DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Supplement 2009

Website: www.aimSciences.org

pp. **181–190**

# COMPARISON AMONG DIFFERENT NOTIONS OF SOLUTION FOR THE $p$-SYSTEM AT A JUNCTION

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Abstract. We consider $n$ tubes exiting a junction and filled with a non viscous isentropic or isothermal fluid. In each tube a copy of the $p$-system in Euler coordinates is considered. The aim of the presentation is to compare three different notions of solutions at the junctions: p-solutions, Q-solutions and P-solutions.

## 1. **Introduction.**

Consider $n$ rectilinear tubes exiting a junction. For $l = 1, \ldots, n$, the direction and section of the $l$th tube are described, respectively, by the direction and the norm of a vector $\nu_l \in \mathbb{R}^3 \setminus \{0\}$. All tubes are filled with the same non-viscous isentropic or isothermal fluid, and friction along the walls is neglected. The resulting system can be modeled through $n$ copies of the one-dimensional $p$-system in Eulerian coordinates:

$$\begin{cases} \partial_t \rho_l + \partial_x q_l = 0, \\ \partial_t q_l + \partial_x \left( \dfrac{q_l^2}{\rho_l} + p(\rho_l) \right) = 0, \end{cases} \qquad \begin{array}{rcl} t & \in & [0, +\infty[\,, \\ x & \in & [0, +\infty[\,, \\ l & \in & \{1, \ldots, n\}\,. \end{array} \tag{1}$$

Here, $t$ is time, and, along the $l$th tube, $x$ is the abscissa, $\rho_l$ is the fluid density, and $q_l$ is the linear momentum density. The pressure law $p = p(\rho)$ is the same along all tubes. The case $n = 2$ has been widely considered in the literature, see [13, 14, 15]. Analogous analytical problems are motivated by traffic flow modeling, see [11].

The aim of this paper is to present and compare three notions of solutions. p-solutions were introduced in [1, 2] and require that the traces of the pressure at the junction be the same for all tubes. This is a standard condition found in engineering manuals and often refers to situations that are either static or with small variations in the speed. Q-solutions, introduced in [6] and slightly modified in [5], impose that the variation in the total linear momentum across the junction be parallel to a given direction dependent on the geometry of the junction itself. This condition does not ensure the uniqueness of the solution to Riemann Problems, see Example 7 below.

2000 *Mathematics Subject Classification.* Primary: 35L65; Secondary: 76N10.

*Key words and phrases.* $p$-System, Hyperbolic Systems of Conservation Laws, Riemann Problem, Compressible Fluids, Gas Dynamics.

Then, P-solutions require that the trace of the flow of the linear momentum be the same for all tubes. As shown in [5], this condition does *not* imply the conservation of the total linear momentum, see also Section 5 below. P-solutions are indirectly justified as limits of Q-solutions, see Proposition 2 below. Indeed, with 2 or 3 pipes, when Q-solutions are unique, they coincide with P-solutions. When Q-solutions are not unique, then P-solutions are the unique limit of Q-solutions. A deeper analysis of these and other definitions of solutions for the Riemann problem for (1) is deferred to [7].

Here, we remark that the present analytical framework fits also to other applications, see for instance [3, 8, 9, 12]. Particularly relevant is the application to networks of open canals, in which $\rho$ plays the role of the water level and $p(\rho) = (1/2)g\rho^2$, $g$ being gravity.

We remark that in the present description, the geometry of the junction has a key role, hidden in the choice of the $x$-axes. Indeed, as shown in Section 5, the variation of the total linear momentum of the fluid crossing the junction is heavily dependent on the relative positions of the pipes exiting the junction.

Remark that the present description neglects the effects of gravity and friction. From an analytical point of view, their consideration amounts to the introduction of suitable source terms along the pipes, whose treatment is standard, see [8].

2. **The Model.** This section is devoted to definitions and results concerning the Riemann problem for (1) at a junction, see [4, 6]. The physics of the fluid is described by its **E**quation **o**f **S**tate, which reduces, in the present isentropic or isothermal setting, to the definition of a pressure law. (Here $\mathbb{R}^+ = [0, +\infty[$ and $\mathring{\mathbb{R}}^+ = ]0, +\infty[$.)

**(EoS):** The pressure law $p = p(\rho)$ satisfies $p \in \mathbf{C^2}\left(\mathbb{R}^+; \mathbb{R}^+\right)$, $p(0) = 0$, $p' > 0$, and $p'' \geq 0$.

