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STABILITY OF LINEAR DYNAMIC EQUATIONS ON TIME SCALES

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Abstract. In this paper we define u_∞-quasisimilarity in order to unify t_∞-quasisimilarity and n_∞-quasisimilarity and then study the stability for solutions of linear dynamic equations on time scales by using the concept of u_∞-quasisimilarity and dynamic inequality.

1. **Introduction.** The concept of kinematic similarity is an effective tool to study the theory of stability of differential systems and difference systems. Markus [20] introduced the notion of kinematic similarity in the set $C(\mathbb{R}_+, \mathbb{R}^n)$ of all $n \times n$ continuous matrices $A(t)$ defined on $\mathbb{R}_+ = [0, \infty)$ and showed that the kinematic similarity is an equivalence relation preserving the type numbers of the linear differential systems. Gohberg et al. [17] studied the problem to classify linear difference systems of the form $x_{n+1} = A_n x_n$, $n \in \mathbb{Z}$, under kinematic similarity.

Conti [13] introduced the notion of t_∞-similarity in $C(\mathbb{R}_+, \mathbb{R}^n)$ and showed that t_∞-similarity is an equivalence relation preserving strict, uniform and exponential stability of linear homogeneous differential systems. Choi et al. [5] studied the variational stability of nonlinear differential systems using the notion of t_∞-similarity. Trench [24] extended this notion to a concept called t_∞-quasisimilarity that is not symmetric or transitive, but still preserves stability properties.

As a discrete analog of Conti's definition of t_∞-similarity Trench [25] defined the notion of summable similarity on pairs of $m \times m$ matrix functions and showed that if A and B are summably similar and the linear difference system $\Delta x(n) = A(n)x(n), n = 0, 1, \cdots,$ is uniformly, exponentially or strictly stable or has linear asymptotic equilibrium, then the linear difference system $\Delta y(n) = B(n)y(n)$ has also the same properties. Also, Choi and Koo [6] introduced the notion of n_∞-similarity in the set of all $m \times m$ invertible matrices and showed that two concepts of global h-stability and global h-stability in variation are equivalent by using the concept of n_∞-similarity and Lyapunov functions. Their approach included most types of stability.

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The theory of time scales (closed subsets of $\mathbb{R}$) was created by Hilger [18] in order to unify the theories of differential equations and of difference equations and in order to extend those theories to other kinds of the so-called "dynamic equations". The two main features of the calculus on time scales are unification and extension of continuous and discrete analysis.

In this paper, we define u_∞-quasisimilarity in order to unify (continuous) t_∞-quasisimilarity and (discrete) n_∞-quasisimilarity and then study h-stability and strong stability for solutions of linear dynamic equations on time scales by using the concept of u_∞-quasisimilarity and Gronwall-type inequalities. This extends a recent result about the strong stability for dynamic equations on time scales by using the concept of u_∞-similarity on time scales in [12].

We mention without proof several foundational notions and results in the calculus on time scales from an excellent introductory text by Bohner and Peterson [4].

A *time scale* $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$, and the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by $\sigma(t) = \inf\{s \in \mathbb{T} \ : \ s > t\}$, while the *graininess* $\mu : \mathbb{T} \to \mathbb{R}_+$ is given by $\mu(t) = \sigma(t) - t$. Assume throughout that $\mathbb{T}$ is unbounded above and the graininess μ is bounded.

A function $f : \mathbb{T} \to \mathbb{R}$ is called *differentiable* (in a point $t \in \mathbb{T}$), if there exists a unique derivative $f^\Delta(t) \in \mathbb{R}$, such that for any $\varepsilon > 0$ the estimate

$$|f^\sigma(t) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s| \text{ for all } s \in U,$$

where $f^\sigma = f \circ \sigma$, holds in a $\mathbb{T}$-neighborhood U of t.

Example. Let δ be a positive constant and $\delta\mathbb{Z} = \{\cdots, -2\delta, -\delta, 0, \delta, 2\delta, \cdots\}$. The derivative $f^\Delta(t) \in \mathbb{R}$ of a function $f : \mathbb{T} \to \mathbb{R}$ read as

$$f^\Delta(t) = f'(t) \ \text{ if } \mathbb{T} = \mathbb{R}, \ f^\Delta(t) = \frac{f(t+\delta) - f(t)}{\delta} \ \text{ if } \mathbb{T} = \delta\mathbb{Z}.$$

A function $f : \mathbb{T} \to \mathbb{R}$ is said to be *rd-continuous* (denoted by $f \in C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$) if

(i) f is continuous at every right-dense point $t \in \mathbb{T}$,

(ii) $\lim_{s \to t^-} f(s)$ exists and is finite at every left-dense point $t \in \mathbb{T}$.

