

BIDIFFERENTIAL GRADED ALGEBRAS AND INTEGRABLE SYSTEMS

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ABSTRACT. In the framework of bidifferential graded algebras, we present universal solution generating techniques for a wide class of integrable systems.

1. **Introduction.** Let \mathcal{A} be a unital associative algebra (over \mathbb{R} or \mathbb{C}) with identity element I , and $\Omega(\mathcal{A}) = \bigoplus_{r \geq 0} \Omega^r(\mathcal{A})$ with $\Omega^0(\mathcal{A}) = \mathcal{A}$ and \mathcal{A} -bimodules $\Omega^r(\mathcal{A})$, $r = 1, 2, \dots$. We call $(\Omega(\mathcal{A}), d, \bar{d})$ a *bidifferential graded algebra* (BDGA), or *bidifferential calculus*, if $(\Omega(\mathcal{A}), d)$ and $(\Omega(\mathcal{A}), \bar{d})$ are both differential graded algebras, which means that $\Omega(\mathcal{A})$ is a graded algebra and the linear maps $d, \bar{d} : \Omega^r(\mathcal{A}) \rightarrow \Omega^{r+1}(\mathcal{A})$ satisfy the graded Leibniz rule (antiderivation property) and

$$d^2 = \bar{d}^2 = 0, \quad d\bar{d} + \bar{d}d = 0. \quad (1)$$

These conditions can be combined into $d_z^2 = 0$, where $d_z := \bar{d} - zd$ with an indeterminate z .¹ In section 2 we connect this structure with “integrable” partial differential (or difference) equations [9, 10, 11, 12, 13, 14, 15, 16, 31, 36, 6, 7, 8, 26, 28, 2, 4, 5, 25, 39, 30]. Although this framework may not be able to cover all possible (in some sense) integrable equations, it has the advantage of admitting *universal* techniques for constructing exact solutions. Whereas previous work concentrated on conservation laws and Bäcklund transformations, the present work addresses Darboux transformations and presents a very effective “linearization approach”, generalizing proposition 5.1 in [22] (see also [32, 3] for related ideas). After collecting some basics in section 2, section 3 addresses universal solution generating techniques. Section 4 then presents some examples. Section 5 contains final remarks.

2. **Dressing bidifferential graded algebras.** In the following, $(\Omega(\mathcal{A}), d, \bar{d})$ denotes a BDGA. Introducing

$$\bar{D} := \bar{d} - A$$

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¹A generalization to an N -differential graded algebra is then obtained if $d_z^2 = 0$ with $d_z = \sum_{n=1}^N z^n d_n$. But this will not be considered in this work.

with a 1-form A , then d and \bar{D} satisfy again the BDGA relations iff

$$\bar{d}A - AA \equiv -\bar{D}^2 = 0 \quad \text{and} \quad dA \equiv -(d\bar{D} + \bar{D}d) = 0. \quad (2)$$

We are interested in cases where these equations are equivalent to a partial differential or difference equation (or a family of such equations), which requires that A depends on a set of independent variables and the differential maps d, \bar{d} involve differential or difference operators. As depicted in the following diagram, we can solve either the first or the second equation.

$$\begin{array}{ccc}
 A = (\bar{d}g)g^{-1} & \xrightarrow{\hspace{2cm}} & d((\bar{d}g)g^{-1}) = 0 \\
 \uparrow \text{red} & \text{“pseudoduality”} & \downarrow \text{red} \\
 \bar{d}A - AA = 0 & \xrightarrow{\text{dashed green}} & dA = 0 \\
 \downarrow \text{black} & & \downarrow \text{red} \\
 \bar{d}d\phi = d\phi d\phi & \xleftarrow{\hspace{2cm}} & A = d\phi
 \end{array}$$

This results in two different equations that are related by a “Miura transformation”

$$(\bar{d}g)g^{-1} = d\phi, \quad (3)$$

and this relationship is sometimes referred to as “pseudoduality”.

The conditions (2) can be combined into

$$D_z^2 = 0 \quad \text{where} \quad D_z = \bar{D} - z d = d_z - A.$$

Such a zero curvature condition is at the roots of the theory of integrable systems. It is the integrability condition of the linear equation

$$D_z W(z) = 0. \quad (4)$$

To get some more information about this equation, let us derive it from

$$\bar{d}d\phi = d\phi d\phi, \quad (5)$$

the equation in the lower left corner of the diagram. Using (1), we write it as

$$d[\bar{d}\phi - (d\phi)\phi] = 0.$$

We shall assume that the first d -cohomology group is trivial, so that d -closed 1-forms are d -exact. Then there is an element $\psi \in \mathcal{A}$ such that

$$\bar{d}\phi - (d\phi)\phi = d\psi. \quad (6)$$

Applying \bar{d} , using (1), (5) and (6), we obtain

$$d[\bar{d}\psi - (d\phi)\psi] = 0,$$

which in turn can be integrated by introduction of a new potential $\chi \in \mathcal{A}$,

$$\bar{d}\psi - (d\phi)\psi = d\chi. \quad (7)$$

This procedure can be iterated and yields the linear equation (4) with

$$W(z) = I + \sum_{n \geq 1} W_n z^{-n}, \quad W_1 = -\phi.$$

Remark 1. A gauge transformation of a BDGA $(\Omega(\mathcal{A}), d, \bar{d})$ is given by $d \mapsto G d G^{-1} = d'$, $\bar{d} \mapsto G \bar{d} G^{-1} = \bar{d}'$ with an invertible map $G : \mathcal{A} \rightarrow \mathcal{A}$. In case of $(\Omega(\mathcal{A}), d, \bar{D})$, choosing G as multiplication by g^{-1} , we obtain the equivalent BDGA $(\Omega(\mathcal{A}), d', \bar{D}')$ with $d' = d + g^{-1} dg$ and $\bar{D}' = \bar{d}$, by use of $A = (\bar{d}g)g^{-1}$.

