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SPIRAL MOTION IN CLASSICAL MECHANICS

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Abstract. We present various models in classical mechanics which exhibit 'exotic' orbits. We give an example of a smooth $|\mathbf{r}|$-independent potential V in dimension three, which exhibits an orbit that spirals as time goes to infinity. This kind of orbits cannot occur for this class of potentials in dimension two [4] or, see below, if $Cr = \{\omega \in S^{n-1} : \nabla V(\omega) = 0\}$, $n \geq 3$, is totally disconnected. In addition, for each $\mu > 2$ we give an example of a potential of the form $V(r, \theta) = O(r^{-\mu})$, in two dimensions, which is not radially symmetric and has a zero-energy orbit that escapes towards infinity in spirals. Zero energy orbits escaping towards infinity in spirals cannot occur for radial potentials with the same rate of decay.

1. **Introduction.** In this note we study some questions concerning the asymptotic behavior, as t goes to infinity, of the solutions $\mathbf{r}(\cdot) : (0, \infty) \to \mathbb{R}^n \setminus \{0\}$ of Newton's equation

$$\frac{d^2\mathbf{r}(t)}{dt^2} = -\nabla V(\mathbf{r}(t)), \tag{1}$$

for various models. First we consider the class of *homogeneous potentials of degree zero* in dimensions greater than or equal three and then the class of potentials *strongly decaying at infinity* in dimension two. A potential V is said to be *strongly decaying at infinity* if $V(\mathbf{r}) = O(|\mathbf{r}|^{-\mu})$, as $r \to \infty$, for some $\mu > 2$. A smooth real-valued function V on $\mathbb{R}^n \setminus \{0\}$, $n \geq 2$, is called homogeneous potential of degree zero if

$$\mathbf{r} \cdot \nabla V(\mathbf{r}) = 0. \tag{2}$$

For brevity, potentials satisfying (2) will be referred to in this paper as *angular potentials.* Note that (2) is equivalent to the fact that V is independent of $|\mathbf{r}|$; that is to say, if $\omega = \mathbf{r}/|\mathbf{r}|$, then $V(\mathbf{r}) = V(\omega)$. In particular, in two dimensions this means that if r and θ are the usual polar coordinates, then V depends only on θ.

The class of potentials satisfying (2) was considered first in [4], where the investigation of the asymptotic behavior of the solutions to (1) was viewed as a source of intuition for the analysis of the quantum scattering for the Schrödinger operator $H = -(1/2)\Delta + V$, which has been developed in [4], [5], and [6]. Let $Cr = \{\omega \in S^{n-1} : \nabla V(\omega) = 0\}$. A result obtained in [4] for the motion of a classical

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particle in two dimensional angular potentials states that if $\mathbf{r}(\cdot):(0,\infty)\to\mathbb{R}^2\setminus\{0\}$ is a solution to (1) and $\omega(t)=\mathbf{r}(t)/|\mathbf{r}(t)|$, then

$$\lim_{t\to\infty}\omega(t)=\omega_0 \tag{3}$$

exists and $\omega_0\in Cr$. For general homogeneous potentials of degree zero it was also proved in [4] that if Cr is finite then the limit (3) exists and belongs to Cr, and a similar result to the two dimensional one was conjectured for $n\geq 3$.

In this note we prove

Theorem 1.1. *Let V be as in (2) and $\mathbf{r}(\cdot):(0,\infty)\to\mathbb{R}^n\setminus\{0\}$ a solution of (1). If $Cr=\{\omega\in S^{n-1}:\nabla V(\omega)=0\}$ is totally disconnected, then*

$$\lim_{t\to\infty}\omega(t)=\omega_0$$

exists and $\omega_0\in Cr$.

Although Theorem 1.1 is just a slight improvement of the result established in [4] for finite Cr in dimension $n\geq 3$, we show that it is optimal in the sense that if Cr has a nontrivial connected component, then the limit (3) may not exists. More precisely we have

Theorem 1.2. *There exists a smooth real valued function V on $\mathbb{R}^3\setminus\{0\}$ satisfying (2) such that the set of critical points of $V|\mathbb{S}^2$ contains the equator of $\mathbb{S}^2$ and (1) has a solution $\mathbf{r}(\cdot):(0,\infty)\to\mathbb{R}^3\setminus\{0\}$ for which $\omega(t)=\mathbf{r}(t)/|\mathbf{r}(t)|$ does not have a limit as t goes to infinity.*

For the class of strongly decaying potentials at infinity, we show that for every $\mu>2$ there is a negative and nonradially symmetric potential in two dimensions satisfying $V(\mathbf{r})=O(|\mathbf{r}|^{-\mu})$, for which (1) possesses a zero energy solution that escapes to infinity in spirals. Moreover, we show that this last cannot occur for radial potentials in this class.

