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SOME ABSTRACT CRITICAL POINT THEOREMS AND APPLICATIONS

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Abstract. Since Palais' pioneer paper in 1963, Condition (C) in both the Palais–Smale version and Cerami's variant has been widely used in order to prove minimax existence theorems for C^1 functionals in Banach spaces.

Here, we introduce a weaker version of these conditions so that a Deformation Lemma still holds and some critical points theorems can be stated. Such abstract results apply to p–Laplacian type elliptic problems.

1. Introduction and main results.

As many problems which arise in nature can be reduced to an Euler–Lagrange equation, whose solutions are critical points of a smooth functional in a suitable Banach space, great interest is devoted to the study of critical points of abstract functionals. Up to now, many existence and multiplicity theorems have already been stated and, here, we go further in this direction and improve some classical results.

Let $(X, \|\cdot\|_X)$ be a Banach space with dual space $(X', \|\cdot\|_{X'})$ while $J : X \to \mathbb{R}$ is a given C^1 functional whose set of critical points is $K_J = \{u \in X : dJ(u) = 0\}$.

In order to study the set K_J, to drop some global compactness assumptions and to guarantee the statement of some minimax theorems, in his pioneer paper [10] Palais introduced the so–called *Condition* (C) (in his setting, X is a complete Riemannian manifold):

"If S is any subset of X on which J is bounded but on which $\|dJ\|_{X'}$ is not bounded away from zero, then there is a critical point of J in the closure of S."

Equivalently,

"if S is a closed subset of X on which J is bounded but $S \cap K_J = \emptyset$, then $\alpha > 0$ exists so that

$$\|dJ(u)\|_{X'} \geq \alpha \qquad \text{for all } u \in S."$$

An almost equivalent way to express Condition (C) but pointing out the role of sequences, is the classical Palais–Smale Condition (see [11]). Here, let us recall the definition of Palais–Smale Condition at a fixed level.

Definition 1.1 (Palais–Smale Condition at level a)**.** The functional J satisfies the *Palais–Smale Condition at level* a ($a \in \mathbb{R}$), briefly $(PS)_a$, if any sequence $(u_n)_n \subset X$ such that

$$\lim_{n\to+\infty} J(u_n) = a \quad \text{and} \quad \lim_{n\to+\infty} \|dJ(u_n)\|_{X'} = 0, \tag{1.1}$$

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converges in X, up to subsequences.

Remark 1.2. Fixing $a \in \mathbb{R}$ so that J satisfies $(PS)_a$, it is easy to prove that the critical point set $K_a^J = \{u \in X : \ J(u) = a,\ dJ(u) = 0\}$ is compact.

More recently, in [8] Cerami has weakened Definition 1.1 by allowing a sequence to go to infinity but only if the gradient of the functional goes to zero "not too slowly". Also with this condition some minimax theorems hold (see, e.g., [3]).

Definition 1.3 (Cerami's variant of Palais–Smale Condition)**.** The functional J satisfies *Cerami's variant of Palais–Smale Condition at level* a ($a \in \mathbb{R}$), briefly $(CPS)_a$, if any sequence $(u_n)_n \subset X$ such that

$$\lim_{n\to+\infty} J(u_n) = a \quad \text{and} \quad \lim_{n\to+\infty} \|dJ(u_n)\|_{X'}(1 + \|u_n\|_X) = 0,$$

converges in X, up to subsequences.

Another way to weaken the Palais–Smale Condition was introduced by Brézis, Coron and Nirenberg in [4] in such a way that the Mountain Pass Theorem still holds (see [4, Theorem 2]).

Definition 1.4. The functional J satisfies (BCN) *Condition at level* a ($a \in \mathbb{R}$), briefly $(BCN)_a$, if the following holds:
"If a sequence $(u_n)_n \subset X$ exists such that (1.1) holds, then a is a critical value."

