DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Supplement 2009

Website: www.aimSciences.org

pp. **143–150**

ON THE MINIMAL TIME NULL CONTROLLABILITY OF THE HEAT EQUATION

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Abstract. We show that the heat equation modelled as $y' = Ay + u$ in $L^1(\Omega)$ is null controllable by controls in $L^\infty(0, T; L^p(\Omega))$ with $1 < p < \infty$. Moreover, the corresponding minimal time function is Hölder continuous.

1. **Introduction.** Let us consider the control process described by the parabolic equation in a domain Ω in $\mathbb{R}^n$

$$\begin{array}{ll} y_t = \Delta y + u & (t,x) \in \mathbb{R}_+ \times \Omega \\ y = 0 & (t,x) \in \mathbb{R}_+ \times \partial\Omega \\ y(0,x) = \xi(x) & x \in \Omega. \end{array} \tag{1}$$

Let $1 \leq p_1 \leq p_2 < \infty$. We are interested in transferring the initial state $\xi \in L^{p_1}(\Omega)$ to the target zero, in minimum time, by means of an admissible control $u(t,x)$ satisfying bounds of the type

$$\int_\Omega |u(t,x)|^{p_2} dx \leq 1. \tag{2}$$

The partial differential equation (1) can be rewritten as an ordinary differential equation in $L^{p_1}(\Omega)$ of the form

$$\begin{aligned} y'(t) &= A_{p_1} y(t) + Bu(t), \quad t \geq 0 \\ y(0) &= \xi, \end{aligned} \tag{3}$$

where A_{p_1} is the Laplace operator with the Dirichlet boundary condition defined on $L^{p_1}(\Omega)$, B is the embedding operator from $L^{p_2}(\Omega)$ to $L^{p_1}(\Omega)$ and the control function u is assumed to be in $L^\infty(0,t;L^{p_2}(\Omega))$, for $t > 0$. It is known that A_{p_1} generates a C_0-semigroup of contractions $\{S_{p_1}(t)\,;\ t \geq 0\}$ on $L^{p_1}(\Omega)$. The mild solution of (3) is

$$y(t,\xi,u) = S_{p_1}(t)\xi + \int_0^t S_{p_1}(t-s)\,Bu(s)\,ds, \quad t \geq 0. \tag{4}$$

2000 *Mathematics Subject Classification.* Primary: 93B05, 49K15; Secondary: 34H05.
Key words and phrases. Null controllability, minimal time function, heat equation.
The authors are supported by CNCSIS grant ID_387/2007.

Let $T>0$. Define the operator $V_T : L^\infty(0,T;L^{p_2}(\Omega)) \to L^{p_1}(\Omega)$ by

$$V_T u = \int_0^T S_{p_1}(s)\, Bu(s)\, ds.$$

Obviously, V_T is a bounded linear operator.

The control system (3) is said to be null-controllable on $[0,T]$ if for each $\xi \in L^{p_1}(\Omega)$ there exists $u \in L^\infty(0,T;L^{p_2}(\Omega))$ such that $y(T,\xi,u)=0$, i.e.,

$$\text{Range}\,(S_{p_1}(T)) \subset \text{Range}\,(V_T).$$

In the particular case when $p_1 = p_2 \in [1,\infty)$, it is known that the control system (3), with B the identity operator, is null controllable on $[0,T]$. In fact, this is true for a general control system of type

$$y' = Ay + u, \tag{5}$$

where A generates a C_0-semigroup on a general Banach space X and u is a control in $L^\infty(0,T;X)$. See, e.g., [6].

One of the aims of this paper is to study the null controllability of the system (3), when the controls are in $L^\infty(0,T;L^{p_2}(\Omega))$, with $p_2 \geq p_1$.

Another problem we are interested in is the regularity of the associated minimal time function.