A function $g : \mathbb{T} \to \mathbb{R}$ is called an *antiderivative* of f on $\mathbb{T}$ if it is differentiable on $\mathbb{T}$ and satisfies $g^\Delta(t) = f(t)$ for $t \in \mathbb{T}$. In this case, we define the Cauchy integral of f as $\int_a^t f(s)\Delta s = g(t) - g(a)$ for $t, a \in \mathbb{T}$.

2. **u_∞-quasisimilarity.** Let $M_n(\mathbb{R})$ be the set of all $n \times n$ matrices over $\mathbb{R}$ and $\mathfrak{M}_n(\mathbb{R})$ the set of all $n \times n$ invertible matrices over $\mathbb{R}$.

We recall some basic facts about linear homogeneous dynamic systems on time scales in [4].

A function $A : \mathbb{T} \to M_n(\mathbb{R})$ is called *regressive* if for each $t \in \mathbb{T}$ the $n \times n$ matrix $I + \mu(t)A(t)$ is invertible, where I is the identity matrix. The class of all regressive and rd-continuous functions A from $\mathbb{T}$ to $M_n(\mathbb{R})$ is denoted by $C_{\mathrm{rd}}\mathcal{R}(\mathbb{T}, M_n(\mathbb{R}))$.

We say that the matrix-valued function A is *differentiable* on $\mathbb{T}$ provided each entry of A is differentiable on $\mathbb{T}$, and in this case we put

$$A^\Delta = (a_{ij}^\Delta)_{1 \leq i,j \leq n}, \text{ where } A = (a_{ij})_{1 \leq i,j \leq n}.$$

Let $t_0 \in \mathbb{T}$. The unique matrix-valued solution of the IVP

$$Y^\Delta = A(t)Y, \ Y(t_0) = I, \tag{1}$$

where $A \in C_{\mathrm{rd}}\mathcal{R}(\mathbb{T}, M_n(\mathbb{R}))$, is called the *the matrix exponential function* and it is denoted by $\Phi_A(t, t_0)$.

Lemma 2.1. [4, Theorem 5.3.] *Suppose that $A, B \in C_{\rm rd}(\mathbb{T}, M_n(\mathbb{R}))$ are differentiable functions, and c is a differentiable scalar function on $\mathbb{T}$. Then*

(i) $A^\sigma(t) = A(t) + \mu(t)A^\Delta(t)$.
(ii) $(AB)^\Delta = A^\Delta B^\sigma + AB^\Delta = A^\sigma B^\Delta + A^\Delta B$.
(iii) $(A^{-1})^\Delta = -(A^\sigma)^{-1}A^\Delta A^{-1} = -A^{-1}A^\Delta(A^\sigma)^{-1}$ *if* AA^σ *is invertible.*
(iv) $(cA)^\Delta = c^\Delta A + c^\sigma A^\Delta = cA^\Delta + c^\Delta A^\sigma$.

Let $C_{\rm rd}I(\mathbb{T}, M_n(\mathbb{R}))$ $(C_{\rm rd}A(\mathbb{T}, M_n(\mathbb{R}))$ be the set of the rd-continuous mappings C from $\mathbb{T}$ to $M_n(\mathbb{R})$ such that $\int_{t_0}^\infty C(t)\Delta t$ $(\int_{t_0}^\infty |C(t)|\Delta t)$ converges, and $C^1_{\rm rd}(\mathbb{T}, \mathfrak{M}_n(\mathbb{R}))$ be the set of the rd-continuous differentiable mappings S from $\mathbb{T}$ to $\mathfrak{M}_n(\mathbb{R})$ such that S and S^{-1} are bounded on $\mathbb{T}_{t_0}$. Here $\mathbb{T}_{t_0} = [t_0, \infty) \cap \mathbb{T}$.

We define the u_∞-quasisimilarity on time scales in order to unify continuous and discrete similarities.