A typical example is the $\gamma$-law, given for fixed $\rho_*, p_* > 0$ and $\gamma \geq 1$ by

$$p(\rho) = p_* \cdot \left(\rho/\rho_*\right)^\gamma \tag{2}$$

A further relevant quantity is the flow of the linear momentum, or *"dynamic pressure"*

$$P(\rho, q) = \frac{q^2}{\rho} + p(\rho)\,.$$

We refer below to $P$ as to the *dynamic pressure*. As is well known (see [10, formula (3.3.21)]), the pair $(E, F)$ plays the role of the (mathematical) entropy-entropy flux pair. Introduce also the following regions, shown in Figure 1, see also Figure 7:

$$\begin{aligned}
A_- &= \left\{\mathbb{R}^+ \times \mathbb{R} \colon \lambda_2(\rho, q) < 0\right\}, & A_0^- &= \left\{\mathbb{R}^+ \times \mathbb{R} \colon \lambda_2(\rho, q) \geq 0,\ q \leq 0\right\},\\
A_+ &= \left\{\mathbb{R}^+ \times \mathbb{R} \colon \lambda_1(\rho, q) > 0\right\}, & A_0^+ &= \left\{\mathbb{R}^+ \times \mathbb{R} \colon \lambda_1(\rho, q) \leq 0,\ q \geq 0\right\},\\
A_0 &= A_0^- \cup A_0^+ .
\end{aligned} \tag{3}$$

Above, as usual, $\lambda_i$ is the $i$th characteristic speed, see (6) for its expression. We shall often refer to $A_0$ as to the *subsonic* region.

Consider $n$ ducts exiting a single junction. Each tube is modeled by $\mathbb{R}^+$, and the junction is at $x = 0$. The $l$th pipe is described by a vector $\nu_l$ parallel to it that exits the junction and whose norm $\|\nu_l\|$ equals the section of the duct, see Figure 2. The standard *"no junction"* situation fits in the present framework setting $n = 2$ and $\nu_1 + \nu_2 = 0$, see Figure 3, left and Example 1 below. An elbow correspond to $n = 2$ and $\nu_1$, $\nu_2$ linearly independent, see Figure 2, left and Example 3 below. Junctions

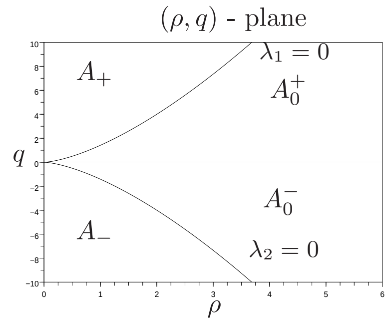


FIGURE 1. The regions defined in (3).

with $n = 3$ are in Figure 2, right, and in Figure 4. This choice makes the geometry of the junction intrinsic to the structure of the model.

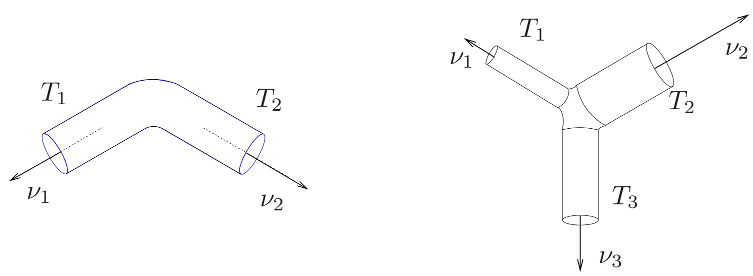


FIGURE 2. Examples of junctions and notation used in Section 2.

Assigning at time $t = 0$ a constant initial state $(\bar\rho_l, \bar q_l) \in \mathbb{R}^+ \times \mathbb{R}$ in each of the $n$ ducts exiting the junction (i.e. for $l \in \{1, \ldots, n\}$), we have a *Riemann problem at the junction*:

$$\begin{cases} \partial_t \rho_l + \partial_x q_l = 0, & t \in \mathbb{R}^+, \\ \partial_t q_l + \partial_x \left( \dfrac{q_l^2}{\rho_l} + p(\rho_l) \right) = 0, & x \in \mathbb{R}^+, \\ & l \in \{1, \ldots, n\}, \\ (\rho_l, q_l)(0, x) = (\bar\rho_l, \bar q_l), & (\bar\rho_l, \bar q_l) \in \mathring{\mathbb{R}}^+ \times \mathbb{R}. \end{cases} \tag{4}$$

Remark that, in any tube, the $x$-axis, as well as the vector $\nu_l$, is directed off from the junction, independently from the direction of the fluid flow, which is given by the sign of the linear momentum density $q_l$.

