^+\Big|_H^2 \le (f(\tau) - M, [\tilde{\theta}_\varepsilon(\tau) - M(\tau+1)]^+)_H \quad \text{for a.e. } \tau \in (0,T].$$

Integrating it over $[0,t]$ with respect to τ,

$$\begin{aligned}&\Big|[\tilde{\theta}_\varepsilon(t) - M(t+1)]^+\Big|_H^2\\ &\quad \le 2\int_0^t \int_\Omega (|f|_{L^\infty(Q)} - M)[\tilde{\theta}_\varepsilon(\tau) - M(\tau+1)]^+ dx d\tau + |[\theta_0 - M]^+|_H^2,\end{aligned}$$

for all $t \in (0,T]$. From the definition of M, we see $[\tilde{\theta}_\varepsilon(t,x) - M(t+1)]^+ = 0$ for a.e. $(t,x) \in Q$. So we get the conclusion. □

Put a convex and compact subset in $L^2(0,T;\boldsymbol{H})$

$$\boldsymbol{X} := \left\{ \boldsymbol{u} \in L^2(0,T;\boldsymbol{H});\ \begin{array}{l} |\boldsymbol{u}|_{L^\infty(0,T;\boldsymbol{H})} \le M_1^* \\ |\boldsymbol{u}|_{L^2(0,T;\boldsymbol{V})} \le M_1^* \\ |\boldsymbol{u}'|_{L^2(0,T;\boldsymbol{V}^*)} \le M_2^* \end{array} \right\},$$

where M_1^* and M_2^* are positive constants, given as:

$$M_1^* := \sqrt{2M_1\Big(M^2(T+1)^2 + |\boldsymbol{v}_0|_{\boldsymbol{H}}^2\Big)}, \quad M_2^* := \sqrt{M_2\Big(M_1^* + (M_1^*)^2 + M^2(T+1)^2\Big)},$$

with the use of positive constants M_1 and M_2 specified by Proposition 3.6. Let $\tilde{\boldsymbol{v}} \in \boldsymbol{X}$, then there exists $\{\tilde{\boldsymbol{v}}_\varepsilon\} \subset L^4(0,T;\boldsymbol{\mathcal{D}}_\sigma(\Omega))$ such that $\tilde{\boldsymbol{v}}_\varepsilon \to \tilde{\boldsymbol{v}}$ in $L^4(0,T;\boldsymbol{Y})$ as $\varepsilon \to 0$. So we find a sequence $\tilde{\theta}_\varepsilon \in W^{1,2}(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega))$ of solutions $\tilde{\theta}_\varepsilon$ to (17) and (18) as in Proposition 3.7. Moreover there exist positive constants M_3, M_4, M_5 and M_6 such that

$$|\tilde{\theta}_\varepsilon(t)|_H^2 \le M_3\Big(|f|_{L^2(0,T;H)}^2 + |G(\tilde{\boldsymbol{v}}_\varepsilon, \tilde{\theta}_\varepsilon)|_{L^2(0,T;H)}^2 + |\theta_0|_H^2 + 1\Big),$$

$$\varphi^t(\tilde{\theta}_\varepsilon(t)) \le M_4\Big(|f|_{L^2(0,T;H)}^2 + |G(\tilde{\boldsymbol{v}}_\varepsilon, \tilde{\theta}_\varepsilon)|_{L^2(0,T;H)}^2 + |\theta_0|_V^2 + 1\Big),$$

for a.e. $t \in [0,T]$,

$$\int_0^T |\partial\varphi^t(\tilde{\theta}_\varepsilon(t))|_H^2 dt \le M_5\Big(|f|_{L^2(0,T;H)}^2 + |G(\tilde{\boldsymbol{v}}_\varepsilon, \tilde{\theta}_\varepsilon)|_{L^2(0,T;H)}^2 + |\theta_0|_V^2 + 1\Big),$$

$$|\tilde{\theta}_\varepsilon'|_{L^2(0,T;H)}^2 \le M_6\Big(|f|_{L^2(0,T;H)}^2 + |G(\tilde{\boldsymbol{v}}_\varepsilon, \tilde{\theta}_\varepsilon)|_{L^2(0,T;H)}^2 + |\theta_0|_V^2 + 1\Big).$$

See Section 5 of [12]. We then can obtain the following uniform estimates independent of $\varepsilon > 0$.