Unluckily, we are not able to prove any of the previous conditions for the non-linear functional

$$F(u) = \int_\Omega A(x, u, \nabla u)dx - \int_\Omega G(x, u)dx, \qquad u \in \mathcal{D} \subset W_0^{1,p}(\Omega), \tag{1.2}$$

which generalizes the model problem

$$f(u) = \int_\Omega \bar{A}(x, u)|\nabla u|^p dx - \int_\Omega G(x, u)dx, \qquad u \in \mathcal{D} \subset W_0^{1,p}(\Omega), \tag{1.3}$$

where Ω is an open bounded domain in $\mathbb{R}^N$ ($N \geq 3$), $p > 1$, and $A : \Omega\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$, respectively $\bar{A} : \Omega \times \mathbb{R} \to \mathbb{R}$, $G : \Omega \times \mathbb{R} \to \mathbb{R}$ are given Carathéodory functions.

Furthermore, in general, even for the simplest model (1.3) with $p = 2$, $G \equiv 0$, $\bar{A}$ smooth and bounded from zero, the functional f is not Gâteaux differentiable in its domain but is differentiable only along directions of $H_0^1(\Omega) \cap L^\infty(\Omega)$.

In order to solve both the lack of regularity and some difficulties in the compactness of the Palais–Smale sequences, different ideas have been used. For example, Arcoya and Boccardo in [2] assume that $u \in W_0^{1,p}(\Omega)$ is a critical point of F if $dF(u)[v] = 0$ for all $v \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$, moreover they construct suitable Palais–Smale sequences which converge to such critical points.

On the contrary, we think it is quite natural to consider F restricted to the Banach space

$$Y = W_0^{1,p}(\Omega) \cap L^\infty(\Omega), \quad \|u\|_Y = \|u\|_p + |u|_\infty, \tag{1.4}$$

where

$$(Y, \|\cdot\|_Y) \quad \text{is continuously embedded in} \quad (W_0^{1,p}(\Omega), \|\cdot\|_p) \tag{1.5}$$

(here, $(W_0^{1,p}(\Omega), \|\cdot\|_p)$ and $(L^\infty(\Omega), |\cdot|_\infty)$ are defined as usual), as classical growth hypotheses imply that F is C^1 in Y. But in this setting both $(PS)_a$ and $(CPS)_a$ require the L^∞–convergence which seems "too much". Hence, we introduce another variant of Condition (C), related to $(CPS)_a$ but similar to $(BCN)_a$.

Definition 1.5. The functional J satisfies a *weak version of Condition* (C) *at level* a ($a \in \mathbb{R}$), briefly $(wC)_a$, if the following holds:
"If a sequence $(u_n)_n \subset X$ exists so that

$$\lim_{n\to+\infty} J(u_n) = a \quad \text{and} \quad \lim_{n\to+\infty} \|dJ(u_n)\|_{X'}(1 + \|u_n\|_X) = 0, \tag{1.6}$$

then a is a critical level for J."

Even if $(wC)_a$ as stated in Definition 1.5 is weaker than more classical Palais–Smale type conditions, it is still enough for stating a Minimum Principle and some critical point theorems.

Theorem 1.6 (Minimum Principle). *If $J \in C^1(X, \mathbb{R})$ is bounded from below in X and $(wC)_\beta$ holds at level $\beta = \inf_X J \in \mathbb{R}$, then J attains its infimum, i.e., $u_0 \in X$ exists such that $J(u_0) = \beta$.*

As for setting the minimax theorems some geometric hypotheses need, in considering (1.5) we can assume X continuously embedded in another Banach space W so that we can weaken also the geometric assumptions of the classical results.

Thus, the following generalization of the Mountain Pass Theorem can be stated (compare with [1, Theorem 2.1]; other versions can be found in [9, 13, 14]).