As usual when considering Riemann problems, we equip $\mathbf{BV}\left(\mathbb{R}^+; (\mathring{\mathbb{R}}^+ \times \mathbb{R})\right)$ with the $\mathbf{L^1_{loc}}$ topology. In order to state various definitions of solutions to (4), we introduce the following conditions on an $n$-tuple $(\rho, q) \equiv \big((\rho_1, q_1), \ldots, (\rho_n, q_n)\big)$ of functions $(\rho_l, q_l) \in \mathbf{C^0}\left(\mathbb{R}^+; \mathbf{BV}(\mathbb{R}^+; (\mathring{\mathbb{R}}^+ \times \mathbb{R}))\right)$, for $l \in \{1, \ldots, n\}$:

**(L):** For all $l = 1, \ldots, n$, the function $(t, x) \to (\rho_l, q_l)(t, x)$ is self-similar and coincides with the restriction to $x \in \mathring{\mathbb{R}}^+$ of the **L**ax solution to the standard

Riemann problem

$$\begin{cases} \partial_t \rho_l + \partial_x q_l = 0, \\ \partial_t q_l + \partial_x \left( \frac{q_l^2}{\rho_l} + p(\rho_l) \right) = 0, \\ (\rho_l, q_l)(0,x) = \begin{cases} (\bar\rho_l, \bar q_l) & \text{if } x > 0, \\ (\rho_l, q_l)(1, 0+) & \text{if } x < 0 . \end{cases} \end{cases}$$

**(M):** **M**ass is conserved at the junction; i.e., for a.e. $t > 0$,

$$\sum_{l=1}^{n} \|\nu_l\| \, q_l(t, 0+) = 0 .$$

**(P):** The trace of the dynamic **P**ressure $P$ is the same along all tubes; i.e., there exists a positive $P_*$ such that for $l = 1, \ldots, n$, and for a.e. $t > 0$

$$P\left(\rho_l(t, 0+), q_l(t, 0+)\right) = P_* .$$

**(Q):** The component of the total linear momentum $q$ orthogonal to $\sum_{l=1}^{n} \nu_l$ is conserved, i.e. for all $t > 0$

$$\forall \, \eta \in \left( \sum_{l=1}^{n} \nu_l \right)^{\perp}, \quad \left( \sum_{l=1}^{n} P\left(\rho_l(t, 0+), q_l(t, 0+)\right) \nu_l \right) \cdot \eta = 0 .$$

**(p):** The trace of the static **p**ressure $p$ is the same along all tubes; i.e., there exists a positive $p_*$ such that for $l = 1, \ldots, n$, and for a.e. $t > 0$

$$p\left(\rho_l(t, 0+)\right) = p_* .$$

Condition **(p)** is often used for gases, although fully justified only in the static case. We include it here for its particular meaning when (1) describes water flowing in open canals. Then, **(p)** amounts to require that the trace of the water level in all canals is the same.

Note that **(Q)** can be rephrased as follows, see [5, p. 1457]:

**(Q):** The variation in the total linear momentum is along the direction $\sum_{l=1}^{n} \nu_l$, i.e. there exists a $\lambda \in \mathbb{R}$ such that

$$\sum_{l=1}^{n} P\left(\rho_l(t, 0+), q_l(t, 0+)\right) \nu_l = \lambda \sum_{l=1}^{n} \nu_l . \tag{5}$$

Further comments on the conservation of the total linear momentum are in Section 5. We are now ready to state some different definitions of solution to (4).

**Definition 2.1.** Fix $(\rho, q) \in \mathbf{C^0}\left(\mathbb{R}^+; \mathbf{BV}(\mathbb{R}^+; (\mathring{\mathbb{R}}^+ \times \mathbb{R})^n)\right)$. We say that:

1. $(\rho, q)$ is a p-solution to (4) if it satisfies **(L)**, **(M)** and **(p)**.
2. $(\rho, q)$ is a P-solution to (4) if it satisfies **(L)**, **(M)** and **(P)**.
3. $(\rho, q)$ is a Q-solution to (4) if it satisfies **(L)**, **(M)** and **(Q)**.

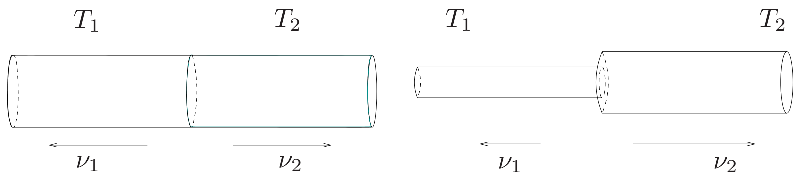


FIGURE 3. Junctions between two tubes with the same section, on the left, and with different sections, on the right.

3. **Comparison Between Conditions (P), (Q) and (p).**

**Example 1** Two pipes with the same section and opposite directions, i.e. $n = 2$ and $\nu_1 + \nu_2 = 0$, see Figure 3, left. Then, **(Q)** reduces to $P\left(\rho_1(t,0+), q_1(t,0+)\right)\nu_1 + P\left(\rho_2(t,0+), q_2(t,0+)\right)\nu_2 = 0$ and it is equivalent to **(P)**.