Lemma 3.7. *There exist positive constants M_7, M_8, M_9, and M_{10} independent of $\varepsilon > 0$ such that*

$$\int_0^T |\partial\varphi^t(\tilde{\theta}_\varepsilon(t))|_H^2 dt \leq M_7, \tag{19}$$

$$|\tilde{\theta}_\varepsilon(t)|_H^2 \leq M_8 \quad \text{for a.e. } t \in (0,T], \tag{20}$$

$$\varphi^t(\tilde{\theta}_\varepsilon(t)) \leq M_9 \quad \text{for a.e. } t \in (0,T], \tag{21}$$

$$|\tilde{\theta}'_\varepsilon|_{L^2(0,T;H)}^2 \leq M_{10}. \tag{22}$$

Proof. By virtue of Lemma 2.2, 3.5, (16), and the Young inequality

$$\begin{aligned}
&\int_0^T |\partial\varphi^t(\tilde{\theta}_\varepsilon(t))|_H^2 dt \\
&\leq M_5\Big(|f|_{L^2(0,T;H)}^2 + |\theta_0|_V^2 + 1\Big) \\
&\quad + c_1 M_5 |\tilde{\boldsymbol{v}}_\varepsilon|_{L^4(0,T;\boldsymbol{Y})}^2 |\tilde{\theta}_\varepsilon|_{L^\infty(Q)} \Big(|\Delta\tilde{\theta}_\varepsilon|_{L^2(0,T;H)} + |h|_{L^2(0,T;H^{3/2}(\Gamma))}\Big) \\
&\leq M_5\Big(|f|_{L^2(0,T;H)}^2 + |\theta_0|_V^2 + 1\Big) + \delta \int_0^T |\partial\varphi^t(\tilde{\theta}_\varepsilon(t))|_H^2 dt \\
&\quad + \frac{2c_1^2 M_5^2 (M_1^*)^4 M^2 (T+1)^2}{4\delta} + 2^{1/4} c_1 M_5 (M_1^*)^2 M(T+1)|h|_{L^2(0,T;H^{3/2}(\Gamma))},
\end{aligned}$$

for each $\delta > 0$. Namely, we obtain (19) where

$$\begin{aligned}
M_7 \quad := \quad & \frac{M_5}{1-\delta}\Big(|f|_{L^2(0,T;H)}^2 + |\theta_0|_V^2 + 1\Big) + \frac{c_1^2 M_5^2 (M_1^*)^2 M^2 (T+1)^2}{2\delta(1-\delta)} \\
& + \frac{2^{1/4} c_1 M_5 (M_1^*)^2 M(T+1)|h|_{L^2(0,T;H^{3/2}(\Gamma))}}{1-\delta}.
\end{aligned}$$

Moreover

$$\begin{aligned}
&|G(\tilde{v}_\varepsilon, \tilde{\theta}_\varepsilon)|_{L^2(0,T;H)}^2 \\
&\leq c_1 |\tilde{\boldsymbol{v}}_\varepsilon|_{L^4(0,T;\boldsymbol{Y})}^2 |\tilde{\theta}|_{L^\infty(Q)} \Big(|\Delta\tilde{\theta}_\varepsilon|_{L^2(0,T;H)} + |h|_{L^2(0,T;H^{3/2}(\Gamma))}\Big) \\
&\leq 2^{1/2} c_1 (M_1^*)^2 M(T+1)\Big(M_7^{1/2} + |h|_{L^2(0,T;H^{3/2}(\Gamma))}\Big) =: M_{11}.
\end{aligned}$$

So

$$M_8 := M_3\Big(|f|_{L^2(0,T;H)}^2 + M_{11} + |\theta_0|_H^2 + 1\Big),$$

$$M_9 := M_4\Big(|f|_{L^2(0,T;H)}^2 + M_{11} + |\theta_0|_V^2 + 1\Big),$$

$$M_{10} := M_6\Big(|f|_{L^2(0,T;H)}^2 + M_{11} + |\theta_0|_V^2 + 1\Big).$$

□

Finally, concluding this section, we have the following proposition.