Finally, we present a brief discussion for potentials of the form $V(\mathbf{r})=-\gamma|\mathbf{r}|^{-\mu}$, with $\gamma>0$ and $\mu>0$.

2. **Preliminaries.** In order to provide the reader with a self-contained context for our exposition, in this section we introduce notation and state some estimates and facts from [4] that will be used in the proofs of Theorems 1.1 and 1.2. If $\mathbf{r}(\cdot):(0,\infty)\to\mathbb{R}^n\setminus\{0\}$ is a solution to (1) we write

$$r(t)=|\mathbf{r}(t)|,\qquad \dot f=\frac{df}{dt},\qquad p(t)=\dot{\mathbf{r}}(t),\qquad \omega(t)=\mathbf{r}(t)/r(t),$$
$$\text{and}\qquad p_\perp(t)=p(t)-(\omega(t)\cdot p(t))\,\omega(t).$$

With this notation at hand we have

Theorem 2.1. *Suppose V satisfies (2). If $\mathbf{r}(\cdot):(0,\infty)\to\mathbb{R}^n\setminus\{0\}$ is a solution to (1), then*

$$\int_0^\infty\frac{|p_\perp(t)|^2}{r(t)}dt<\infty,\qquad \textit{and}\qquad \int_0^\infty\frac{|\nabla V(\omega(t))|^2}{r(t)}dt<\infty. \tag{4}$$

In addition, for some $\lambda\in[0,\infty)$ we have

$$\dot r(t)\nearrow\lambda, \tag{5}$$

as $t\nearrow\infty$.

Proof. See [4]. □

Before we turn to the proof of Theorem 1.1 we prove an elementary topology lemma.

Lemma 2.2. *If K is a compact totally disconnected subset of a separable metric space (X, d), then for every x and y in K, $x \neq y$, there exist disjoint open subsets U and V of X such that $x \in U$, $y \in V$, and $K \subset U \cup V$.*

Proof. Let P be the set of all condensation points of K; that is to say, $x \in P$ if $N \cap K$ is uncountable for every open neighborhood N of x. Note that P is a totally disconnected perfect set; moreover, $C = K \setminus P$ is countable since otherwise, by the separability of X, there would be $a \in X \setminus P$ and a closed ball $B[a, \delta] \subset X \setminus P$, with $\delta > 0$, such that $K_1 = B[a, \delta] \cap K$ is uncountable. Thus, since K_1 is compact, K_1 would have a condensation point of K which is not in P. Now, let $x, y \in K = P \cup C$ such that $x \neq y$. If either x or y belongs to C, say $x \in C$, then there is $\delta \in (0, d(x, y))$ such that $B(x, \delta) \subset X \setminus P$. Since C is countable, there is $r \in (0, \delta)$ such that $C \cap \partial B[x, r] = \emptyset$. Hence $U = B(x, r)$ and $V = X \setminus B[x, r]$ satisfy the desired conditions. Suppose now that both x and y belong to P. Since P is homeomorphic to the Cantor set [3], there are open sets U_1 and U_2 in X such that $x \in U_1$, $y \in U_2$, $(P \cap U_1) \cap (P \cap U_2) = \emptyset$, and $P \subset U_1 \cup U_2$. It follows that $P_1 = P \cap U_1$ and $P_2 = P \cap U_2$ are compact and disjoint and therefore $\alpha = \inf\{d(x, y) : x \in P_1, y \in P_2\} > 0$. Thus, since C is countable, there is $r \in (0, \alpha)$ such that $C \cap \{z \in X : d(z, P_1) = r\} = \emptyset$. Therefore $U = \{z \in X : d(z, P_1) < r\}$ and $V = \{z \in X : d(z, P_1) > r\}$ enjoy the desired properties. □

Now we are ready to prove Theorem 1.1.