Definition 2.2. Let $A, B \in C_{\rm rd}\mathcal{R}(\mathbb{T}, M_n(\mathbb{R}))$. A function B is u_∞-*quasisimilar* to a function A if there exists an $S \in C^1_{\rm rd}(\mathbb{T}, \mathfrak{M}_n(\mathbb{R}))$ such that

$$S^\Delta + S^\sigma B - AS := F_0 \tag{2}$$

is in $C_{\rm rd}I(\mathbb{T}, M_n(\mathbb{R}))$, and either $F_0 \in C_{\rm rd}A(\mathbb{T}, M_n(\mathbb{R}))$ or there is a positive integer k such that $F_1, \cdots, F_k$ defined by

$$\begin{aligned} Q_i(t) &:= \int_t^\infty F_{i-1}(s)\Delta s, \\ F_i(t) &:= Q_i(\sigma(t))B(t) - A(t)Q_i(t),\ 1 \le i \le k \end{aligned}$$

are in $C_{\rm rd}I(\mathbb{T}, M_n(\mathbb{R}))$ and $F_k \in C_{\rm rd}A(\mathbb{T}, M_n(\mathbb{R}))$.

Remark 1. If $\mathbb{T} = \mathbb{R}$, then the u_∞-quasisimilarity implies the t_∞-quasisimilarity in [24]. Also if $\mathbb{T} = \mathbb{Z}$, then the u_∞-quasisimilarity means that Assumption 1 holds in [25], which in turn means n_∞-quasisimilarity. Note that u_∞-quasisimilarity is not an equivalence relation. If $k = 0$, then u_∞-quasisimilarity reduces to (continuous) t_∞-similarity in [5, 13, 14, 19, 24] or (discrete) n_∞-similarity in [6, 7, 8, 25]. Furthermore, if $F_0 = 0$ in Definition 2.2, then u_∞-quasisimilarity becomes kinematically similar in [2].

We consider linear dynamic systems

$$x^\Delta(t) = A(t)x(t), \tag{3}$$

and

$$y^\Delta(t) = B(t)y(t), \tag{4}$$

where $A, B \in C_{\rm rd}\mathcal{R}(\mathbb{T}, M_n(\mathbb{R}))$.

Lemma 2.3. *Suppose that B is u_∞-quasisimilar to A. Let*

$$\Gamma_0 = I \text{ and } \Gamma_i = I + S^{-1}\sum_{j=1}^{i} Q_j,\ 0 \le i \le k. \tag{5}$$

Then there is $t_0 \in \mathbb{T}$ such that

$$\begin{aligned} \Gamma_k(t)\Phi_B(t) = S^{-1}(t)\Phi_A(t)[&\Phi_A^{-1}(\tau)S(\tau)\Gamma_k(\tau)\Phi_B(\tau) \\ &+ \int_\tau^t \Phi_A^{-1}(\sigma(s))F_k(s)\Phi_B(s)\Delta s] \end{aligned} \tag{6}$$

for all $t \ge \tau \ge t_0$ with $t, \tau \in \mathbb{T}$. Here $\Phi_A(t) = \Phi_A(t, t_0)$ and $\Phi_B(t) = \Phi_B(t, t_0)$.

Proof. Since $\lim_{t\to\infty} Q_i(t) = 0, 1 \le i \le k$, and S^{-1} are bounded, it follows that $\Gamma_k^{-1}(t)$ exists and is bounded for t sufficiently large, say $t \ge t_0$, and $\lim_{t\to\infty} \Gamma_k^{-1}(t) = I$. In view of Lemma 2.1, we obtain

$$\begin{aligned}
&(\Phi_A^{-1}(t)S(t)\Phi_B(t))^\Delta = \Phi_A^{-1}(\sigma(t))(S(t)\Phi_B(t))^\Delta + (\Phi_A^{-1}(t))^\Delta S(t)\Phi_B(t)\\
&= \Phi_A^{-1}(\sigma(t))(S(\sigma(t))\Phi_B^\Delta(t) + S^\Delta(t)\Phi_B(t)) - \Phi_A^{-1}(\sigma(t))\Phi_A^\Delta(t)\Phi_A^{-1}(t)S(t)\Phi_B(t)\\
&= \Phi_A^{-1}(\sigma(t))(S(\sigma(t))B(t)\Phi_B(t) + S^\Delta(t)\Phi_B(t)) - \Phi_A^{-1}(\sigma(t))A(t)S(t)\Phi_B(t)\\
&= \Phi_A^{-1}(\sigma(t))[S(\sigma(t))B(t) + S^\Delta(t) - A(t)S(t)]\Phi_B(t) = \Phi_A^{-1}(\sigma(t))F_0(t)\Phi_B(t),
\end{aligned}$$

where $F_0(t) = S^\Delta(t) + S(\sigma(t))B(t) - A(t)S(t)$. Integrating this and multiplying the result by $S^{-1}(t)\Phi_A(t)$ yields

$$\Phi_B(t) = S^{-1}(t)\Phi_A(t)[\Phi_A^{-1}(\tau)S(\tau)\Phi_B(\tau) + \int_\tau^t \Phi_A^{-1}(\sigma(s))F_0(s)\Phi_B(s)\Delta s], t, \tau \ge t_0.$$

The estimate (6) holds for $k = 0$.