Let $U_{ad} = \left\{ u \in L^\infty(0,\infty;L^{p_2}(\Omega));\ \|u(t)\|_{L^{p_2}(\Omega)} \leq 1 \text{ a.e.} \right\}$ be the admissible set of controls. For $t \geq 0$, denote by $R(t)$ the set of all initial states in $L^{p_1}(\Omega)$ which can be transferred to zero during $[0,t]$ by admissible controls, i.e.,

$$R(t) = \{\xi \in L^{p_1}(\Omega);\ y(t,\xi,u)=0 \text{ for some } u \in U_{ad}\}.$$

Let $R = \cup_{t\geq 0} R(t)$ and define the minimal time function $\mathcal{T} : L^{p_1}(\Omega) \to [0,\infty]$ by

$$\begin{array}{ll} \mathcal{T}(\xi) = \inf\{t;\ \xi \in R(t)\} & \text{for } \xi \in R, \\ \mathcal{T}(\xi) = +\infty & \text{for } \xi \notin R. \end{array} \tag{6}$$

When $p_1 = p_2 \in [1,\infty]$ and B is the identity operator, because $S_{p_1}(t)$ is a C_0-semigroup of contractions, we have that $R = L^{p_1}(\Omega)$ and

$$|\mathcal{T}(x_1) - \mathcal{T}(x_2)| \leq \|x_1 - x_2\|,\ \forall x_1, x_2 \in L^{p_1}(\Omega),$$

that is, the minimal time function $\mathcal{T}(\cdot)$ is Lipschitz continuous on $L^{p_1}(\Omega)$. In fact, for a general system of type (5) it is known that the minimal time function is locally Lipschitz continuous on the reachable set. Moreover, if the semigroup generated by A is a contraction semigroup, as in our case, then $R = X$ (see [10]) and the minimal time function is globally Lipschitz (see [3, 6]).

In this paper we obtain some regularity properties for the minimal time function, in case where the controls are taken from $L^\infty(0,T;L^{p_2}(\Omega))$, with $p_2 \geq p_1$.

Finally, we study the null controllability with vanishing energy of system (3).

2. **The null-controllability problem.** In order to solve our null-controllability problem, we shall consider an abstract framework where our problem will be a particular case.

Consider the linear control system

$$y'(t) = Ay(t) + Bu(t),\quad t \geq 0, \tag{7}$$

where $A : D(A) \subset X \to X$ generates a C_0-semigroup $\{S(t);\ t \geq 0\}$ on X and $B : U \to X$ is a bounded linear operator, U being a Banach space. The following characterization of the null controllability of (7) was proved in [13] in the hypothesis that both spaces X and U where reflexive. Here we relax the conditions on X to be a general Banach space, requiring that the reflexive Banach space U be separable. Here and hereafter we denote by $\mathcal{B}_Y(0,1)$ the unit closed ball in the Banach space Y.

Theorem 2.1. *Let X and U be Banach spaces. Assume that U is reflexive and separable. Then the system (7) is null-controllable with respect to $L^\infty(0,T;U)$ if and only if there exists $\alpha(T)>0$ such that*

$$\alpha(T)\left\|S^*(T)x^*\right\|_{X^*} \leq \int_0^T \left\|B^*S^*(t)x^*\right\|_{U^*} dt, \tag{8}$$

for all $x^* \in X^*$.

Proof. Since (8) is equivalent to

$$S(T)\left(\mathcal{B}_X(0,\alpha(T))\right) \subset \text{closure} V_T\left(\mathcal{B}_{L^\infty(0,T;U)}(0,1)\right) \tag{9}$$

(see, e.g., [5, Theorem 2.2]), in order to complete the proof, we only have to show that $V_T\left(\mathcal{B}_{L^\infty(0,T;U)}(0,1)\right)$ is a closed set.

Let $(u_n)_n \subset \mathcal{B}_{L^\infty(0,T;U)}(0,1)$ be such that $\lim_{n\to\infty} V_T(u_n)=y$, $y\in X$. There exists $u \in \mathcal{B}_{L^\infty(0,T;U)}(0,1)$ such that, on a subsequence, we have that $(u_n)_n$ is weakly star convergent to u in $L^\infty(0,T;U)$. On the other hand, let us observe that $V_T^*x^* = B^*S^*(\cdot)x^*$ and belongs to $L^1(0,T;U^*)$ for any $x^*\in X^*$, because U^* is a separable Banach space (where strong and weak measurability coincide). See, e.g., [7]. Hence, for any $x^*\in X^*$,

$$\langle V_T(u_n), x^*\rangle = \langle u_n, V_T^*x^*\rangle \to \langle u, V_T^*x^*\rangle = \langle V_T(u), x^*\rangle .$$

In conclusion, we obtain that $y=V_T(u)$ with $u\in\mathcal{B}_{L^\infty(0,T;U)}(0,1)$. This completes the proof. □

The following theorem gives a sufficient condition for the null-controllability of the system (7).