Proposition 4. *For each given $\tilde{\mathbf{v}} \in L^2(0,T;\boldsymbol{V}) \cap L^\infty(0,T;\boldsymbol{H})$ there exists a unique solution $\tilde{\theta}$ of the following evolution equation:*

$$\tilde{\theta}'(t) + \partial\varphi^t(\tilde{\theta}(t)) + \partial I_{K(t)}(\tilde{\theta}(t)) + G(\tilde{\mathbf{v}}(t), \tilde{\theta}(t)) \ni f(t) \quad \text{in } H \text{ a.e. } t \in (0,T], \tag{23}$$

$$\tilde{\theta}(0) = \theta_0 \quad \text{in } H. \tag{24}$$

Moreover, the following uniform estimates hold:

$$\int_0^T |\partial\varphi^t(\tilde{\theta}(t))|_H^2 dt \leq M_7, \tag{25}$$

$$|\tilde{\theta}(t)|_H^2 \le M_8 \quad \textit{for a.e. } t \in (0,T], \tag{26}$$

$$\varphi^t(\tilde{\theta}(t)) \le M_9 \quad \textit{for a.e. } t \in (0,T], \tag{27}$$

$$|\tilde{\theta}'|_{L^2(0,T;H)}^2 \le M_{10}, \tag{28}$$

$$|G(\tilde{\boldsymbol{v}},\tilde{\theta})|_{L^2(0.T;H)}^2 \le M_{11}. \tag{29}$$

Proof. Let us consider the limiting $\varepsilon \to 0$. Uniform estimates in Lemma 3.9 imply that we have some subsequences for $\varepsilon > 0$, which we also denote by $\{\tilde{\theta}_\varepsilon\}$ for simplicity, and there exists a function $\tilde{\theta} \in W^{1,2}(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega))$ such that

$$\tilde{\theta}_\varepsilon \to \tilde{\theta} \quad \text{weakly-} * \text{ in } L^\infty(0,T;V),$$
$$\partial\varphi^{\{\cdot\}}(\tilde{\theta}_\varepsilon) = -\Delta\tilde{\theta}_\varepsilon \to -\Delta\tilde{\theta} \quad \text{weakly in } L^2(0,T;H),$$
$$\tilde{\theta}_\varepsilon' \to \tilde{\theta}' \quad \text{weakly in } L^2(0,T;H),$$

and thanks to the Aubin compact theorem, see Simon [24]

$$\tilde{\theta}_\varepsilon \to \tilde{\theta} \quad \text{in } L^2(0,T;V) \cap C([0,T];H) \quad \text{as } \varepsilon \to 0,$$

namely $-\Delta\tilde{\theta}(t) = \partial\varphi^t(\tilde{\theta}(t))$ in H a.e. $t \in (0,T]$. Moreover, $\{\nabla\tilde{\theta}_\varepsilon\}$ is bounded in $\boldsymbol{L}^4(Q)$ so from Lemma 3.5,

$$\tilde{\boldsymbol{v}}_\varepsilon \cdot \nabla\tilde{\theta}_\varepsilon \to \tilde{\boldsymbol{v}} \cdot \nabla\tilde{\theta} \quad \text{weakly in } L^2(0,T;H) \quad \text{as } \varepsilon \to 0.$$

In the variational inequality for $\tilde{\theta}_\varepsilon$

$$\int_0^T (\tilde{\theta}_\varepsilon'(t) + \partial\varphi^t(\tilde{\theta}_\varepsilon(t)) + \tilde{\boldsymbol{v}}_\varepsilon(t)\cdot\nabla\tilde{\theta}_\varepsilon(t) - f(t), \tilde{\theta}_\varepsilon(t) - z)_H dt \le 0 \quad \text{for all } z \in K(t),$$

letting $\varepsilon \to 0$, we see that $\tilde{\theta}$ satisfies (23) and (24). Moreover from the strong convergence of $\tilde{\theta}_\varepsilon(t)$, from the lower semi-continuity of φ^t, from the weakly convergences of $\{\tilde{\theta}_\varepsilon\}$, and from Lemma 3.4 we obtain (25)–(29). The uniqueness proof is essentially same as Theorem 2.3. □

4. **Proof of the main theorem.** In the last section we prove the main theorem. We define a mapping $S_1 : \boldsymbol{X} \to L^2(0,T;\boldsymbol{H})$, by assigning to each $\boldsymbol{u} \in \boldsymbol{X}$ the solution $S_1(\boldsymbol{u}) := \boldsymbol{v}$, where $\boldsymbol{v}$ is obtained from Proposition 3.6 with given $\tilde{\theta}$, in which $\tilde{\theta}$ is obtained from Proposition 3.10 with given $\tilde{\boldsymbol{v}} = \boldsymbol{u}$. We see that for each T, $S_1(\boldsymbol{X}) \subset \boldsymbol{X}$, namely S_1 maps $\boldsymbol{X}$ into itself. Now we show the continuity of S_1 in $L^2(0,T;\boldsymbol{H})$.