Now, we show the result by finite induction on i. Clearly (6) holds for $k = 0$. Suppose that (6) is true for $k \ge 1$. Then

$$\begin{aligned}
\Gamma_i(t)\Phi_B(t) = S^{-1}(t)\Phi_A(t)[\Phi_A^{-1}(\tau)S(\tau)\Gamma_i(\tau)\Phi_B(\tau)\\
+ \int_\tau^t \Phi_A^{-1}(\sigma(s))F_i(s)\Phi_B(s)\Delta s], 1 \le i \le k-1,\ t, \tau \ge t_0.
\end{aligned} \tag{7}$$

Then we have

$$\begin{aligned}
& (\Phi_A^{-1}(t)Q_{i+1}(t)\Phi_B(t))^\Delta \\
= & \Phi_A^{-1}(\sigma(t))(Q_{i+1}(t)\Phi_B(t))^\Delta + (\Phi_A^{-1}(t))^\Delta Q_{i+1}(t)\Phi_B(t)\\
= & \Phi_A^{-1}(\sigma(t))(Q_{i+1}(\sigma(t))\Phi_B^\Delta(t) + Q_{i+1}^\Delta(t)\Phi_B(t))\\
- & \Phi_A^{-1}(\sigma(t))\Phi_A^\Delta(t)\Phi_A^{-1}(t)Q_{i+1}(t)\Phi_B(t)\\
= & \Phi_A^{-1}(\sigma(t))[Q_{i+1}(\sigma(t))B(t) - A(t)Q_{i+1}(t) - F_i(t)]\Phi_B(t)\\
= & \Phi_A^{-1}(\sigma(t))[F_{i+1}(t) - F_i(t)]\Phi_B(t).
\end{aligned}$$

Solving this for $\Phi_A^{-1}(\sigma(t))F_i(t)\Phi_B(t)$ and integrating yields

$$\begin{aligned}
\int_\tau^t \Phi_A^{-1}(\sigma(s))F_i(s)\Phi_B(s)\Delta s = -\Phi_A^{-1}(t)Q_{i+1}(t)\Phi_B(t) + \Phi_A^{-1}(\tau)Q_{i+1}(\tau)\Phi_B(\tau)\\
+ \int_\tau^t \Phi_A^{-1}(\sigma(s))F_{i+1}(s)\Phi_B(s)\Delta s.
\end{aligned} \tag{8}$$

Substituting (8) into (7) and using (5) yields

$$\begin{aligned}
&[\Gamma_i(t) + S^{-1}(t)Q_{i+1}(t)]\Phi_B(t) = S^{-1}(t)\Phi_A(t)[\Phi_A^{-1}(\tau)S(\tau)(\Gamma_i(\tau)\\
&+S^{-1}(\tau)Q_{i+1}(\tau))\Phi_B(\tau) + \int_\tau^t \Phi_A^{-1}(\sigma(s))F_{i+1}(s)\Phi_B(s)\Delta s].
\end{aligned}$$

Hence we have

$$\begin{aligned}
\Gamma_{i+1}(t)\Phi_B(t) \quad &= S^{-1}(t)\Phi_A(t)[\Phi_A^{-1}(\tau)S(\tau)\Gamma_{i+1}(\tau)\Phi_B(\tau)\\
&+ \textstyle\int_\tau^t \Phi_A^{-1}(\sigma(s))F_{i+1}(s)\Phi_B(s)\Delta s],\ t, \tau \ge t_0.
\end{aligned}$$

This completes the finite induction. □

We can obtain the following corollary as a special case of Lemma 2.3.

Corollary 1. *Suppose that B is u_∞-quasisimilar to A on $\mathbb{T}$.*

(i) If $\mathbb{T}=\mathbb{R}$, then there is $t_0 \in \mathbb{R}$ such that

$$\Gamma_k(t)\Phi_B(t) = S^{-1}(t)\Phi_A(t)[\Phi_A^{-1}(\tau)S(\tau)\Gamma_k(\tau)\Phi_B(\tau) + \int_\tau^t \Phi_A^{-1}(s)F_k(s)\Phi_B(s)ds],\ t \geq \tau \geq t_0 \in \mathbb{R}. \tag{9}$$

(ii) If $\mathbb{T}=\delta\mathbb{Z}$ with a positive constant δ, then there is $t_0 \in \delta\mathbb{Z}$ such that

$$\Gamma_k(t)\Phi_B(t) = S^{-1}(t)\Phi_A(t)[\Phi_A^{-1}(\tau)S(\tau)\Gamma_k(\tau)\Phi_B(\tau) + \delta\sum_{s=\tau}^{t-\delta} \Phi_A^{-1}(s+\delta)F_k(s)\Phi_B(s)],\ t \geq \tau \geq t_0 \in \delta\mathbb{T}. \tag{10}$$

3. **Stability of linear dynamic equations on time scales.** Pinto introduced the notion of h-stability which is an extension of the notions of exponential stability and uniform Lipschitz stability of differential equations in [22] and difference equations in [21].