Remark 2. We can consider a simultaneous dressing of d and \bar{d} by introducing $D = d - B$ in addition to $\bar{D} = \bar{d} - A$. Then $(\Omega(\mathcal{A}), D, \bar{D})$ is a BDGA iff

$$dB = B B, \quad \bar{d}A = A A, \quad dA + \bar{d}B = A B + B A.$$

Solving the first two conditions by setting $A = (\bar{d}g)g^{-1}$, $B = (dh)h^{-1}$, the third (multiplied from the left by h^{-1} , from the right by h) becomes $d[(\bar{d}J)J^{-1}] = 0$ with $J = h^{-1}g$. An equivalent form is $\bar{d}[(dJ^{-1})J] = 0$. This generalizes Yang's gauge in the case of the (anti-) self-dual Yang-Mills equation (see also [38]).

3. Solution generating techniques.

3.1. Bäcklund transformation (BT). An elementary BT is given by

$$D'_z = G(z)D_z G(z)^{-1}, \quad D'_z := d_z - A',$$

where $G(z) = I + F z^{-1}$ [14]. This is equivalent to

$$dF = A - A', \quad \bar{d}F = A'F - FA. \quad (8)$$

Using $A = d\phi$, the first equation can be integrated,

$$F = \phi - \phi' - C \quad \text{where} \quad dC = 0,$$

and from the second equation we obtain the elementary BT

$$\bar{d}(\phi' - \phi + C) = (d\phi')(\phi' - \phi + C) - (\phi' - \phi + C)d\phi. \quad (9)$$

Alternatively, using $A = (\bar{d}g)g^{-1}$, the second of equations (8) is solved by

$$F = g' \mathcal{K} g^{-1} \quad \text{where} \quad \bar{d}\mathcal{K} = 0,$$

and the first of equations (8) becomes

$$d(g' \mathcal{K} g^{-1}) = (\bar{d}g')g^{-1} - (\bar{d}g)g^{-1},$$

which is the elementary BT for the pseudodual equation. Using the Miura transformation (3), this can be integrated and yields

$$\phi' - \phi + C = g' \mathcal{K} g^{-1}. \quad (10)$$

This equation connects the two elementary BTs.

3.2. Darboux transformation (DT). The linear system²

$$\bar{d}\psi = (d\phi)\psi + (d\psi)\Delta, \quad (11)$$

has the following integrability condition,

$$\left(\bar{d}d\phi - (d\phi)^2\right)\psi - (d\psi)(\bar{d}\Delta - (d\Delta)\Delta) = 0,$$

which reduces to (5) if

$$\bar{d}\Delta = (d\Delta)\Delta. \quad (12)$$

Let θ be an invertible solution of (11) with a solution Δ' of (12), hence

$$\bar{d}\theta = (d\phi)\theta + (d\theta)\Delta'. \quad (13)$$

As a consequence,

$$\bar{d}(\theta\Delta'\theta^{-1}) = (d\phi')\theta\Delta'\theta^{-1} - \theta\Delta'\theta^{-1}d\phi,$$

²Instead of (11), we may consider $\bar{d}\psi = (d\phi)\psi + d(\psi\Delta)$, which results from (7) by setting $\chi = \psi\Delta$. In this case we have to impose $\bar{d}\Delta = \Delta d\Delta$ in order to obtain (5) as integrability condition. Some of the following formulae, also in section 3.4, then have to be modified accordingly. One can prove that the two possibilities are in fact equivalent.

where

$$\phi' := \phi + \theta \Delta' \theta^{-1} - C' \quad \text{with} \quad dC' = 0. \quad (14)$$

This is in accordance with (9), i.e. ϕ' is related to ϕ by an elementary BT. Hence, any solution ϕ of (5) and any invertible solution θ of the linear equation (13) determine a new solution ϕ' of (5) via (14). This is an abstraction of what is known as a Darboux transformation (see e.g. [34]). Introducing

$$\psi' = (\psi \Delta - \theta \Delta' \theta^{-1} \psi) \mathcal{M}, \quad (15)$$

where \mathcal{M} satisfies

$$\bar{d}\mathcal{M} = (d\mathcal{M}) \Delta, \quad [\Delta, \mathcal{M}] = 0, \quad (16)$$

it follows that ψ' satisfies (11) with ϕ replaced by ϕ' , i.e.