Lemma 4.1. *S_1 is continuous on $\boldsymbol{X}$ with respect to the topology of $L^2(0,T;\boldsymbol{H})$.*

Proof. Let us take any convergent sequence $\{\boldsymbol{u}_k\} \subset \boldsymbol{X}$ to a limit $\boldsymbol{u} \in L^2(0,T;\boldsymbol{H})$ as $k \to +\infty$. Then $\boldsymbol{u} \in \boldsymbol{X}$ immediately follows from the compactness of $\boldsymbol{X}$. Firstly, from the boundedness we have a subsequence $\{\boldsymbol{u}_{k_m}\} \subset \{\boldsymbol{u}_k\}$

$$\boldsymbol{u}_{k_m} \to \boldsymbol{u} \quad \text{weakly in } L^4(0,T;\boldsymbol{Y}) \quad \text{as } m \to +\infty.$$

The corresponding $\tilde{\theta}_{k_m}$ of the unique solution of Proposition 3.10 with given $\tilde{\boldsymbol{v}} = \boldsymbol{u}_{k_m}$ is bounded in the sense of (25)–(29). So we have a subsequence, which we also denote by $\{\tilde{\theta}_{k_m}\}$ for simplicity, to find $\tilde{\theta} \in W^{1,2}(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega))$ such that

$$\tilde{\theta}_{k_m} \to \tilde{\theta} \quad \text{weakly-} * \text{ in } L^\infty(0,T;V),$$
$$\partial\varphi^{\{\cdot\}}(\tilde{\theta}_{k_m}) = -\Delta\tilde{\theta}_{k_m} \to -\Delta\tilde{\theta} \quad \text{weakly in } L^2(0,T;H),$$
$$\tilde{\theta}_{k_m}' \to \tilde{\theta}' \quad \text{weakly in } L^2(0,T;H),$$

and thanks to the Aubin compactness theorem

$$\tilde{\theta}_{k_m} \to \tilde{\theta} \quad \text{in } L^2(0,T;V) \text{ and } C([0,T];H) \quad \text{as } m \to +\infty,$$

namely $-\Delta\tilde{\theta}(t) = \partial\varphi^t(\tilde{\theta}(t))$ in H a.e. $t \in (0,T]$. Moreover, $\{\boldsymbol{u}_{k_m}\}$ and $\{\nabla\tilde{\theta}_{k_m}\}$ are bounded in $\boldsymbol{L}^4(Q)$ so from Lemma 3.5 and p.12, Lemma 1.3 of Lions [19]

$$\boldsymbol{u}_{k_m} \cdot \nabla\tilde{\theta}_{k_m} \to \boldsymbol{u} \cdot \nabla\tilde{\theta} \quad \text{weakly in } L^2(0,T;H) \quad \text{as } m \to +\infty.$$

So, we see that $\tilde{\theta}$ satisfies

$$\tilde{\theta}'(t) + \partial\varphi^t(\tilde{\theta}(t)) + \partial I_{K(t)}(\tilde{\theta}(t)) + G(\boldsymbol{u}(t), \tilde{\theta}(t)) \ni f(t) \quad \text{in } H \quad \text{a.e. } t \in (0,T],$$

$$\tilde{\theta}(0) = \theta_0 \quad \text{in } H.$$

Namely, $\tilde{\theta}$ is the solution corresponding with $\boldsymbol{u}$. Secondly, the corresponding $\{\tilde{\boldsymbol{v}}_{k_m}\}$ of the unique solution of Proposition 3.6 with given $\tilde{\theta} = \tilde{\theta}_{k_m}$ is bounded in the sense of (14) and (15). So Lemma 3.8 means that we have a subsequence, which we also denote by $\{\tilde{\boldsymbol{v}}_{k_m}\}$ for simplicity, to find a function $\tilde{\boldsymbol{v}}^* \in W^{1,2}(0,T;\boldsymbol{V}^*) \cap L^\infty(0,T;\boldsymbol{H}) \cap L^2(0,T;\boldsymbol{V})$ such that