Choi et al. and DaCunha gave the characterizations of the various types of stability for solutions of dynamic systems on time scales in [9, 10, 11, 16].

We consider the dynamic system

$$x^\Delta = F(t,x),\ x(t_0) = x_0, \tag{11}$$

where $F \in C_{\mathrm{rd}}(\mathbb{T}\times\mathbb{R}^n, \mathbb{R}^n)$ with $F(t,0)=0$ and x^Δ is the delta derivative of $x:\mathbb{T}\to\mathbb{R}^n$ with respect to $t \in \mathbb{T}$. We assume that the solutions x of (11) exist and are unique for $t \geq t_0$. The norm of an $n\times n$ matrix M is defined to be $|M| = \max_{1\leq j\leq n}|M^j|$ for the j-th column M^j of M.

We recall the notion of h-stability of dynamics equations on time scales in [9].

Definition 3.1. System (11) is said to be *h-stable* if there exists a positive bounded function $h:\mathbb{T}\to\mathbb{R}$ and a constant $c\geq 1$ such that

$$|x(t,t_0,x_0)| \leq c|x_0|h(t)h(t_0)^{-1},\ t\geq t_0$$

for $|x_0|$ small enough (here $h(t)^{-1} = \frac{1}{h(t)}$).

Lemma 3.2. [9, Lemma 2.3] *If (3) is h-stable if and only if there exists a positive function h defined on $\mathbb{T}$ and a constant $c\geq 1$ such that*

$$|\Phi_A(t,t_0)| \leq ch(t)h(t_0)^{-1},\ t\geq t_0 \in \mathbb{T}$$

where $\Phi_A(t,t_0)$ is a matrix exponential function for (3).

Theorem 3.3. *Suppose that system (3) is h-stable and B is u_∞-quasisimilar to A with $\int_{t_0}^\infty \frac{h(t)}{h(\sigma(t))}|F_k(t)|\Delta t < \infty$ for each $t_0 \in \mathbb{T}$. Then (4) is also h-stable.*

Proof. Since (3) is h-stable, there exists a positive bounded function h defined on $\mathbb{T}$ and a constant $c\geq 1$ such that

$$|\Phi_A(t,t_0)| \leq ch(t)h(t_0)^{-1},\ t\geq t_0 \in \mathbb{T}.$$

Since B is u_∞-quasisimilar to A, from Lemma 2.3 and 3.2, and by virtue of the boundedness of $S(t), S^{-1}(t), \Gamma_k(t)$ and $\Gamma_k^{-1}(t)$ there are positive constants c_1 and

c_2 such that

$$\begin{aligned}|\Phi_B(t,\ t_0)| &\le |\Gamma_k^{-1}(t)||S^{-1}(t)|[|\Phi_A(t,\ t_0)||S(t_0)||\Gamma_k(\tau)| \\ &+ \int_{t_0}^{t}|\Phi_A(t,\ \sigma(s))||F_k(s)||\Phi_B(s,\ t_0)|\Delta s] \\ &\le c_1c_2h(t)h(t_0)^{-1}+c_1c_2\int_{t_0}^{t}h(t)h(\sigma(s))^{-1}|F_k(s)||\Phi_B(s,\ t_0)|\Delta s,\end{aligned}$$

where $\Phi_B(t,t_0)$ is a matrix exponential function for (4). Dividing by $h(t)$ on both sides, we have

$$\frac{|\Phi_B(t,\ t_0)|}{h(t)} \le c_1c_2h(t_0)^{-1}+c_1c_2\int_{t_0}^{t}\frac{h(s)}{h(\sigma(s))}|F_k(s)|\frac{|\Phi_B(s,\ t_0)|}{h(s)}\Delta s.$$