$$\bar{d}\psi' = (d\phi') \psi' + (d\psi') \Delta.$$

Now we can iterate this procedure. Let $\theta_k, k = 1, \dots, n$, be invertible solutions of $\bar{d}\theta_k = (d\phi) \theta_k + (d\theta_k) \Delta_k$, and \mathcal{M}_k satisfy (16) with Δ_k . Set $\psi_{[1]} = \psi, \theta_{[1]} = \theta_1$,

$$\psi_{[k+1]} = (\psi_{[k]} \Delta - \theta_{[k]} \Delta_k \theta_{[k]}^{-1} \psi_{[k]}) \mathcal{M} \quad \text{with} \quad \theta_{[k]} = \psi_{[k]} \Big|_{\psi \rightarrow \theta_k, \Delta \rightarrow \Delta_k, \mathcal{M} \rightarrow \mathcal{M}_k}$$

Then $\psi_{[n+1]}$ satisfies $\bar{d}\psi_{[n+1]} = (d\phi_{[n+1]}) \psi_{[n+1]} + (d\psi_{[n+1]}) \Delta$ with the following solution of (5),

$$\phi_{[n+1]} = \phi + \sum_{k=1}^n (\theta_{[k]} \Delta_k \theta_{[k]}^{-1} - C_k).$$

If $\psi_{[n+1]}$ is invertible, then it solves (18) below (see the next subsection).

3.3. Modified Miura transformation. If ψ in (11) is invertible, we have

$$[\bar{d}g - (dg) \Delta] g^{-1} = d\phi, \quad (17)$$

writing g instead of ψ . The integrability condition is

$$d([\bar{d}g - (dg) \Delta] g^{-1}) = 0, \quad (18)$$

a *modified pseudodual* of (5), related by the *modified Miura transformation* (17).³ (18) corresponds to

$$A = [\bar{d}g - (dg) \Delta] g^{-1},$$

which reduces the two equations (2) to a single one since $\bar{d}A - A A = (dA) g \Delta g^{-1}$. We note that (17) is equivalent to

$$\bar{d}g^{-1} + g^{-1} d\phi' = d(\Delta g^{-1}) \quad \text{where} \quad \phi' = \phi + g \Delta g^{-1}. \quad (19)$$

³If $\Delta = \lambda I$ with $\lambda \in \mathbb{C}$, the modification can be absorbed by a redefinition $\bar{d}' := \bar{d} - \lambda d$ of \bar{d} .

3.4. Binary Darboux transformation (bDT). (5) is also integrability condition of

$$\bar{d}\tilde{\psi} = -\tilde{\psi} d\phi + \tilde{\Delta} d\tilde{\psi} \quad \text{where} \quad \bar{d}\tilde{\Delta} = \tilde{\Delta} d\tilde{\Delta}. \quad (20)$$

Combining this with (11), we get

$$\bar{d}(\tilde{\psi}\psi) = \tilde{\Delta} (d\tilde{\psi})\psi + \tilde{\psi} (d\psi)\Delta.$$

Introducing $\Omega(\tilde{\psi}, \psi)$ via

$$\tilde{\Delta}\Omega(\tilde{\psi}, \psi) - \Omega(\tilde{\psi}, \psi)\Delta = \tilde{\psi}\psi, \quad (21)$$

the previous equation is satisfied if

$$\bar{d}\Omega(\tilde{\psi}, \psi) = (d\Omega(\tilde{\psi}, \psi))\Delta - (d\tilde{\Delta})\Omega(\tilde{\psi}, \psi) + (d\tilde{\psi})\psi. \quad (22)$$

Now let $\Theta = (\theta_1, \dots, \theta_N)$ and $\tilde{\Theta} = (\tilde{\theta}_1, \dots, \tilde{\theta}_N)^T$ be solutions of

$$\bar{d}\Theta = (d\phi)\Theta + (d\Theta)\Delta, \quad \bar{d}\tilde{\Theta} = -\tilde{\Theta}d\phi + \tilde{\Delta}d\tilde{\Theta}, \quad (23)$$

with matrices $\Delta, \tilde{\Delta}$ where $\bar{d}\Delta = (d\Delta)\Delta$, $\bar{d}\tilde{\Delta} = \tilde{\Delta}d\tilde{\Delta}$, and Ω a matrix such that

$$\tilde{\Delta}\Omega - \Omega\Delta = \tilde{\Theta}\Theta \quad \text{and} \quad \bar{d}\Omega = (d\Omega)\Delta - (d\tilde{\Delta})\Omega + (d\tilde{\Theta})\Theta \quad (24)$$

(or equivalently $\bar{d}\Omega = \tilde{\Delta}d\Omega - \Omega d\Delta - \tilde{\Theta}d\Theta$) holds.⁴ It follows that

$$\Theta' = \Theta\Omega^{-1}\tilde{\mathbf{N}}^{-1}, \quad \tilde{\Theta}' = \mathbf{N}^{-1}\Omega^{-1}\tilde{\Theta},$$

satisfy (23) with Δ and $\tilde{\Delta}$ exchanged and ϕ replaced by

$$\phi' = \phi - \Theta\Omega^{-1}\tilde{\Theta} = \phi - \Theta'\tilde{\mathbf{N}}\tilde{\Theta} = \phi - \Theta\mathbf{N}\tilde{\Theta}', \quad (25)$$

if the matrices $\mathbf{N}, \tilde{\mathbf{N}}$ are invertible and satisfy $[\Delta, \mathbf{N}] = [\tilde{\Delta}, \tilde{\mathbf{N}}] = 0$, $\bar{d}\mathbf{N} = d(\Delta\mathbf{N})$ and $\bar{d}\tilde{\mathbf{N}} = d(\tilde{\mathbf{N}}\tilde{\Delta})$. In particular, ϕ' is again a solution of (5).