**Example 2** Two pipes with different sections and opposite directions, say $\nu_1 = (-1,0,0)$ and $\nu_2 = (2,0,0)$, see Figure 3, right. Since $\nu_1 + \nu_2 = (1,0,0)$, condition **(Q)** does not imply any constraint on the values $P\left(\rho_1(t,0+), q_1(t,0+)\right)$ and on $P(\rho_2(t,0+), q_2(t,0+))$. In this case, **(Q)** is not equivalent to **(P)**.

**Example 3** An elbow consisting of two non collinear pipes with the same section, say $\nu_1 = (-1,-1,0)$ and $\nu_2 = (1,-1,0)$, see Figure 2, left. Now, $\nu_1+\nu_2 = (0,-2,0)$; hence condition **(Q)** reads $P\left(\rho_1(t,0+), q_1(t,0+)\right) = P\left(\rho_2(t,0+), q_2(t,0+)\right)$ and **(Q)** is equivalent to **(P)**.

**Example 4** A T-junctions between three tubes with the same section, say $\nu_1 = (-1,0,0)$, $\nu_2 = (1,0,0)$ and $\nu_3 = (0,-1,0)$, see Figure 4, left. We have $\nu_1+\nu_2+\nu_3 = \nu_3$, hence **(Q)** implies $P\left(\rho_1(t,0+), q_1(t,0+)\right) = P\left(\rho_2(t,0+), q_2(t,0+)\right)$, but it does not give any constraint on the dynamic pressure $P\left(\rho_3(t,0+), q_3(t,0+)\right)$ in tube 3. Therefore, in this situation, **(Q)** is not equivalent to **(P)**.

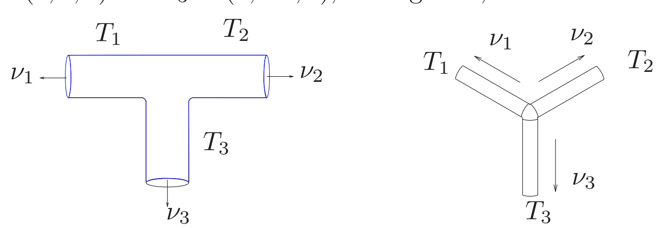


FIGURE 4. Junctions among three tubes with the same section. Left, a T-junction and, right, 3 tubes at an angle $2\pi/3$.

**Example 5** Three coplanar tubes with the same section at an angle $2\pi/3$, i.e. $\nu_1 = (-\sqrt{3}/2, 1/2, 0)$, $\nu_2 = (\sqrt{3}/2, 1/2, 0)$ and $\nu_3 = (0,-1,0)$, see Figure 4, right. Now, $\nu_1+\nu_2+\nu_3 = 0$, hence **(Q)** implies $P\left(\rho_1(t,0+), q_1(t,0+)\right) = P\left(\rho_2(t,0+), q_2(t,0+)\right)$ and $P\left(\rho_1(t,0+), q_1(t,0+)\right) = P\left(\rho_3(t,0+), q_3(t,0+)\right)$. Therefore, **(Q)** $\Longleftrightarrow$ **(P)**.

**Example 6** Three coplanar tubes with the same section and with $\nu_1, \nu_2, \nu_3$ pairwise linearly independent. Fix $\bar q > 0$ and densities $\bar\rho_1, \bar\rho_2$ such that $\bar\rho_1 > \bar\rho_2$ and $P(\bar\rho_1, \bar q) = P(\bar\rho_2, \bar q)$, see Figure 5. The Cauchy problem for (1) with initial datum

$$(\rho_1, q_1)(x) = \begin{cases} (\bar\rho_2, \bar q_2) \text{ if } x \in [\varepsilon, +\infty[ \\ (\bar\rho_1, \bar q_1) \text{ if } x \in [0, \varepsilon[ \end{cases} \qquad \begin{array}{rcl} (\rho_2, q_2)(x) & = & (\bar\rho_1, \bar q_1) \\ (\rho_3, q_3)(x) & = & (\bar\rho_1, -2\bar q_1) \end{array}$$

is solved by a stationary p-solution, for any $\varepsilon > 0$. At the limit $\varepsilon \to 0+$, the above initial datum converges to the Riemann datum which does not admit a stationary p-solution. For further details see [4, Section 2.2].

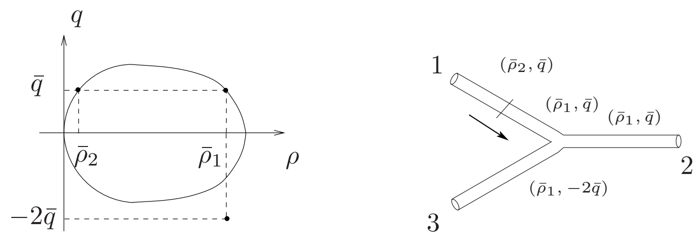


FIGURE 5. Notation for Example 6.