Theorem 2.2. *Assume $U\subset X$ continuously, i.e. there exists $k>0$ such that $\|x\|_X \leq k\|x\|_U$, for any $x\in U$. Assume further that U is reflexive and separable. Let $T>0$ and B the embedding operator from U to X.*

Suppose that there is $f(\cdot)\in L^\infty(\tau,T)$, for some $\tau\in(0,T)$, such that

$$\left\|S^*(t)x^*\right\|_{X^*} \leq f(t)\left\|B^*x^*\right\|_{U^*}, \ \forall x^*\in X^*, \ \text{a.e. on } (\tau,T). \tag{10}$$

Then, the system (7) is null-controllable on $[0,T]$ with respect to $L^\infty(0,T;U)$.

Proof. For the proof, we shall use Theorem 2.1, more exactly, we shall prove the existence of $\alpha(T)>0$ such that (8) holds true.

To this aim, observe that for any $x^*\in X^*$ we have

$$\begin{aligned}\left\|S^*(T)x^*\right\|_{X^*} &= \frac{1}{T-\tau}\int_0^{T-\tau}\left\|S^*(T-t)S^*(t)x^*\right\|_{X^*} dt \\ &\leq \frac{1}{T-\tau}\int_0^{T-\tau} f(T-t)\left\|B^*S^*(t)x^*\right\|_{U^*} dt.\end{aligned}$$

Hence,

$$\begin{aligned}\|S^*(T)\,x^*\|_{X^*} &\leq \frac{1}{T-\tau}\,\|f\|_{L^\infty(\tau,T)}\int_0^{T-\tau}\|B^*S^*(t)\,x^*\|_{U^*}\,dt\\ &\leq \frac{1}{T-\tau}\,\|f\|_{L^\infty(\tau,T)}\int_0^{T}\|B^*S^*(t)\,x^*\|_{U^*}\,dt.\end{aligned}$$

The proof is completed. □

Now, let us return to the control system (3). First, we shall present some regularity results concerning the semigroup generated by the Laplace operator on $L^p(\Omega)$, for $p \geq 1$, useful in our study.

Regarding the dual semigroup of $S_p(t)$, $p \geq 1$, we have that $S_p^*(t)\,h = S_q(t)\,h$, for any $h \in L^q(\Omega)$, where $\frac{1}{p}+\frac{1}{q}=1$.

Lemma 2.3. [14, Lemma 7.2.1] *Let Ω be a nonempty, bounded and open subset in $\mathbb{R}^n$, whose boundary is of class C^1 and let $p \in [1,\infty]$. Let A_p be the Laplace operator subjected to the Dirichlet boundary condition on $L^p(\Omega)$, with the convention that, for $p=\infty$, $L^p(\Omega)$ is replaced with $C_0\left(\overline{\Omega}\right)$. Let $\{S_p(t);\ t \geq 0\}$ be the C_0-semigroup generated by A_p on $L^p(\Omega)$. Then, for each $p, r \in [1,\infty]$, each $\xi \in C_0\left(\overline{\Omega}\right)$ and each $t \geq 0$ we have $S_p(t)\,\xi = S_r(t)\,\xi$.*

By this Lemma, we can denote the C_0-semigroup generated by the Laplace operator subjected to the Dirichlet boundary condition on any of the spaces $L^p(\Omega)$, by the same symbol $\{S(t);\ t \geq 0\}$.

Theorem 2.4. [14, Theorem 7.2.6] *Let Ω be a nonempty, bounded and open subset in $\mathbb{R}^n$, whose boundary is of class C^1. Let $p \geq 1$ and let $\{S(t);\ t \geq 0\}$ be the C_0-semigroup generated by the Laplace operator subjected to the Dirichlet boundary condition on $L^p(\Omega)$. Then, for each $1 \leq p \leq r \leq \infty$, each $\xi \in L^p(\Omega)$ and each $t > 0$, we have*

$$\|S(t)\,\xi\|_{L^r(\Omega)} \leq (4\pi t)^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\|\xi\|_{L^p(\Omega)}.$$