In view of the Gronwall's inequality on time scale in [4], we have

$$\begin{aligned}\frac{|\Phi_B(t,\ t_0)|}{h(t)} &\le \frac{c_1c_2}{h(t_0)}e_{p(t)}(t,\ t_0)=\frac{c_1c_2}{h(t_0)}\exp\left(\int_{t_0}^{t}\xi_{\mu(s)}(p(s))\Delta s\right) \\ &= \begin{cases}\frac{c_1c_2}{h(t_0)}\exp(\int_{t_0}^{t}\frac{1}{\mu(s)}\mathrm{Log}(1+\mu(s)p(s))\Delta s) & \text{if } \mu\neq 0 \\ \frac{c_1c_2}{h(t_0)}\exp(\int_{t_0}^{t}p(s)\Delta s) & \text{if } \mu=0\end{cases} \\ &\le \frac{c_1c_2}{h(t_0)}\exp\left(\int_{t_0}^{t}p(s)\Delta s\right)\le\frac{c_1c_2}{h(t_0)}\exp\left(\int_{t_0}^{\infty}p(s)\Delta s\right),\end{aligned}$$

where $p(t)=c_1c_2\frac{h(t)}{h(\sigma(t))}|F_k(t)|$ and the cylinder transformation $\xi_\mu(z)$ is given by

$$\xi_\mu(z)=\begin{cases}\frac{1}{\mu}\mathrm{Log}(1+\mu z) & \text{if } \mu\neq 0\\ z & \text{if } \mu=0.\end{cases}$$

Thus we have

$$|\Phi_B(t,\ t_0)|\le dh(t)h(t_0)^{-1},\ t\ge t_0,$$

where $d=c_1c_2\exp(\int_{t_0}^{\infty}p(s)\Delta s)$ is a positive constant. Hence (4) is h-stable by Lemma 3.2. This completes the proof. □

Remark 2. If $h(t)$ is a positive bounded function on $\mathbb{T}$, then $\frac{h(t)}{h(\sigma(t))}$ is not bounded in general. For example, see [7, Remark 3.1] for $\mathbb{T}=\mathbb{Z}_+$.

Corollary 2. *Suppose that B is u_∞-quasisimilar to A.*

(i) If (3) is h-stable with bounded function $\frac{h(t)}{h(\sigma(t))}$ on $\mathbb{T}$, then (4) is h-stable.
(ii) If (3) is h-stable with $h(t)=e_{-\lambda}(t,a_0)$ for some positive constant λ with $-\lambda\in\mathcal{R}^+$ in Theorem 3.3 and $\int_{t_0}^{\infty}\frac{|F(t)|}{1-\mu(t)\lambda}\Delta t<\infty$ for each $t_0\in\mathbb{T}$. Then (4) is also uniformly exponentially stable.
(iii) If (3) is h-stable with a constant function h, then (4) is uniformly stable.

We can obtain the following results as special case of Theorem 3.3.

Remark 3. Suppose that B is u_∞-quasisimilar to A.

(i) If system (3) is uniformly stable (or stable), then system (4) is also uniformly stable (or stable).
(ii) If $\mathbb{T}=\mathbb{R}$ and (3) is uniformly exponentially stable, then (4) is also uniformly exponentially stable [24, Theorem 1].

(iii) If $\mathbb{T} = \mathbb{Z}$ and (3) is uniformly exponentially stable, then (4) is also uniformly exponentially stable [25, Theorem 3].

The notion of strong stability of differential equations was introduced by Ascoli [3]. Also, Agarwal [1] studied the various types of stability of solutions of difference equations as in the continuous case. Furthermore, Choi and Koo [12] introduced the notion of u_∞-similarity and investigated the strong stability of linear dynamic equations on time scales by using the notion of u_∞-similarity.

Now, we recall notion of the strong stability of linear dynamic systems on time scales in [10, 12].

Definition 3.4. The solution $x(t)$ of (11) is said to be *strongly stable* if, for each $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that, for any solution $\bar{x}(t) = x(t, t_0, \bar{x}_0)$ of (11), the inequalities $t_1 \geq t_0$ and $|\bar{x}(t_1) - x(t_1)| \leq \delta$ imply $|\bar{x}(t) - x(t)| < \varepsilon$ for all $t \geq t_0 \in \mathbb{T}$.

We note that strong stability implies uniform stability which, in turn, leads to stability [1, 12, 15].

Restrictive stability of differential systems is related to strong stability in [23].

Definition 3.5. System (3) is said to be *restrictively stable* if it is stable and its adjoint system

$$x^\Delta = (\ominus A)^*(t)x, \tag{12}$$

where A^* is the conjugate transpose of the matrix A and $\ominus A = -[I + \mu(t)A(t)]^{-1}$, is also stable.

Choi et al. [12] and Aulbach et al. [2] gave the characterization of strong stability for linear dynamic system (3) and its continuous version was presented in [23].