The following result presents general relations between conditions **(P)** and **(Q)**, see also, in a slightly different setting, [6, Theorem 3.5].

**Proposition 1.** *Consider a junction between n pipes. Then:*

*(i) Condition **(P)** implies condition **(Q)**.*
*(ii) If $\nu_1, \ldots, \nu_n$ are linearly independent, then condition **(Q)** is equivalent to **(P)**.*
*(iii) If $n \in \{2, 3, 4\}$ and $\nu_1, \ldots, \nu_n$ are linearly dependent, then **(P)** and **(Q)** are equivalent if and only if $\sum_{l=1}^{n} \nu_l = 0$ and $\dim \operatorname{span}\{\nu_1, \ldots \nu_n\} = n - 1$.*
*(iv) If $n \geq 5$, then condition **(Q)** does not imply condition **(P)**.*

*Proof.* Consider the different statements separately.

(i) If condition **(P)** holds, then $\sum_{l=1}^{n} P\left(\rho_l(t, 0+), q_l(t, 0+)\right) \nu_l = P_* \sum_{l=1}^{n} \nu_l$ and condition **(Q)** is clearly satisfied.
(ii) Use now **(Q)** rewritten in the form (5). If $\nu_1, \ldots, \nu_n$ are linearly independent, then the previous condition becomes $\sum_{l=1}^{n} \left[P\left(\rho_l(t, 0+), q_l(t, 0+)\right) - \lambda\right] \nu_l = 0$, and the statement easily follows.
(iii) The 1D vector space $P_1 = \ldots = P_n$ in $\mathbb{R}^n$ always solves the linear system $\sum_l P_l \nu_l = \lambda \sum_l \nu_l$. No other solution exists as soon as $\dim \operatorname{span}\{\nu_1, \cdots, \nu_n\} = n - 1$, completing the proof.
(iv) Simply note that **(P)** gives $n - 1$ independent conditions, while **(Q)** gives at most 3 independent conditions. □

4. **Non uniqueness of Q-subsonic solution.** In this section, we show that the concept of Q-solution, in general, is not sufficient to isolate a unique solution to Riemann problems. Let us consider the following example.

**Example 7** Fix $\gamma = 1$, $c = 1$ and two pipes with different sections and opposite directions, say $\nu_1 = (-1, 0, 0)$ and $\nu_2 = (2, 0, 0)$, see Figure 3, right. Consider the Riemann Problem (4) with the initial conditions $(\bar{\rho}_1, \bar{q}_1) = (1.3, 0)$, $(\bar{\rho}_2, \bar{q}_2) = (1, 0)$. The constant functions $(\rho_l(t, x), q_l(t, x)) = (\bar{\rho}_l, \bar{q}_l)$ $(l \in \{1, 2\})$ provide a Q-subsonic solution to (4). Moreover, there are infinitely many Q-subsonic solutions to (4). Indeed, their traces at $x = 0$ solve the system

$$\begin{cases} q_1(t, 0+) = L_2^-\left(\rho_1(t, 0+); 1.3, 0\right), \\ q_2(t, 0+) = L_2^-\left(\rho_2(t, 0+); 1, 0\right), \\ q_1(t, 0+) = -2q_2(t, 0+). \end{cases}$$

Instead, the monotonicity of the reversed Lax curves of the second family in the subsonic region $A_0$ implies that there are at most one P-solution and one p-solution in $A_0$. The traces of a P-solution are given by

$$\begin{cases} q_1(t,0+) = L_2^-\left(\rho_1(t,0+);1.3,0\right), \\ q_2(t,0+) = L_2^-\left(\rho_2(t,0+);1,0\right), \\ q_1(t,0+) = -2q_2(t,0+), \\ P(\rho_1(t,0+),q_1(t,0+)) = P\left(\rho_2(t,0+),q_2(t,0+)\right), \end{cases}$$

while the traces of a p-solution are given by

$$\begin{cases} q_1(t,0+) = L_2^-\left(\rho_1(t,0+);1.3,0\right), \\ q_2(t,0+) = L_2^-\left(\rho_2(t,0+);1,0\right), \\ q_1(t,0+) = -2q_2(t,0+), \\ \rho_1(t,0+) = \rho_2(t,0+). \end{cases}$$