Let q_1, q_2 be the conjugates of p_1, p_2, i.e. $\frac{1}{q_i}+\frac{1}{p_i}=1$, $i=1,2$. Using Theorem 2.4 for $q_2 \leq q_1$ we obtain

$$\|S^*(t)\,h\|_{L^{q_1}(\Omega)} \leq (4\pi t)^{-\frac{n}{2}\left(\frac{1}{q_2}-\frac{1}{q_1}\right)}\|h\|_{L^{q_2}(\Omega)},\ \forall h \in L^{q_1}(\Omega),\ \forall t > 0.$$

Denote $C = \frac{n}{2}\left(\frac{1}{q_2}-\frac{1}{q_1}\right) = \frac{n}{2}\left(\frac{1}{p_1}-\frac{1}{p_2}\right) > 0$ and let $f(t) = (4\pi t)^{-C}$, for $t > 0$. Obviously $f(\cdot) \in L^\infty(\tau, T)$ for any $\tau \in (0,T)$.

Now, we can state the following result regarding the null-controllability of (3).

Theorem 2.5. *Let Ω be a nonempty, bounded and open subset in $\mathbb{R}^n$, whose boundary is of class C^1. Let $T > 0$ and $1 \leq p_1 \leq p_2 < \infty$, $p_2 > 1$. Then, the system (3) is null-controllable on $[0,T]$ with respect to $L^\infty(0,T;L^{p_2}(\Omega))$.*

Remark 1. Obviously, if the system is null controllable with respect to $L^\infty(0,T;L^{p_2}(\Omega))$ then it is null controllable with respect to $L^p(0,T;L^{p_2}(\Omega))$, for any $p \in [1,\infty)$.

3. **The minimal time function.** In this section we use the previous results in order to obtain estimates for the minimal time function.

From the previous section, taking $\tau = \frac{T}{2}$, we have

$$\|S^*(T)x^*\|_{(L^{p_1}(\Omega))^*} \leq 2^{C+1}(4\pi)^{-C}T^{-1-C}\int_0^T \|B^*S^*(t)x^*\|_{(L^{p_2}(\Omega))^*}dt,$$

for any $x^* \in (L^{p_1}(\Omega))^*$ and for any $T > 0$. This inequality can be rewritten as

$$\alpha(T)\|S^*(T)x^*\|_{(L^{p_1}(\Omega))^*} \leq \int_0^T \|B^*S^*(t)x^*\|_{(L^{p_2}(\Omega))^*}dt, \tag{11}$$

where

$$\alpha(T) = 2^{C-1}\pi^C T^{C+1}. \tag{12}$$

For $p_2 \in (1, \infty)$ the space $L^{p_2}(\Omega)$ is a reflexive and separable Banach space and, by Theorem 2.1, we have that (11) is equivalent to the inclusion

$$S(T)\left(\mathcal{B}_{L^{p_1}(\Omega)}(0, \alpha(T))\right) \subset V_T\left(\mathcal{B}_{L^\infty(0,T;L^{p_2}(\Omega))}(0,1)\right),\ \forall T > 0. \tag{13}$$

At this point let us remark that (13) gives precise estimates for the minimum time function as the following theorem, taken from [1], shows.

Theorem 3.1. *Let X and U be Banach spaces. Suppose there exists a function α : $[0, \infty) \to \mathbb{R}$ strictly increasing, continuous, with $\alpha(0) = 0$ and $\lim_{t\to\infty}\alpha(t) = \infty$, such that*

$$S(T)(\mathcal{B}_X(0, \alpha(T))) \subset V_T\left(\mathcal{B}_{L^\infty(0,T;U)}(0,1)\right) \tag{14}$$

is verified for any $T \geq 0$. Then $\mathcal{T}(x) \leq \alpha^{-1}(\|x\|)$ for any $x \in X$.

Returning to our problem, taking $\alpha(\cdot)$ given by (12), we have that it is strictly increasing, continuous, $\alpha(0) = 0$, (13) holds for any $T > 0$ and $\alpha^{-1}(r) = \lambda^{-\frac{1}{C+1}}r^{\frac{1}{C+1}}$, where $\lambda = 2^{C-1}\pi^C$.

Applying Theorem 3.1 we get

$$\mathcal{T}(x) \leq \lambda^{-\frac{1}{C+1}}\|x\|^{\frac{1}{C+1}}, \tag{15}$$

for any $x \in R$. Moreover, we have $R = L^{p_1}(\Omega)$ because $S(t)$ is a contraction semigroup (see [10]), hence (15) holds for any $x \in L^{p_1}(\Omega)$.