If ψ and $\tilde{\psi}$ are solutions of (11) and (20), respectively, and if $\omega, \tilde{\omega}$ satisfy

$$\begin{aligned} \tilde{\Delta}\omega - \omega\Delta &= \tilde{\psi}\Theta, & \tilde{\Delta}\tilde{\omega} - \tilde{\omega}\Delta &= \tilde{\Theta}\psi, \\ \bar{d}\omega &= (d\omega)\Delta - (d\tilde{\Delta})\omega + (d\tilde{\psi})\Theta = \tilde{\Delta}d\omega - \omega d\Delta - \tilde{\psi}d\Theta, \\ \bar{d}\tilde{\omega} &= (d\tilde{\omega})\Delta - (d\tilde{\Delta})\tilde{\omega} + (d\tilde{\Theta})\psi = \tilde{\Delta}d\tilde{\omega} - \tilde{\omega}d\Delta - \tilde{\Theta}d\psi, \end{aligned}$$

then one verifies by direct calculation that

$$\psi' = \psi - \Theta'\tilde{\mathbf{N}}\tilde{\omega} = \psi - \Theta\Omega^{-1}\tilde{\omega}, \quad \tilde{\psi}' = \tilde{\psi} - \omega\mathbf{N}\tilde{\Theta}' = \tilde{\psi} - \omega\Omega^{-1}\tilde{\Theta},$$

satisfy again (11), respectively (20), with ϕ replaced by ϕ' defined above. If ψ is invertible, it is a solution of (18) and then also ψ' , if invertible. Correspondingly, $\tilde{\psi}^{-1}$ and then also $\tilde{\psi}'^{-1}$ solves $d([\bar{d}g - d(g\tilde{\Delta})]g^{-1}) = 0$.

3.5. A linearization approach. Let us consider (6) in the form

$$\bar{d}\Phi = (d\Phi)Q\Phi + d\Psi,$$

where $dQ = 0$. The reason for the introduction of Q will be given below. Setting $\Psi = \Phi R$ with a d-constant R , this becomes

$$\bar{d}\Phi = (d\Phi)(Q\Phi + R). \quad (26)$$

Next we express Φ as

$$\Phi = YX^{-1}, \quad (27)$$

⁴We note that the first of equations (24) is a rank one condition.

and impose the constraint

$$RX + QY = XP \tag{28}$$

with some P . Multiplying (26) by X from the right, leads to

$$\bar{d}Y - \Phi \bar{d}X = (dY)P - \Phi(dX)P,$$

which is a consequence of the two linear equations

$$\bar{d}Y = (dY)P, \quad \bar{d}X = (dX)P. \tag{29}$$

The following theorem is now easily verified.⁵

Theorem 3.1. *Let X, Y solve the linear equations (29) and the constraint (28) with d -constant Q, R and some P , and let X be invertible. Then $\Phi = YX^{-1}$ solves*

$$\bar{d}d\Phi = d\Phi Q d\Phi. \tag{30}$$

Let Φ take values in the algebra of $M \times N$ matrices over \mathcal{A} . The other objects above are then also matrices with appropriate dimensions. If Q has *rank one* over \mathcal{A} , i.e. $Q = VU^T$ with (d - and \bar{d} -) constant vectors U, V having entries in \mathcal{A} , then

$$\phi = U^T \Phi V$$

solves (5) if Φ solves (30). The above theorem provides us with a method to construct exact solutions of the nonlinear equation (30) from solutions of linear equations, and the last argument shows how these generate exact solutions of (5). Since M and N can be chosen freely, in this way we obtain exact solutions of (5) involving an arbitrarily large number of parameters. This partly explains the existence of infinite families of solutions like multi-solitons.

More generally, if $Q = VU^T$ with constant $M \times m$ matrix U and $N \times m$ matrix V , then $\phi = U^T \Phi V$ solves (5) in the algebra of $m \times m$ matrices (with entries in \mathcal{A}) if Φ solves (30).

A somewhat weaker version of the theorem is obtained by extending (28) to

$$HZ = ZP \quad \text{where} \quad Z = \begin{pmatrix} X \\ Y \end{pmatrix}, \quad H = \begin{pmatrix} R & Q \\ S & L \end{pmatrix}, \tag{31}$$

with constant matrices L and S . This imposes the additional equation $SX + LY = YP$ on X and Y . Together with (28) it implies the algebraic Riccati equation

$$S + L\Phi - \Phi R - \Phi Q \Phi = 0. \tag{32}$$

The two equations (29) combine to

$$\bar{d}Z = dZP. \tag{33}$$

The equations for Z are form-invariant (with the same P) under a transformation

$$Z = \Gamma Z', \quad H = \Gamma H' \Gamma^{-1}, \tag{34}$$

with a (d - and \bar{d} -) constant matrix Γ . Such a transformation relates solutions of two versions of (30) corresponding to two different Q 's. One can therefore use the theorem with a simple form of H (our choice of H') and then apply a transformation to generate a solution associated with a more complicated choice of H . Choosing

$$H' = \begin{pmatrix} R & 0 \\ 0 & L \end{pmatrix} \quad \text{or} \quad H' = \begin{pmatrix} L & I_N \\ 0 & L \end{pmatrix}, \quad \text{and} \quad \Gamma = \begin{pmatrix} I_N & -K \\ 0 & I_M \end{pmatrix} \tag{35}$$

(I_N is the $N \times N$ unit matrix, $M = N$ in the second case), yields the next result.