Proof. Following [4], we begin by showing that

$$\lim_{t \to \infty} \nabla V(\omega(t)) = 0. \tag{6}$$

A calculation yields

$$\frac{d}{dt}\left(|\nabla V(\omega(t))|^2\right) = 2 \sum_{j,k} \partial_j V(\omega(t)) \partial_k \partial_j V(\omega(t)) \frac{p_\perp(t)_k}{r(t)}.$$

Using (4), it follows from this last expression and the estimate

$$\int_0^\infty \frac{|\partial_j V(\omega(t)) p_\perp(t)_k|}{r(t)} dt \le \left(\int_0^\infty \frac{|p_\perp(t)_k|^2}{r(t)} dt\right)^{1/2} \left(\int_0^\infty \frac{|\partial_j V(\omega(t))|^2}{r(t)} dt\right)^{1/2}$$

that $\frac{d}{dt}\left(|\nabla V(\omega(t))|^2\right)$ is integrable on $(0, \infty)$. Hence $\alpha = \lim_{t \to \infty} |\nabla V(\omega(t))|^2$ exists and

$$\alpha = |\nabla V(\omega(0))|^2 + \int_0^\infty \frac{d}{dt}\left(|\nabla V(\omega(t))|^2\right) dt.$$

If $\alpha > 0$, then there exists $T > 0$ such that $|\nabla V(\omega(t))|^2 \ge \alpha/2$ for all $t \ge T$. Therefore, since (5) implies that

$$r(t) \le r(1) + \lambda(t - 1), \tag{7}$$

we find that

$$\int_0^\infty \frac{|\nabla V(\omega(t))|^2}{r(t)} dt \geq \frac{\alpha}{2} \int_T^\infty \frac{dt}{r(t)} = \infty,$$

which contradicts the second estimate in (4). Thus $\alpha = 0$. Suppose now that Cr is totally disconnected and that $\omega(t)$ does not converge as $t \to \infty$. Then there is a sequence t_k going to infinity for which $\omega(t_k)$ does not converge as $k \to \infty$. By compactness of $\mathbb{S}^{n-1}$, the sequence $\omega(t_k)$ necessarily possesses at least two different limit points ω_0, ω_1. Thus, by the assumption on Cr and Lemma 2.2 there exist two disjoint open sets U and V in $\mathbb{S}^{n-1}$ such that $\omega_0 \in U$, $\omega_1 \in V$, and $Cr \subset U \cup V$. Hence we can pick two sequences σ_k and τ_k, both diverging to infinity, such that for all k we have $\sigma_k < \tau_k < \sigma_{k+1}$ with $\omega(\sigma_k) \in U$ and $\omega(\tau_k) \in V$. For every k let

$$\upsilon_k = \sup\{t > \sigma_k : \omega(s) \in U, \text{ for } s \in [\sigma_k, t]\}.$$

By continuity of $\omega(t)$ we have that $\sigma_k < \upsilon_k < \tau_k$ and $\omega(\upsilon_k) \in \partial U$. By compactness of ∂U, the sequence $\omega(\upsilon_k)$ has a limit point $\omega_2 \in \partial U$. Hence by (6) and the continuity of ∇V we find that $\nabla V(\omega_2) = 0$, which is impossible since $Cr \subset U \cup V$. □

3. **Spiral motion in angular potentials.** In this section we prove Theorem 1.2. To this end we exhibit a smooth function V, homogeneous of degree zero on $\mathbb{R}^3 \setminus \{0\}$, such that the equator of $\mathbb{S}^2$ is a subset of the set Cr of critical points of $V|\mathbb{S}^2$, and such that the corresponding Newton's equation has a solution $\mathbf{r}(\cdot)$ for which $\omega(t)$ spirals towards the equator of $\mathbb{S}^2$ and therefore the limit (3) does not exist.

The idea of finding an example of the type given below was suggested by the two dimensional gradient vector field constructed in [7], which has an orbit that spirals towards the unit circle, and the way of defining V was suggested to J. Cruz by I. Herbst in a private conversation.