In this example numerical simulations for P-solutions give $\rho_1(t,0+) = 1.06885$, $q_1(t,0+) = -0.20926$, $\rho_2(t,0+) = 1.09977$, $q_2(t,0+) = 0.104629$, see Figure 6, left. Numerical simulations for p-solution give $\rho_1(t,0+) = \rho_2(t,0+) = 1.092$, $q_1(t,0+) = -0.190394$, $q_2(t,0+) = 0.0961389$, see Figure 6, right. Proposition 1 and Example 7

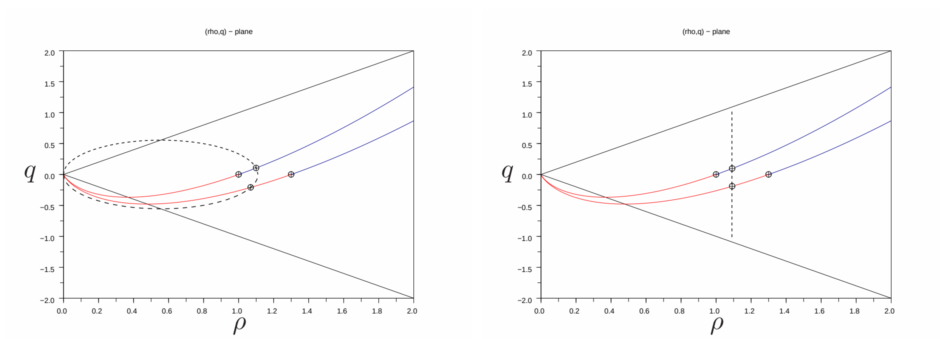


FIGURE 6. Left: the P-solution and, dashed, the line of constant $P$. Right: the p-solution and, dashed, the line of constant $p$.

above provide a physical justification of the definition of P-solutions in the cases $n = 2, 3$. Indeed, P-solutions are characterized as either coinciding with Q-solutions or as being the unique limits of Q-solutions whenever the latter are not unique.

**Proposition 2.** *Let $n = 2, 3$, consider a junction $\mathcal{J} = (\nu_1, \ldots, \nu_n)$ and fix initial states $(\bar\rho_1, \bar q_1), \ldots, (\bar\rho_n, \bar q_n) \in \mathring{A}_0$. Assume that (4) admits a P-solution in $A_0$. Then, only the following two cases are possible:*
***either** (4) at $\mathcal{J}$ has a unique Q-solution which is also a P-solution,*
***or** there exists a sequence of junctions $\mathcal{J}^h = (\nu_1^h, \ldots, \nu_n^h)$ such that (4) at $\mathcal{J}^h$ admits a unique Q-solution which is also a P-solution and the sequence of these Q-solutions converges in $\mathbf{L^1_{loc}}$ to the unique P-solution of (4) at $\mathcal{J}$.*

*Proof.* Preliminarily, note that if $\nu_1, \ldots, \nu_n$ are linearly independent, then Proposition 1 ensures that P-solutions and Q-solutions of (4) at $\mathcal{J}$ coincide. Moreover, they are unique by [4, Theorem 1].

If the Q-solution to (4) at $\mathcal{J}$ is unique, then by (i) in Proposition 1, it coincides with the P-solution. Assume now that there is more than one Q-solution. Then, $\nu_1, \ldots, \nu_n$ are linearly dependent. Let $\mathcal{J}^h = (\nu_1^h, \ldots, \nu_n^h)$ be any sequence of linearly independent vectors converging to $\nu_1, \ldots, \nu_n$. The observation above applies to (4) at $\mathcal{J}^h$. The convergence of the solutions to (4) at $\mathcal{J}^h$ is immediate by **(P)**. $\square$

5. **Variation of the total linear momentum.** Consider, for every $l \in \{1, \ldots, n\}$, a function $(\rho_l, q_l) \in \mathbf{C^0}\left(\mathbb{R}^+; \mathbf{BV}(\mathbb{R}^+; \mathbb{R}^+ \times \mathbb{R})\right)$ such that $\left(\rho_l(t,x), q_l(t,x)\right) \in A_0$ for every $t > 0$ and for a.e. $x \in \mathbb{R}^+$ and such that it is a distributional solution to the $p$-system (1). Moreover, let us assume that $\lim_{x\to+\infty}\left(\rho_l(t,x), q_l(t,x)\right) = (0,0)$ for $l \in \{1, \ldots, n\}$ and $t > 0$.