Further, because $S(t)$ is a semigroup of contractions, the local properties of the minimal time function become global (see [6]) and applying the abstract results from [6, Theorem 2.1] in our case, we obtain:

Theorem 3.2. *Let Ω be a nonempty, bounded and open subset in $\mathbb{R}^n$, whose boundary is of class C^1. Assume $1 \leq p_1 \leq p_2 < \infty$, $p_2 > 1$. Then, there exists $k > 0$, depending on p_1, p_2 and n, such that*

$$\mathcal{T}(x) \leq k\|x\|^{\frac{1}{\frac{n}{2}\left(\frac{1}{p_1}-\frac{1}{p_2}\right)+1}},$$

for any $x \in L^{p_1}(\Omega)$.
Moreover, $\mathcal{T}$ is Hölder continuous, i.e.

$$|\mathcal{T}(x_1) - \mathcal{T}(x_2)| \leq k\|x_1 - x_2\|^{\frac{1}{\frac{n}{2}\left(\frac{1}{p_1}-\frac{1}{p_2}\right)+1}},$$

for any $x_1, x_2 \in L^{p_1}(\Omega)$.

4. **Controllability with vanishing energy.** In [11], the authors introduce an interesting concept, null controllability with vanishing energy, for linear control systems in Hilbert spaces, and give characterizations for it. Further results in this direction, when X is a Banach space and U is a Hilbert space, are given in [12]. In both [11] and [12] the control space is $L^2(0,T;U)$. Here we shall present some results concerning a related concept considering the control space $L^\infty(0,T;U)$.

Definition 4.1. We say that the system (7) is null controllable with vanishing energy, in short NCVE, if for all $x \in X$ and all $\varepsilon > 0$ there exists a time $t > 0$ and a function $u \in L^\infty(0,t;U)$ satisfying $\|u\| \le \varepsilon$ and such that $y(t,x,u) = 0$.

Proposition 1. *The system (7) is NCVE if and only if there exists $\widetilde{\varepsilon} > 0$ such that for all $x \in X$ there exist a time $t > 0$ and a function $u \in L^\infty(0,t;U)$ satisfying $\|u\| \le \widetilde{\varepsilon}$ and such that $y(t,x,u) = 0$.*

Proof. Let $\varepsilon > 0$ and $x \in X$. For $z = \dfrac{\widetilde{\varepsilon}}{\varepsilon} x$ there exist $t > 0$ and $u \in L^\infty(0,t;U)$, with $\|u\| \le \widetilde{\varepsilon}$ such that

$$S(t)(z) + \int_0^t S(t-s)\,Bu(s)\,ds = 0,$$

so, we have that $y(t,x,v) = 0$, where $v = \dfrac{\varepsilon}{\widetilde{\varepsilon}} u \in L^\infty(0,t;U)$ and $\|v\| \le \varepsilon$. □

Let $\rho > 0$ and denote by R_ρ the set of all points $x \in X$ such that $y(t,x,u) = 0$, for some $t > 0$ and $u \in L^\infty(0,t;U)$ with $\|u\| \le \rho$. Then, Definition 4.1 gives the following result.

Proposition 2. *The system (7) is NCVE if and only if for each $\rho > 0$ the reachable set R_ρ is the whole space.*

Remark 2. Proposition 1 says that if for some $\rho^* > 0$ we have $R_{\rho^*} = X$ then, for all $\rho > 0$, the reachable set R_ρ is the whole space.

The next result is related to Theorem 3.1 from [12] and gives a necessary condition for the NCVE. See also Theorem 3 in [4]. The proof is standard and is based on the Baire Category Theorem.

Theorem 4.2. *Assume that X is a reflexive Banach space and, for some Banach space F, $U = F^*$. Assume further that U is separable. If the system (7) is NCVE, then there exists $t > 0$ such that the system is null controllable in time t, i.e., for all $x \in X$ there exists $u \in L^\infty(0,t;U)$ such that $y(t,x,u) = 0$.*