Proof. Let T be a homogeneous function of degree zero of class C^∞ on $\mathbb{R}^3 \setminus \{0\}$, and consider the function

$$S(x, y, z) = rT(x, y, z), \tag{8}$$

where $r = \sqrt{x^2 + y^2 + z^2}$. Thus S is a smooth homogeneous function of degree one on $\mathbb{R}^3 \setminus \{0\}$, and therefore

$$V(\mathbf{r}) = -\frac{1}{2}|\nabla S(\mathbf{r})|^2 \tag{9}$$

is a smooth homogeneous function of degree zero on the same set. Moreover, if $\mathbf{r}(t)$ satisfies

$$\dot{\mathbf{r}} = \nabla S(\mathbf{r}), \tag{10}$$

setting $p = \dot{\mathbf{r}}$, $q = \mathbf{r}$, and $H(p, q) = p^2/2 + V(q)$, by Hamilton's equations we have

$$\ddot{\mathbf{r}} = \dot{p} = -\frac{\partial H}{\partial q} = -\nabla V(\mathbf{r}).$$

Hence $\mathbf{r}(t)$ is a solution to Newton's equation (1) and, using (8) and (10), we find that

$$\frac{d\omega}{dt} = \frac{\dot{\mathbf{r}}}{r} - \frac{\mathbf{r} \cdot \dot{\mathbf{r}}}{r^3}\mathbf{r} = \frac{\nabla S}{r} - \frac{S}{r^3}\mathbf{r} = \nabla T(\omega).$$

To construct our example we will show that, for an appropriate T, the dynamical system

$$\dot{\omega} = \nabla T \tag{11}$$

on $\mathbb{S}^2$ has an orbit $\omega(t)$ that spirals towards the equator of $\mathbb{S}^2$ as t goes to infinity. To this end we consider the function f given in cylindrical coordinates by

$$\begin{aligned} f(\rho,\theta,z) &= e^{-r^2/z^2}\sin\left(\frac{r}{r-\rho}-\theta\right), \quad z\neq 0; \\ &= 0, \qquad z=0, \end{aligned}$$

where r is as in (8) and

$$x=\rho\cos\theta, \qquad y=\rho\sin\theta.$$

Note that f is homogeneous of degree zero but fails to be differentiable for $\rho = 0$. So, we consider the function $T = \phi f$, where ϕ is a suitable smooth cut-off function on the unit sphere such that, for sufficiently small $\epsilon > 0$, say $0 < \epsilon < 1/16$, ϕ vanishes if $|z| \geq 1-\epsilon$ and equals 1 if $|z| \leq 1-2\epsilon$. For this choice of T, the equator of $\mathbb{S}^2$ is contained in the set of critical points of V. Now we show that (11) has an orbit $\omega(t)$ on the Northern hemisphere of $\mathbb{S}^2$ which spirals towards the equator as t goes to infinity.

Proceeding as in [7] we consider the open set U on $\mathbb{S}^2$ given by

$$U=\left\{(\rho,\theta,z)\in\mathbb{S}^2 : \theta>0, z>0, \quad \text{and} \quad T(\rho,\theta,z)<0\right\}.$$

Note that if I is the piece of the meridian $\theta = 0$ given by

$$I=\{(\rho,0,z)\in\mathbb{S}^2 : 1-\frac{1}{\pi}\leq\rho\leq 1-\frac{1}{2\pi}\},$$

then U is a spiral-like region on $\mathbb{S}^2$ which lies between I and the spirals

$$A_1=\left\{(\rho,\theta,z)\in\mathbb{S}^2 : \rho=1-\frac{1}{\theta+\pi}, \theta\geq 0\right\}$$

and

$$A_2=\left\{(\rho,\theta,z)\in\mathbb{S}^2 : \rho=1-\frac{1}{\theta+2\pi}, \theta\geq 0\right\}.$$

Note also that, for $i = 1, 2$, A_i is contained in $T^{-1}(0)$ and thus for any $q \in A_i$ we have that $\nabla T(q)$ is perpendicular to A_i and points towards the exterior of U. Moreover, if $L(p)$ is the part of the level set of $p \in I$ that is contained in U, then $L(p)$ approaches the equator of $\mathbb{S}^2$ as p approaches either of the ends of I. Thus, if we fix $q_i \in A_i$, then the negative orbit of q_i necessarily intersects I at a point $p_i = (\rho_i, \theta_i, z_i)$ such that

$$a\equiv 1-\frac{1}{\pi}\leq\rho_1<\rho_2\leq 1-\frac{1}{2\pi}\equiv b.$$

This implies that the set

$$J=\{\rho\in[a,b] : \text{the positive orbit of } (\rho,0,z) \text{ exits } U \text{ transversally to } A_2\ \}$$

is nonempty and bounded below by ρ_1. It can easily be verified that if $\rho_0 = \inf J$ and $P = (\rho_0, 0, z)$, then $P \in I$ and its positive orbit must stay in U and spiral towards the equator of $\mathbb{S}^2$. □