In this section, we want to study the variation of the total linear momentum

$$Q(t) = \sum_{l=1}^{n} \int_0^{+\infty} q_l(t,x)\, \nu_l \, dx$$

along the different concepts of solution at the junction. Fix $0 < t_1 < t_2$. We have

$$\begin{aligned}
& Q(t_2) - Q(t_1) \\
= & \sum_{l=1}^{n} \int_0^{+\infty} \left[q_l(t_2,x) - q_l(t_1,x)\right] \nu_l \, dx \\
= & \sum_{l=1}^{n} \int_{t_1}^{t_2} \left[P\left(\rho_l(t,0+), q_l(t,0+)\right) - P\left(\rho_l(t,+\infty), q_l(t,+\infty)\right)\right] \nu_l \, dt \\
= & \sum_{l=1}^{n} \int_{t_1}^{t_2} P\left(\rho_l(t,0+), q_l(t,0+)\right) \nu_l \, dt\,.
\end{aligned}$$

If $\left((\rho_1,q_1), \ldots, (\rho_n,q_n)\right)$ is a P-solution, then, by **(P)**

$$Q(t_2) - Q(t_1) = \left(\int_{t_1}^{t_2} P_*(t)\, dt\right) \sum_{l=1}^{n} \nu_l.$$

If $\left((\rho_1,q_1), \ldots, (\rho_n,q_n)\right)$ is a $Q$-solution, then, by (5)

$$Q(t_2) - Q(t_1) = \left(\int_{t_1}^{t_2} \lambda(t)\, dt\right) \sum_{l=1}^{n} \nu_l.$$

If $\left((\rho_1,q_1), \ldots, (\rho_n,q_n)\right)$ is a p-solution, then, by direct computations

$$Q(t_2) - Q(t_1) = \left(\int_{t_1}^{t_2} p(\rho_1(t,0+))\, dt\right) \sum_{l=1}^{n} \nu_l + \int_{t_1}^{t_2} \frac{1}{\rho_1(t,0+)} \sum_{l=1}^{n} q_l^2(t,0+)\nu_l.$$

Thus, the following proposition holds.

**Proposition 3.** *Assume that $\left((\rho_1,q_1), \ldots, (\rho_n,q_n)\right)$ is a subsonic solution to the p-system (1) such that for every $l \in \{1, \ldots, n\}$,*

1. $(\rho_l, q_l) \in \mathbf{C^0}\left(\mathbb{R}^+; \mathbf{BV}(\mathbb{R}^+; \mathbb{R}^+ \times \mathbb{R})\right)$*;*
2. $\left(\rho_l(t,x), q_l(t,x)\right) \in A_0$ *for every* $t > 0$ *and for a.e.* $x \in \mathbb{R}^+$*;*
3. $\lim_{x\to+\infty}\left(\rho_l(t,x), q_l(t,x)\right) = (0,0)$ *for every* $t > 0$.

*If $\left((\rho_1,q_1), \ldots, (\rho_n,q_n)\right)$ is a P-solution or a Q-solution, then the variation of the total linear momentum is directed along $\sum_{l=1}^{n} \nu_l$. If it is a p-solution, then the variation of the total linear momentum is not, in general, directed along $\sum_{l=1}^{n} \nu_l$.*

In the present irrotational setting, it is mandatory that the variation of the total linear momentum be a linear combination of $\nu_1, \ldots, \nu_n$. This condition is satisfied by all solutions defined above. In cases **(P)** and **(Q)**, the two variations in $Q$ are parallel. Examples 8, 9 and 10 below show that in simple situations with $n = 2$ also p-solution yield a variation of the total linear momentum parallel to that in cases **(P)** and **(Q)**. When $n = 3$, the variation of $Q$ for p-solutions may qualitatively differ from that of P- and Q-solutions.

**Example 8** Two pipes with the same section and opposite directions, i.e. $n = 2$ and $\nu_1 + \nu_2 = 0$, see Figure 3, left. In this case, the variation of the total linear momentum is clearly null for $P$ and $Q$ solutions. The condition of mass conservation implies that $q_1^2(t, 0+) = q_2^2(t, 0+)$ for a.e. $t > 0$ and so we deduce that the variation of the total linear momentum is also null for p-solutions.

**Example 9** Two pipes with the same section but different directions, say $\nu_1 = (-1, -1, 0)$ and $\nu_2 = (1, -1, 0)$, see Figure 2, left. Now, $\nu_1 + \nu_2 = (0, -2, 0)$. Conservation of mass at the junction implies that $q_1^2(t, 0+) = q_2^2(t, 0+)$ for a.e. $t > 0$. Hence the variation of the total linear momentum for p-solutions is exactly the same of that for P-solutions, which coincide with Q-solutions.

**Example 10** Two pipes with different sections and opposite directions, say $\nu_1 = (-1, 0, 0)$ and $\nu_2 = (2, 0, 0)$, see Figure 3, right. Since $\nu_1 + \nu_2 = (1, 0, 0)$, then the variation of the total linear momentum for P and Q solution is not null and is directed along the vector $(1, 0, 0)$. For p-solutions, instead, the conservation of mass implies that the variation of the total linear momentum is given by $Q(t_2) - Q(t_1) = \left(\int_{t_1}^{t_2} \left[p(\rho_1(t, 0+)) + \frac{5q_1^2(t,0+)}{4\rho_1(t,0+)}\right] dt\right)(\nu_1 + \nu_2)$. Therefore it has the same direction as that of P-solutions, but bigger norm.