⁵The proof of the theorem does *not* use $\bar{d}^2 = 0$.

Corollary 3.2. *Let X' and Y' solve*

$$\bar{d}X' = (dX')P, \quad \bar{d}Y' = (dY')P, \quad LY' = Y'P$$

and

$$RX' = X'P \quad \text{respectively} \quad LX' + Y' = X'P.$$

Then $\Phi = Y'(X' - KY')^{-1}$ with a constant matrix K solves (30) with

$$Q = RK - KL \quad \text{respectively} \quad Q = I + [L, K]. \quad (36)$$

Remark 3. Solutions of the modified pseudodual equation (18) are obtained as follows. Let G be an $m \times N$ matrix solution of

$$\bar{d}(GX) = d(GX)P \quad \text{and} \quad (GX)P - \Delta(GX) = \mathcal{C}X,$$

where \mathcal{C} and Δ satisfy $d\mathcal{C} = 0$ and $\bar{d}\Delta = (d\Delta)\Delta$. Requiring (27), (28) and (29), one finds that G solves

$$\bar{d}G + GQd\Phi = d(\Delta G).$$

With Q as specified above, it follows that $g = (GV)^{-1}$, provided the inverse exists, solves (19) with $\phi' = U^T\Phi V$. As a consequence, g solves (18).

Remark 4. A relation with the bDT is established as follows. If the algebraic Riccati equation (32) holds, assuming d -constant L and R we have in addition to (26) also $\bar{d}\Phi = (L - \Phi Q)d\Phi$. Now let $Q = VU^T$. Setting $\Theta = U^T\Phi$ and $\tilde{\Theta} = \Phi V$, we obtain (23) with $\Delta = R$ and $\tilde{\Delta} = L$. With the identification $\Omega = \Phi$, we find that (24) holds if $S = 0$. Furthermore, (25) yields $\phi' = 0$.

4. Examples. In some examples presented below, the graded algebra will be taken of the form $\Omega(\mathcal{A}) = \mathcal{A} \otimes_{\mathbb{C}} \bigwedge(\mathbb{C}^n)$ where $\bigwedge(\mathbb{C}^n)$ is the exterior algebra of \mathbb{C}^n . It is then sufficient to define the maps d and \bar{d} on \mathcal{A} . They extend to $\Omega(\mathcal{A})$ in an obvious way, treating elements of $\bigwedge(\mathbb{C}^n)$ as constants. ξ_1, \dots, ξ_n denotes a basis of $\bigwedge^1(\mathbb{C}^n)$.

4.1. Self-dual Yang-Mills (sdYM) equation. Let \mathcal{A} be the algebra of smooth complex functions of complex variables y, z and their complex conjugates \bar{y}, \bar{z} . Let

$$df = \mp f_y \xi_1 + f_z \xi_2, \quad \bar{d}f = f_{\bar{z}} \xi_1 + f_{\bar{y}} \xi_2$$

for $f \in \mathcal{A}$. This determines a BDGA. Then (5), for an $m \times m$ matrix ϕ with entries in \mathcal{A} , is equivalent to

$$\phi_{\bar{y}y} \pm \phi_{\bar{z}z} + [\phi_y, \phi_z] = 0, \quad (37)$$

which is a potential form of the (Euclidean or split signature) sdYM equation (see e.g. [33]). Writing J instead of g , the Miura transformation (3) becomes $J_{\bar{y}}J^{-1} = \phi_z$ and $J_{\bar{z}}J^{-1} = \mp\phi_y$, and the pseudodual of (37) takes the form

$$(J_{\bar{y}}J^{-1})_y \pm (J_{\bar{z}}J^{-1})_z = 0,$$

which is another well-known potential form of the sdYM equation.

Turning to DTs, we choose $\Delta, \tilde{\Delta}$ constant. Then (11) reads

$$\begin{aligned} \psi_{\bar{z}} &= \mp(\phi_y \psi + \psi_y \Delta) = J_{\bar{z}}J^{-1} \psi \mp \psi_y \Delta, \\ \psi_{\bar{y}} &= \phi_z \psi + \psi_z \Delta = J_{\bar{y}}J^{-1} \psi + \psi_z \Delta. \end{aligned}$$

For an invertible solution θ of this system, a new solution is given by (14), respectively $J' = (\theta \Delta \theta^{-1} - C') J \mathcal{K}^{-1}$ via (10). (20) becomes

$$\begin{aligned} \tilde{\psi}_{\bar{z}} &= \pm(\tilde{\psi} \phi_y - \tilde{\Delta} \tilde{\psi}_y) = -\tilde{\psi} J_{\bar{z}} J^{-1} \mp \tilde{\Delta} \tilde{\psi}_y, \\ \tilde{\psi}_{\bar{y}} &= -\tilde{\psi} \phi_z + \tilde{\Delta} \tilde{\psi}_z = -\tilde{\psi} J_{\bar{y}} J^{-1} + \tilde{\Delta} \tilde{\psi}_z. \end{aligned}$$

(21) reads $\tilde{\Delta} \Omega - \Omega \Delta = \tilde{\psi} \psi$ and (22) takes the form

$$\Omega_z = \mp(\Omega_y \Delta + \tilde{\psi}_y \psi), \quad \Omega_{\bar{y}} = \Omega_z \Delta + \tilde{\psi}_z \psi.$$

In this way one recovers corresponding formulae in [35]. Corollary 3.2 provides a more easily applied construction of exact solutions (see also [22]).