$$\tilde{\boldsymbol{v}}_{k_m} \to \tilde{\boldsymbol{v}}^* \quad \text{weakly-}* \text{ in } L^\infty(0,T;\boldsymbol{H}),$$

$$\tilde{\boldsymbol{v}}_{k_m} \to \tilde{\boldsymbol{v}}^* \quad \text{weakly in } L^2(0,T;\boldsymbol{V}),$$

$$\tilde{\boldsymbol{v}}'_{k_m} \to (\tilde{\boldsymbol{v}}^*)' \quad \text{weakly in } L^2(0,T;\boldsymbol{V}^*) \quad \text{as } m \to +\infty.$$

Furthermore, using the Aubin compactness theorem again, we have

$$\tilde{\boldsymbol{v}}_{k_m} \to \tilde{\boldsymbol{v}}^* \quad \text{in } L^2(0,T;\boldsymbol{H}),$$

and then $\langle A(\tilde{\boldsymbol{v}}_{k_m}(t) - \tilde{\boldsymbol{v}}^*(t)), \boldsymbol{z}\rangle_{\boldsymbol{V}^*,\boldsymbol{V}} = a(\tilde{\boldsymbol{v}}_{k_m}(t) - \tilde{\boldsymbol{v}}^*(t), \boldsymbol{z})$ means that

$$A\tilde{\boldsymbol{v}}_{k_m} \to A\tilde{\boldsymbol{v}}^* \quad \text{weakly in } L^2(0,T;\boldsymbol{V}^*),$$

$$P\boldsymbol{g}(\tilde{\theta}_{k_m}) \to P\boldsymbol{g}(\tilde{\theta}) \quad \text{in } C([0,T];\boldsymbol{H}) \quad \text{as } m \to +\infty.$$

Therefore, we conclude from p.196, Lemma 3.2 of Temam [25] that

$$B(\tilde{\boldsymbol{v}}_{k_m}, \tilde{\boldsymbol{v}}_{k_m}) \to B(\tilde{\boldsymbol{v}}^*, \tilde{\boldsymbol{v}}^*) \quad \text{weakly in } L^2(0,T;\boldsymbol{V}^*) \quad \text{as } m \to +\infty,$$

and $\tilde{\boldsymbol{v}}^*$ is the unique solution of

$$(\tilde{\boldsymbol{v}}^*)'(t) + A\tilde{\boldsymbol{v}}^*(t) + B(\tilde{\boldsymbol{v}}^*(t), \tilde{\boldsymbol{v}}^*(t)) = P\boldsymbol{g}(\tilde{\theta}(t)) \quad \text{in } \boldsymbol{V}^* \text{ a.e. } t \in (0,T],$$

$$\tilde{\boldsymbol{v}}^*(0) = \boldsymbol{v}_0 \quad \text{in } \boldsymbol{H}.$$

Namely

$$\tilde{\boldsymbol{v}}_{k_m} = S_1(\boldsymbol{u}_{k_m}) \to \tilde{\boldsymbol{v}}^* = S_1(\boldsymbol{u}) \quad \text{in } L^2(0,T;\boldsymbol{H}) \quad \text{as } m \to +\infty.$$

Thus, the map S_1 is continuous in $L^2(0,T;\boldsymbol{H})$. □

Proof of the Main Theorem. By virtue of Lemma 4.1, the Schauder fixed point theorem can be applied to conclude that the map S_1 has a fixed point $\boldsymbol{v} \in \boldsymbol{X}$. Let θ be the corresponding solution of Proposition 3.10. Then, $\{\theta, \boldsymbol{v}\}$ is the solution of the problem (P). □

Acknowledgments. We would like to thank the referees very much for their valuable comments and suggestions.

REFERENCES

[1] H. Attouch and A. Damlamian, *Problèmes d'évolution dans les Hilberts et applications*, J. Math. Pures. Appl.(9), **54**(1975), 53–74.

[2] V. Barbu, "Nonlinear Semigroups and Differential Equations in Banach Spaces", Noordhoff, Leyden, 1976.

[3] M. Biroli, *Sur les inèquations paraboliques avec convexe dépendant du temps: solution forte et solution faible*, Riv. Mat. Univ. Parma (3), **3**(1974), 33–72.