Lemma 3.6. *For each fixed $t_0 \in \mathbb{T}$ the following statements are equivalent:*

(i) (3) is strongly stable.
(ii) (3) is restrictively stable.
(iii) There exists a positive constant M such that

$$|\Phi_A(t, t_0)| \leq M, \quad |\Phi_A^{-1}(t, t_0)| \leq M, \; t \geq t_0 \in \mathbb{T}. \tag{13}$$

(iv) System (3) is kinematically similar to $x^\Delta = 0$ on $\mathbb{T}_{t_0}$.
(v) There exists a positive constant M such that

$$|\Phi_A(t, \tau)| \leq M, \quad t, \tau \in \mathbb{T}_{t_0}. \tag{14}$$

For an example to illustrate Lemma 3.6, see [10, 12].

We need the following comparison Lemma to prove Theorem 3.8.

Lemma 3.7. [12, Lemma 4.12] *Suppose $u, u^\Delta, a, b \in C_{\rm rd}(\mathbb{T}, \mathbb{R})$ and $a \geq 0$. Let $\tau \in \mathbb{T}$. Then*

$$u^\Delta(t) \geq -a(t)u(t) + b(t) \text{ for all } t \leq \tau,$$

implies

$$u(t) \leq u(\tau)e_a(\tau, t) + \int_\tau^t b(s)e_a(s, t)\Delta s \text{ for all } t \leq \tau.$$

Theorem 3.8. *Assume that B is u_∞-quasisimilar to A and (3) is strongly stable. Then (4) is also strongly stable.*

Proof. Suppose that (3) is strongly stable. Then there exists a positive constant M such that

$$|\Phi_A(t,\tau)| \le M, \ \ t,\tau \in \mathbb{T}_{t_0}.$$

Thus it suffices to show that $|\Phi_B(t,\tau)|$ is also bounded for each $t,\tau \ge t_0$. First, from the proof of Theorem 3.6 with $h(t) = c$, we obtain

$$|\Phi_B(t,\tau)| \le M_1, \ t \ge \tau \ge t_0,$$

where $M_1 = c_1c_2c\exp(c_1c_2c\int_{t_0}^{\infty}|F(t)|\Delta t)$ is a positive constant and $\Phi_B(t,t_0)$ is a matrix exponential function for (4).

Next, we show that $|\Phi_B(t,\tau)|$ is also bounded for each $t_0 \le t \le \tau$. If follows from Lemma 2.3 that

$$\begin{aligned}\Phi_B(t)\Phi_B^{-1}(\tau) \ \ &= \Gamma_k^{-1}(t)S^{-1}(t)\Phi_A(t)[\Phi_A^{-1}(\tau)S(\tau)\Gamma_k(\tau)\\ &\quad + \textstyle\int_\tau^t \Phi_A^{-1}(\sigma(s))F_k(s)\Phi_B(s,\tau)\Delta s]\end{aligned}$$

for each $t_0 \le t \le \tau$. From this and the strong stability of (3), there exist $\alpha > 0, \beta > 0$ such that

$$\begin{aligned}&|\Gamma_k^{-1}(t)S^{-1}(t)\Phi_A(t,\tau)S(\tau)\Gamma_k(\tau)| \le \alpha, \ t \le \tau,\\ &|\Gamma_k^{-1}(t)S^{-1}(t)\Phi_A(t,\sigma(s))| \le \beta, \ t \le s \le \tau.\end{aligned}$$

Hence, we obtain

$$|\Phi_B(t,\tau)| \ \ \le \ \ \alpha + \beta\int_t^\tau |F_k(s)||\Phi_B(s,\tau)|\Delta s = v(\tau,t), \ t_0 \le t \le \tau,$$

where $v(\tau,t) = \alpha + \beta\int_t^\tau |F_k(s)||\Phi_B(s,\tau)|\Delta s$. We have

$$v^{\Delta_t}(\tau,t) = -\beta|F_k(t)||\Phi_B(t,\tau)| \ge -\beta|F_k(t)|v(\tau,t), \ t_0 \le t \le \tau.$$

Putting $a(t) = \beta|F_k(t)|$ and from Lemma 3.7, we have

$$v(\tau,t) \le \alpha e_a(\tau,t), \ \ t_0 \le t \le \tau.$$

From the explicit presentation of the exponential function in [4], we obtain

$$\begin{aligned}v(\tau,t) \ \ &\le \ \ \alpha e_a(\tau,t) \le \alpha\exp(\beta\int_t^\tau |F_k(s)|\Delta s)\\ &\le \ \ \alpha\exp(\beta\int_{t_0}^\infty |F_k(s)|\Delta s) = M_2, \ t_0 \le t \le \tau,\end{aligned}$$

where $M_2 = \alpha\exp(\beta\int_{t_0}^\infty |F_k(s)|\Delta s)$. Hence this estimation holds for each $t,\tau \ge t_0$. This completes the proof. □

We can obtain the following results as the special cases of Theorem 3.8.