4.2. Pseudodual chiral model hierarchy. Let \mathfrak{M} be a space with coordinates x_1, x_2, \dots . On smooth functions on \mathfrak{M} we define

$$df = \sum_{n \geq 1} f_{x_n} dx_n, \quad \bar{d}f = \sum_{n \geq 1} f_{x_{n+1}} dx_n.$$

Hence d is the ordinary exterior derivative. Let \mathcal{A} be the algebra of $m \times m$ matrices of smooth functions and $\Omega(\mathcal{A}) = \mathcal{A} \otimes_{C^\infty(\mathfrak{M})} \Lambda(\mathfrak{M})$, where $\Lambda(\mathfrak{M})$ is the algebra of differential forms on \mathfrak{M} . This determines a BDGA $(\Omega(\mathcal{A}), d, \bar{d})$ and (5) reproduces the hierarchy of the $gl(m, \mathbb{C})$ “pseudodual chiral model” in $2 + 1$ dimensions,

$$\phi_{x_{n+1}, x_m} - \phi_{x_{m+1}, x_n} = [\phi_{x_n}, \phi_{x_m}], \quad m, n = 1, 2, \dots$$

The first (non-trivial) equation is a well-known reduction of the sdYM equation. ϕ can be restricted to any Lie subalgebra of $gl(m, \mathbb{C})$, but corresponding conditions then have to be imposed on the solution generating methods. The method of section 3.5 has been applied in [22]. In the $su(m)$ case, a variant of corollary 3.2 has been used in particular to construct multiple lump solutions.

4.3. The potential KP (pKP) equation. On smooth functions of x, y, t we define

$$df = [\partial_x, f] \xi_1 + \frac{1}{2}[\partial_y + \partial_x^2, f] \xi_2, \quad \bar{d}f = \frac{1}{2}[\partial_y - \partial_x^2, f] \xi_1 + \frac{1}{3}[\partial_t - \partial_x^3, f] \xi_2.$$

Besides smooth functions of x, y, t with values in some associative algebra, \mathcal{A} must also contain powers of the partial derivative operator ∂_x . Then (5) becomes the (noncommutative) pKP equation, and (18) with $\Delta = -\partial_x$ the (noncommutative) mKP equation. Concerning DTs, we set $\Delta = \Delta' = \tilde{\Delta} = C' = -\partial_x$, and $\mathcal{M} = \mathbf{N} = \tilde{\mathbf{N}} = I$. Then (14) and (15) take the form $\phi' = \phi + \theta_x \theta^{-1}$ and $\psi' = \psi_x - \theta_x \theta^{-1} \psi$, respectively. Equation (11) becomes

$$\psi_y = \psi_{xx} + 2 \phi_x \psi, \quad \psi_t = \psi_{xxx} + 3 \phi_x \psi_x + \frac{3}{2}(\phi_y + \phi_{xx}) \psi,$$

a familiar Lax pair for the pKP equation. The same equations, with ψ replaced by g , are obtained from (17), which means that the DT $\psi \mapsto \psi'$ acts on mKP solutions. Turning to bDTs, (20) reads

$$\tilde{\psi}_y = -\tilde{\psi}_{xx} - 2 \tilde{\psi} \phi_x, \quad \tilde{\psi}_t = \tilde{\psi}_{xxx} + 3 \tilde{\psi}_x \phi_x - \frac{3}{2} \tilde{\psi}(\phi_y - \phi_{xx}).$$

(21) becomes $\Omega_x = -\tilde{\psi} \psi$, and (22) yields

$$\Omega_y = \tilde{\psi}_x \psi - \tilde{\psi} \psi_x, \quad \Omega_t = -\tilde{\psi}_{xx} \psi + \tilde{\psi}_x \psi_x - \tilde{\psi} \psi_{xx} - 3 \tilde{\psi} \phi_x \psi.$$

These are well-known formulae, see e.g. [34, 24].

The ξ_1 -part of (33) is $Z_y - Z_{xx} = 2Z_x(P + \partial_x)$. Choosing $P = -I_N \partial_x$, this is the heat equation $Z_y = Z_{xx}$. The ξ_2 -part of (33) then becomes the second heat hierarchy equation, $Z_t = Z_{xxx}$. Setting $R = \tilde{R} - I_N \partial_x$ in (28), turns it into

$$X_x = \tilde{R} X + Q Y . \tag{38}$$

Now theorem 3.1 expresses a result for the pKP equation [18, 19] that extends to the whole pKP hierarchy, see the next subsection.