**Example 11** A T-junctions between three tubes with the same section, say $\nu_1 = (-1, 0, 0)$, $\nu_2 = (1, 0, 0)$ and $\nu_3 = (0, -1, 0)$, see Figure 4, left. We have $\nu_1 + \nu_2 + \nu_3 = \nu_3$. The variation of the total linear momentum for a p-solution is given by $Q(t_2) - Q(t_1) = \left(\int_{t_1}^{t_2} P(\rho_3(t, 0+), q_3(t, 0+))\, dt\right)\nu_3 + \left(\int_{t_1}^{t_2} \frac{q_1^2(t,0+) - q_2^2(t,0+)}{\rho_3(t,0+)}\, dt\right)\nu_1$ and so it has, in general, a different direction of the variation of the total linear momentum of P and Q-solutions.

**Example 12** Three coplanar tubes with the same section at an angle of $2\pi/3$, say $\nu_1 = (-\sqrt{3}/2, 1/2, 0)$, $\nu_2 = (\sqrt{3}/2, 1/2, 0)$ and $\nu_3 = (0, -1, 0)$, see Figure 4, right. Now, $\nu_1 + \nu_2 + \nu_3 = 0$, hence the variation of the total linear momentum for P and Q-solutions vanishes. For a p-solution, we have $Q(t_2) - Q(t_1) = \int_{t_1}^{t_2} \frac{1}{\rho_1(t,0+)} \sum_{l=1}^{3} q_l^2(t, 0+)\nu_l\, dt$, which is different from 0, unless $q_1^2(t, 0+) = q_2^2(t, 0+) = q_3^2(t, 0+)$. The last condition, coupled with the conservation of mass, implies that $Q(t_2) = Q(t_1)$ if and only if $q_1(t, 0+) = q_2(t, 0+) = q_3(t, 0+) = 0$.

6. **Properties of the $p$-system.** We recall here basic formulas of the $p$-system (1) valid for a pressure law satisfying **(EoS)**. Throughout, $c(\rho) = \sqrt{p'(\rho)}$ denotes the sound speed. Let $\lambda_i$ be the $i$th eigenvalue corresponding to the $i$th right eigenvector $r_i$ of the Jacobian of the flow $f(\rho, q) = [q \quad q^2/\rho + p(\rho)]^T$. We have

$$\begin{aligned} \lambda_1(\rho, q) &= \tfrac{q}{\rho} - c(\rho)\,, & \lambda_2(\rho, q) &= \tfrac{q}{\rho} + c(\rho)\,, \\ r_1(\rho, q) &= \begin{bmatrix} \rho \\ q - \rho c(\rho) \end{bmatrix}, & r_2(\rho, q) &= \begin{bmatrix} \rho \\ q + \rho c(\rho) \end{bmatrix}, \\ \nabla\lambda_1 \cdot r_1 &= -c(\rho) - \rho c'(\rho), & \nabla\lambda_2 \cdot r_2 &= c(\rho) + \rho c'(\rho)\,. \end{aligned} \tag{6}$$

The speeds of $1, 2$-shock waves between $(\rho_o, q_o)$ and the state at density $\rho$ are

$$\Lambda_{\frac{1}{2}}(\rho, \rho_o, q_o) = \frac{q_o}{\rho_o} \mp \sqrt{\frac{\rho}{\rho_o} \cdot \frac{p(\rho) - p(\rho_o)}{\rho - \rho_o}}.$$

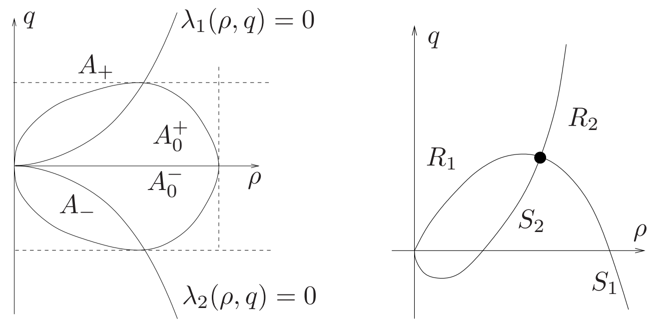


FIGURE 7. Left: the regions $A_-$, $A_0^\pm$, $A_+$ and a level curve of the dynamic pressure. Right: the Lax forward curves for (1).

The forward $1, 2$-Lax curves are in Figure 7, right. Further analytical expressions are found, for instance, in [4, Section 4].

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Received July 2008; revised March 2009.

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