[4] H. Brezis, *Problèmes unilatéraux* J. Math. Pures Appl., **51**(1972), 1–168.

[5] H. Brezis, *Un problème d'évolution avec contraintes unilatérales dépendent du temps*, C. R. Acad. Sci. Paris, **274**(1972), 310–313.

[6] H. Brezis, "Opérateurs Maximaux Monotones et Semi-Groupes de Contractions dans les Especes de Hilbert", North-Holland, Amsterdam-London-New York, 1973.

[7] H. Brezis, M. G. Crandall and A. Pazy, *Perturbations of nonlinear maximal monotone sets in Banach space*, Comm. Pure. Appl. Math., **23**(1970), 123–144.

[8] J. I. Diaz and G. Galiano, *Existence and uniqueness of solutions of the Boussinesq system with nonlinear thermal diffusion*, Topol. Methods Nonlinear Anal., **11**(1998), 59–82.

[9] G. Duvaut and J.L. Lions, "Inequalities in mechanics and physics", Springer, New York, 1972.

[10] D. Fujiwara and H. Morimoto, *An L^r-theorem of the Helmholtz decomposition of vector fields*, J. Fac. Sci. Univ. Tokyo Sect. IA Math., **24**(1977), 685–700.

[11] T. Fukao and M. Kubo, *Nonlinear degenerate parabolic equations for a thermohydraulic model*, in "Discrete Contin. Dyn. Syst. Supplement", Am. Inst. Math. Sci., (2007), 399–408.

[12] T. Fukao and M. Kubo, *Time-dependent double obstacle problem in thermohydraulics*, in GAKUTO Internat. Ser. Math. Sci. Appl., **Vol.29**, Gakkōtosho, Tokyo, (2008), 73–92.

[13] N. Kenmochi, *Solvability of nonlinear evolution equations with time-dependent constraints and applications*, Bull. Fac. Edu., Chiba Univ., **30**(1981), 1–87.

[14] N. Kenmochi, *On the quasi-linear heat equation with time-dependent obstacles*, Nonlinear Anal., **5**(1981), 71–80.

[15] D. Kinderlehrer and G. Stampacchia, "An introduction to variational inequalities and their applications", Academic Press, New York, 1980.

[16] M. Kubo, *Characterization of a class of evolution operators generated by time-dependent subdifferentials*, Funkcial. Ekvac., **32**(1989), 301–321.

[17] M. Kubo, *Weak solutions of a thermohydraulics model with a general nonlinear heat flux*, in GAKUTO Internat. Ser. Math. Sci. Appl., **Vol.23**, Gakkōtosho, Tokyo, (2005), 163–178.

[18] M. Kubo, *Second order evolution equations with time-dependent subdifferentials*, J. Evol. Equ., **7**(2007), 701–717.

[19] J.L. Lions, "Quelques méthodes de résolution des problèmes aux limites non linéaires," Études Mathématiques, Dunod Gauthier-Villas, Paris, 1968.

[20] S.A. Lorca and J.L. Boldrini, *The initial value problem for a generalized Boussinesq model*, Nonlinear Anal., **36**(1999), 457–480.

[21] H. Morimoto, *Nonstationary Boussinesq equations*, J. Fac. Sci., Univ. Tokyo., Sec. IA. Math., **39**(1992), 61–75.

[22] J.F. Rodrigues, "Obstacle problems in mathematical physics", North-Holland, Amsterdam-New York, 1987.

[23] K. Shirakawa, A. Ito, N. Yamazaki and N. Kenmochi, *Asymptotic stability for evolution equations governed by subdifferentials*, in GAKUTO Internat. Ser. Math. Sci. Appl., **Vol.11**, Gakkōtosho, Tokyo, (1998), 287–310.

[24] J. Simon, *Compact sets in the spaces $L^p(0,T;B)$*, Ann. Mate. Pura. Appl., **146** (1987), 65–96.

[25] R. Temam, "Navier-Stokes equations. Theory and numerical analysis", North-Holland, Amsterdam-New York, 1977.

[26] N. Yamazaki, Periodic behaviour of solutions of time-dependent double obstacle problems, Adv. Math. Sci. Appl., **9**(1999), 885-906.

Received July 2008; revised February 2009.

E-mail address: `fukao@kyokyo-u.ac.jp`
E-mail address: `kubo.masahiro@nitech.ac.jp`