4.4. Kadomtsev-Petviashvili hierarchy. On smooth functions of variables x and $\mathbf{t} = (t_1, t_2, \dots)$ we define

$$df = [\mathbb{E}_\lambda, f] \xi_1 + [\mathbb{E}_\mu, f] \xi_2, \quad \bar{d}f = [(\lambda^{-1} - \partial_x)\mathbb{E}_\lambda, f] \xi_1 + [(\mu^{-1} - \partial_x)\mathbb{E}_\mu, f] \xi_2$$

where \mathbb{E}_λ is the Miwa shift operator with an indeterminate λ , i.e. $\mathbb{E}_\lambda f = f_{[\lambda]}\mathbb{E}_\lambda$ where $f_{\pm[\lambda]}(x, \mathbf{t}) = f(x, \mathbf{t} \pm [\lambda])$ with $[\lambda] = (\lambda, \lambda^2/2, \lambda^3/3, \dots)$. Furthermore, ∂_x is the partial derivative operator with respect to x . Let \mathcal{A} contain the algebra of $m \times m$ matrices of smooth functions. The above expressions for $df, \bar{d}f$ require that \mathcal{A} also contains the Miwa shift operators and powers of ∂_x . (5) is equivalent to the following functional representation [1, 17] of the (matrix) potential KP hierarchy,

$$\begin{aligned} (\phi_{-[\lambda]} - \phi_{-[\mu]})_x &= (\mu^{-1} - \phi + \phi_{-[\mu]})_{-[\lambda]}(\lambda^{-1} - \phi + \phi_{-[\lambda]}) \\ &\quad - (\lambda^{-1} - \phi + \phi_{-[\lambda]})_{-[\mu]}(\mu^{-1} - \phi + \phi_{-[\mu]}) . \end{aligned}$$

In particular, $\phi_{t_1} = \phi_x$. The linear system (29), which is (33), takes the form

$$(Z - Z_{-[\lambda]})(P + \partial_x - \lambda^{-1}) + Z_x = 0 .$$

Choosing $P = -I_N \partial_x$ and applying a Miwa shift, this reduces to

$$\lambda^{-1}(Z - Z_{-[\lambda]}) = Z_x ,$$

which is the linear heat hierarchy $Z_{t_n} = \partial_x^n(Z)$, $n = 2, 3, \dots$. Choosing moreover $R = \tilde{R} - I_N \partial_x$, (28) takes the form (38). Now theorem 3.1 reproduces theorem 4.1 in [18]. See also [21, 19, 20] for exact solutions obtained in this way.

4.5. 2-dimensional Toda lattice (2dTL) equation. On smooth functions of x, y and an additional discrete variable, we set

$$df = [\Lambda, f] \xi_1 + [\partial_y, f] \xi_2, \quad \bar{d}f = [\partial_x, f] \xi_1 - [\Lambda^{-1}, f] \xi_2,$$

where Λ is the shift operator in the discrete variable. \mathcal{A} must also contain powers of Λ . Now (5) leads to the noncommutative 2dTL equation

$$\tilde{\phi}_{xy} = (\tilde{\phi}^+ - \tilde{\phi})(I + \tilde{\phi}_y) - (I + \tilde{\phi}_y)(\tilde{\phi} - \tilde{\phi}^-) \quad \text{where } \tilde{\phi} := \phi \Lambda \tag{39}$$

and $\tilde{\phi}^+ = \Lambda(\tilde{\phi})$, $\tilde{\phi}^- = \Lambda^{-1}(\tilde{\phi})$. For a commutative algebra \mathcal{A} , this takes the form

$$(\log(1 + v))_{xy} = v^+ - 2v + v^-$$

in terms of $v = \tilde{\phi}_y$. Choosing $\Delta = \nu \Lambda^{-1}$ with a constant ν , (18) takes the form

$$(g_x g^{-1})_y = g^+ g^{-1} (I - \nu g_y g^{-1}) - (I - \nu g_y g^{-1}) g (g^-)^{-1} . \tag{40}$$

If $g = e^q$ with a scalar function q , this is the *modified Toda equation* [27]

$$q_{xy} = (1 - \nu q_y)(e^{q^+ - q} - e^{q - q^-}) .$$

Inspection of (11) suggests $\Delta = \nu \Lambda^{-1}$, which turns it into $\psi_x = (\tilde{\phi}^+ - \tilde{\phi})\psi + \nu(\psi^+ - \psi)$ and $\nu\psi_y = -\tilde{\phi}_y\psi^- + (\psi - \psi^-)$. Setting $\mathcal{M} = \mathcal{M}' = \Delta' = \Lambda^{-1}$ and $C' = 0$ in section 3.2, (14) and (15) yield the familiar DT $\tilde{\phi}' = \tilde{\phi} + \theta(\theta^-)^{-1}$,

$\psi' = \nu \psi - \theta (\theta^-)^{-1} \psi^-$. For the iterated transformation (with $\Delta_k = \mathcal{M}_k = \Lambda^{-1}$), we obtain in quasideterminant notation (see also [23, 29])

$$\begin{aligned} \tilde{\phi}_{[N+1]} &= \tilde{\phi} + \sum_{k=1}^N \theta_{[k]} (\theta_{[k]}^-)^{-1} \quad \text{where} \quad \theta_{[k+1]} = \left| \begin{array}{ccc} \theta_1 & \cdots & \boxed{\theta_{k+1}} \\ \theta_1^- & \cdots & \theta_{k+1}^- \\ \vdots & \ddots & \vdots \\ \theta_1^{(k-)} & \cdots & \theta_{k+1}^{(k-)} \end{array} \right|, \\ \psi_{[N+1]} &= \left| \begin{array}{cccc} \theta_1 & \cdots & \theta_N & \boxed{\nu^N \psi} \\ \theta_1^- & \cdots & \theta_N^- & \nu^{N-1} \psi^- \\ \vdots & \ddots & \vdots & \vdots \\ \theta_1^{(N-)} & \cdots & \theta_N^{(N-)} & \psi^{(N-)} \end{array} \right|. \end{aligned}$$