4. **Spiral motion in negative rapidly decaying potentials.** For $\mu > 2$, let S be the function on $\mathbb{R}^2 \setminus \{0\}$ given in polar coordinates by $S(r,\theta) = r^{(2-\mu)/2} \sin(\log r - \theta)$. It is easily verified that if we set $V(\mathbf{r}) = -|\nabla S(r,\theta)|^2/2$, with $\mathbf{r} = r(\cos\theta, \sin\theta)$, then $V(\mathbf{r}) = O(r^{-\mu})$. Moreover, using the same strategy of the previous section, it is easy to see that if $\mathbf{r}(t)$ satisfies $\dot{\mathbf{r}} = \nabla S(\mathbf{r})$, then $\mathbf{r}$ is a zero energy solution of (1) and there is $P \in [\pi, 2\pi] \times \{0\}$ whose positive orbit escapes to infinity inside the spiral region

$$U = \{(r,\theta) : e^{\pi+\theta} < r < e^{2\pi+\theta}\}.$$

The relevance of this example lies in the fact that zero energy orbits escaping towards infinity in spirals cannot occur for radial potentials of the form $V(r) = O(r^{-\mu})$, with $\mu > 2$. In fact, if V is radial then the angular momentum L is preserved and if $\mathbf{r} = r(\cos\theta, \sin\theta)$ is a zero energy orbit of (1), then $\dot{\theta} = Lr^{-2}$ and, by conservation of energy, $\dot{r}^2 = -2V(r) - L^2 r^{-2}$. Thus, if $L = 0$ the orbit is a straight line and if $L \neq 0$ the orbit is bounded, since $r^2\dot{r}^2 \leq (C/r^{\mu-2}) - L^2$. On the other hand, it follows from a well known fact [8] that if V is a smooth potential on $\mathbb{R}^n$, $n \geq 2$, such that $V(\mathbf{r}) = O(r^{-\mu})$, with $\mu > 2$, and $\mathbf{r}(t)$ is a positive energy orbit of (1), then $\lim_{t\to\infty} \omega(t)$ exists.

We find interesting to note that for $V(\mathbf{r}) = -\gamma|\mathbf{r}|^{-\mu}$, the only value of $\mu > 0$ for which (1) may have a zero energy orbit that escapes to infinity in spirals is $\mu = 2$. This is so because, as we showed above, for $\mu > 2$, the zero energy orbits of (1) are either bounded or straight lines; and for $\mu \in (0,2)$, it has been proved in [2] that the zero energy solutions of (1) which escape to infinity are given implicitly by

$$r(t)^{2-\mu} = \frac{2}{1 + \cos(\pi - (2-\mu)\theta(t))}.$$

Note that, for fixed μ, the number of times that the orbit winds around the origin is finite. It can actually be proved [2] that $\lim_{t\to\infty} \theta(t) = 2\pi/(2-\mu)$ and thus $\lim_{t\to\infty} \omega(t)$ exists. Note also that the number of times that these orbits wind around the origin increases to infinity as $\mu \nearrow 2$. The spirals obtained in this limiting situation are known as *Cote's spirals* [1]. Since for $\mu = 2$ we have $\dot{\theta} = L/r^2$ and $r^2\dot{r}^2 = 2\gamma - L^2$, then an orbit that escapes towards infinity may exist only if $c = 2\gamma - L^2 > 0$. In this case, there are constants α and β such that $r(t)^2 = 2\sqrt{c}t + \alpha$ and $\theta(t) = (L/2\sqrt{c})\log(2\sqrt{c}t + \alpha) + \beta$. Furthermore, for $\mu \in (0,2)$, it has been proved in [2] that if V is a potential on $\mathbb{R}^n$, $n \geq 2$, that satisfies

$$V(\mathbf{r}) = -\gamma|\mathbf{r}|^{-\mu} + O(|\mathbf{r}|^{-\mu-\epsilon}),$$

where $\gamma > 0$, $\epsilon > 0$, and $\mathbf{r}(t)$ is a solution to (1) such that $|\mathbf{r}(t)| \to \infty$ as $t \to \infty$, then $\lim_{t\to\infty} \omega(t)$ exists, for any $\lambda \geq 0$. Finally, the same reference provides an example of a negative potential in two dimensions that is not radially symmetric, satisfies $V(x) = O(|x|^{-\mu})$, as $|x| \to \infty$, and exhibits an orbit that is a logarithmic spiral.

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