Remark 4. Suppose that B is u_∞-quasisimilar to A on $\mathbb{T}$ and system (3) is strongly stable.

(i) If $\mathbb{T} = \mathbb{R}$, then u_∞-quasisimilarity becomes t_∞-quasisimilarity and differential system (4) is also strongly stable in [24, Theorem 1].

(ii) If $\mathbb{T} = \mathbb{Z}$, then u_∞-quasisimilarity becomes n_∞-quasisimilarity and difference system (4) is also strongly stable in [25, Theorem 4].

Remark 5. Suppose that B is u_∞-quasisimilar to A with $k = 0$ in Definition 2.2, i.e., B is u_∞-similar to A.

(i) Then system (3) is strongly stable if and only if system (4) is also strongly stable in [12, Theorem 4.13].

(ii) If $\mathbb{T} = \mathbb{R}$ (or $\mathbb{Z}$), then u_∞-similarity means t_∞-similarity (or n_∞-similarity), then (3) is strongly stable if and only if (4) is also strongly stable in [23, Theorem 21] (or [8, Lemma 4.4]).

Remark 6. We note that Lemma 2.3 and Theorem 3.3 hold under the condition that the matrix-valued function B in Definition 2.2 and linear system (4) is rd-continuous (not necessarily regressive) [11], but the regressive assumption on B is essential for the strong stability in Theorem 3.8 [12].

We recall the following example about the various types of stability for solutions of linear dynamic systems in [10].

Example. We consider the linear dynamic system

$$x^\Delta = A(t)x = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix} x,\ x(t_0) = x_0 \in \mathbb{R}^2,\ t \geq t_0 \in \mathbb{T}, \tag{15}$$

where $A(t) = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}$ and $\mu(t) \neq \frac{1}{2}$ is bounded for $t \in \mathbb{T}$. Then the matrix exponential function $\Phi_A(t, t_0)$ of (15) is given by

$$\Phi_A(t, t_0) = \begin{pmatrix} 1 & 0 \\ 0 & e_{-2}(t, t_0) \end{pmatrix},\ t \geq t_0 \in \mathbb{T},$$

where the generalized exponential function $e_{-2}(t, t_0)$ is given by

$$e_{-2}(t, t_0) = \exp \int_{t_0}^{t} \frac{1}{\mu(\tau)} \mathrm{Log}(1 - 2\mu(\tau))\Delta\tau.$$

We easily see that $e_{-2}(t, 0)$ is given by

$$e_{-2}(t, 0) = \begin{cases} e^{-2t}, & t \in \mathbb{R}, \\ (1 - 2\delta)^{\frac{t}{\delta}}, & t \in \delta\mathbb{Z}, \\ \prod_{\tau \in q^{\mathbb{N}_0} \cap [0,t)} (1 + (1 - q)2\tau), & t \in q^{\mathbb{N}_0}, \\ (-e^2)^k e^{-2t}, & t \in \cup_{k=0}^{\infty} [2k, 2k + 1], \end{cases}$$

respectively. Thus we obtain the following various types of stability results for (15) and $e_{-2}(t, 0)$:

(i) If $\mathbb{T} = \mathbb{R}$, then (15) is h-stable but not strongly stable.

(ii) If $\mathbb{T} = \mathbb{Z}$, then (15) is strongly stable but not asymptotically stable.

(iii) If $\mathbb{T} = \delta\mathbb{Z}$ with $0 < \delta < 1$ and $\delta \neq \frac{1}{2}$, then (15) is neither asymptotically stable nor strongly stable. However $e_{-2}(t, t_0)$ goes to zero as $t \to \infty$.

(iv) If $\mathbb{T} = \delta\mathbb{Z}$ with $\delta > 1$, then (15) is neither asymptotically stable nor strongly stable.

(v) If $\mathbb{T} = q^{\mathbb{N}_0}$ with $q > \frac{3}{2}$, then (15) is unbounded and $e_{-2}(t, t_0)$ is oscillatory.

(vi) If $\mathbb{T} = \cup_{k=0}^{\infty} [2k, 2k + 1]$, then (15) is bounded and $e_{-2}(t, t_0)$ goes to zero as $t \to \infty$.

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