If $\psi_{[N+1]}$ is invertible, then it solves the noncommutative (or non-Abelian) modified 2dTL equation (40), and for $\nu = 0$ it solves the ordinary noncommutative 2dTL equation. In a similar way, one recovers the bDT in [29]. In the approach of section 3.5, setting $X = \Lambda \tilde{X}$, $R = \tilde{R} \Lambda^{-1}$, $P = \Lambda^{-1}$, we have

$$\tilde{\Phi} = \Phi \Lambda = Y \tilde{X}^{-1}, \quad \tilde{Z}_x = \tilde{Z}^+ - \tilde{Z}, \quad \tilde{Z}_y = \tilde{Z}^- - \tilde{Z}, \quad \tilde{Z} = \begin{pmatrix} \tilde{X} \\ Y \end{pmatrix},$$

and $\tilde{R} \tilde{X} + QY = \tilde{X}^+$. Extending the last relation to $H \tilde{Z} = \tilde{Z}^+$, we obtain

$$\tilde{Z}_n = e^{x(H-I_{N+M})+y(H^{-1}-I_{N+M})} H^n \tilde{Z}_0,$$

assuming H invertible. Using (34) with (35), and imposing $\text{rank}(Q) = m$ on Q then given by (36), one can now compute explicit solutions of (39) in the algebra of $m \times m$ matrices.

4.6. Lotka-Volterra (LV) lattice equation. Let

$$df = [\Lambda^2, f] \xi_1 + [\Lambda, f] \xi_2, \quad \bar{d}f = \dot{f} \xi_1 + [\Lambda^{-1}, f] \xi_2,$$

with the shift operator Λ and $\dot{f} = f_t$. Introducing $a = \varphi^+ - \varphi - I$ where $\varphi = \phi \Lambda^2$, (5) becomes the (noncommutative) LV lattice equation

$$\dot{a} = a^+ a - a a^-.$$

In terms of $b = g^+ g^{-1}$ and $c = \dot{g} g^{-1}$, the modified Miura transformation (17) with $\Delta = \lambda \Lambda^{-2}$ reads

$$a = -b^- - \lambda (b b^- - b^-), \quad c = 2 - \lambda + (\lambda - 1)(b + b^-) - \lambda b b^-.$$

Since $\dot{b} = c^+ b - b c$ by definition of b and c , we obtain

$$\dot{b} = (\lambda - 1)(b^+ b - b b^-) - \lambda (b^+ b^2 - b^2 b^-),$$

a noncommutative version of the *modified LV lattice* [37]. For a DT, appropriate choices are $\Delta = \lambda \Lambda^{-2}$ and $\mathcal{M} = -\Delta^{-1}$. In the approach of section 3.5, we set

$$X = \Lambda^2 \tilde{X}, \quad R = \tilde{R} \Lambda^{-2}, \quad P = -\tilde{P}^{-1} \Lambda^{-2},$$

with constant \tilde{P} . Then (28) takes the form

$$\tilde{R} \tilde{X} + QY = -\tilde{X}^{++} \tilde{P}^{-1}, \quad (41)$$

and (29) leads to

$$\dot{\tilde{X}} = -(\tilde{X}^{++} - \tilde{X}) \tilde{P}^{-1}, \quad (\tilde{X}^+ - \tilde{X}) = (\tilde{X}^+ - \tilde{X})^- \tilde{P},$$

and the same equations for Y . These equations are solved by

$$\tilde{X}_n = A + B\tilde{P}^n e^{t(\tilde{P}^{-1} - \tilde{P})}, \quad Y_n = C + D\tilde{P}^n e^{t(\tilde{P}^{-1} - \tilde{P})},$$

with constant matrices A, B, C, D . Inserting this in (41), we find the constraints

$$QC + \tilde{R}A = -A\tilde{P}^{-1}, \quad QD + \tilde{R}B = -B\tilde{P}.$$

If $Q = VU^t$, then via $\varphi_n = U^t Y_n \tilde{X}_n^{-1} V$ and $a_n = \varphi_{n+1} - \varphi_n - 1$ we obtain solutions of the scalar LV lattice equation. An extension of (41) in the sense of (31) turns out to be too restrictive.

5. Final remarks. A BDGA formulation of an integrable system is a zero curvature condition for a (generalized) connection that is *linear* in the “spectral” parameter, a situation well known from the (anti-) self-dual Yang-Mills system. This restriction is what essentially enabled us to work out calculations, that appeared in the literature for specific integrable systems, in a universal way. Although many integrable models indeed fit into this framework, it is not clear to what extent this actually covers the existing variety of integrable systems. The existence of a BDGA formulation does not mean that there is a Lax pair (in standard form) that is linear in the spectral parameter. Typically a Lax pair can be derived from (11) and then will turn out to exhibit a *nonlinear* parameter dependence. Because of page limitations the examples in this work were not treated in detail. We plan to elaborate some of them in more detail elsewhere.

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