Proof. For $n \ge 1$, let X_n be the set of all $x \in X$ such that $y(n,x,u) = 0$ for some $u \in L^\infty(0,n;U)$ with $\|u\| \le 1$. Clearly, $X = \cup X_n$. We show that X_n are closed sets. To this end, fix $n \ge 1$ and take a sequence (x_k) in X_n such that $x_k \to x$. We have to show that $x \in X_n$. Let $u_k \in L^\infty(0,n;U)$ with $\|u_k\| \le 1$ such that $y(n,x,u_k) = 0$. Since

$$L^\infty(0,n;U) = L^1(0,n;F)^*, \tag{16}$$

we can select a subsequence of (u_k) (denoted with the same name) which is $L^1(0,n;F)$ -weakly convergent to $u \in L^\infty(0,n;U)$ and $\|u\| \le 1$ holds true. Applying a functional $x^* \in X^*$ to both sides of

$$S(n)x + \int_0^n S(n-s)Bu_k(s)\,ds = 0$$

we get

$$\langle x^*, S(n)x\rangle + \int_0^n \langle B^*S^*(n-s)x^*, u_k(s)\rangle\, ds.$$

Taking limits, we obtain $S(n)x + \int_0^n S(n-s)Bu\, ds = 0$, as claimed. By the Baire Category Theorem, there exist n_0 and $x_0 \in X_{n_0}$ such that $X_{n_0} - x_0$ is a neighborhood of the origin. By linearity, all points of $X_{n_0} - x_0$ can be transferred to 0 by controls in $L^\infty(0, n; U)$. Again by linearity it follows that all points of X can be transferred to 0 and the proof is complete. □

Remark 3. Since F^* is separable, F is separable too, so $L^1(0, n; F)$ is separable. This allowed us to work with subsequences in the above proof. The equality in (16) holds true also in case U is reflexive with $F = U^*$. The result of Theorem 4.2 is also true in this case, although $L^1(0, n; F)$ is not necessarily separable, working with generalized subsequences instead of subsequences. The reader can find details in [7, Section 12.9], [8, Section 3.1].

Remark 4. The reflexivity of X can be dropped in Theorem 4.2, working with the Phillips adjoint semigroup $S^{\odot}(t)$ instead of $S^*(t)$ (which may be not strongly continuous). See, e.g., [8, Section 3.1] for a similar discussion.

Let us denoted by $\mathcal{E}(t, x, 0)$ the minimum energy to bring x to the origin of X in time t, i.e.

$$\mathcal{E}(t, x, 0) = \inf_{u \in \mathcal{M}(t,x,0)} \|u\|_{L^\infty(0,t;U)},$$

where $\mathcal{M}(t, x, 0)$ is the (possibly empty) set of controls $u \in L^\infty(0, t; U)$ such that $y(t, x, u) = 0$.

The following theorem is a version of a result from [1] and gives estimates for the minimum energy related to estimates of the minimal time function.

Theorem 4.3. *Suppose that there exists a function $\beta : [0, \infty) \to \mathbb{R}$ strictly increasing, continuous, with $\beta(0) = 0$ and $\lim_{s\to\infty} \beta(s) = \infty$, such that $\mathcal{T}(x) \le \beta(\|x\|)$ for any $x \in X$. Then, for each $x \in X$ and each $t > 0$, we have*

$$\mathcal{E}(t, x, 0) \le \frac{1}{\beta^{-1}(t)} \|x\|.$$

Returning to our system (3), we can state the following result.

Theorem 4.4. *Let Ω be a nonempty, bounded and open subset in $\mathbb{R}^n$, whose boundary is of class C^1. Let $T > 0$ and $1 \le p_1 \le p_2 < \infty$, $p_2 > 1$. Then system (3) is NCVE.*

Proof. Let us define

$$\beta(r) = k\, r^{\frac{1}{C+1}}, \text{ for any } r \ge 0,$$

where $C = \frac{n}{2}\left(\frac{1}{p_1} - \frac{1}{p_2}\right)$ as before. Obviously, this function is strictly increasing, continuous, $\beta(0) = 0$ and $\lim_{s\to\infty} \beta(s) = \infty$. Moreover, from the previous section we have that $\mathcal{T}(x) \le \beta(\|x\|)$, for any $x \in L^{p_1}(\Omega)$. Using Theorem 4.3 we obtain

$$\mathcal{E}(t, x, 0) \le k^{C+1} t^{-C-1} \|x\|,$$

for any $x \in L^{p_1}(\Omega)$ and any $t > 0$. Now, since $\lim_{t\to\infty} \mathcal{E}(t, x, 0) = 0$ for any $x \in X$, we get that the system (3) is NCVE. □

Acknowledgements. The authors are very grateful to the referee for valuable suggestions for improving the paper.

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Received June 2008; revised May 2009.

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