Theorem 1.7 (Mountain Pass Theorem). *Let $J \in C^1(X, \mathbb{R})$ be such that $J(0) = 0$ and $(wC)_a$ holds at each level $a \in \mathbb{R}$. Moreover, assume that there exist another Banach space $(W, \|\cdot\|_W)$ and r, $\varrho > 0$, $e \in X$ such that $X \hookrightarrow W$ continuously and*

$$u \in X, \ \|u\|_W = r \quad \Longrightarrow \quad J(u) \geq \varrho,$$

$$\|e\|_W > r \qquad \textit{and} \qquad J(e) < \varrho.$$

Then, J has a Mountain Pass critical point $u^* \in X$ such that $J(u^*) \geq \varrho$.

Furthermore, with the stronger assumption that J is symmetric, some multiplicity results can be stated (see, e.g., [3, Theorem 2.4], [15], [16, Theorem 6.5 in Chapter II]).

Theorem 1.8. *Let $J \in C^1(X, \mathbb{R})$ be an even functional such that $J(0) = 0$ and $(wC)_a$ holds at each level $a \in \mathbb{R}_+$. Assume that a Banach space $(W, \|\cdot\|_W)$ exists such that $X \hookrightarrow W$ continuously. Moreover, assume $\varrho > 0$ exists so that:*

(A_ϱ) there exist two closed subspaces E_ϱ and Z_ϱ of X such that

$$E_\varrho + Z_\varrho = X, \qquad \operatorname{codim} Z_\varrho \ < \ \dim E_\varrho \ < \ +\infty,$$

and J satisfies the following assumptions:

(i) there exist a constant $r > 0$ such that

$$u \in Z_\varrho, \quad \|u\|_W = r \qquad \Longrightarrow \qquad J(u) \geq \varrho;$$

(ii) there exists $R > 0$ such that

$$u \in E_\varrho, \quad \|u\|_X \geq R \qquad \Longrightarrow \qquad J(u) \leq 0$$

(hence, $\sup_{u\in E_\varrho} J(u) < +\infty$).

Then, the functional J possesses at least a pair of symmetric critical points in X whose corresponding critical level belongs to $[\varrho, \varrho_1]$, with $\varrho_1 \geq \sup_{u\in E_\varrho} J(u) > \varrho$.

Corollary 1.9. *Let $J \in C^1(X, \mathbb{R})$ be an even functional such that $J(0) = 0$ and $(wC)_a$ holds at each level $a \in \mathbb{R}_+$. Assume that a Banach space $(W, \|\cdot\|_W)$ exists such that $X \hookrightarrow W$ continuously, and condition (A_ϱ) holds for all $\varrho > 0$.*
Then, the functional J possesses a sequence of critical points $(u_k)_k \subset X$ such that $J(u_k) \nearrow +\infty$ as $k \nearrow +\infty$.

Remark 1.10. Even if $(wC)_a$ at level a is weaker than $(CPS)_a$, it is still enough for stating some existence results but it is not sufficient for generalizing multiplicity theorems. In fact, a multiplicity result such as Corollary 1.9 can be stated only because we are able to find a strictly increasing sequence of critical levels so the corresponding critical points are different. But, in general, $(wC)_a$ does not allow us to distinguish different critical points at the same critical level.

The rest of this paper is essentially divided into two parts: in Section 2 we prove the abstract theorems stated in this introduction, while in Section 3 we apply such theorems to some model problems.

2. **Proofs of the abstract theorems.** Throughout this section, let us assume that $(X, \|\cdot\|_X)$ is a Banach space while $J : X \to \mathbb{R}$ is a given C^1 functional. Moreover, let $(W, \|\cdot\|_W)$ be a Banach space such that $X \hookrightarrow W$ continuously.

For simplicity, here and in the following we name $(CPS)_a$*–sequence* each sequence $(u_n)_n \subset X$ which satisfies (1.6).

So, fixing $a \in \mathbb{R}$, $(wC)_a$, as defined in Definition 1.5, reduces to the following statement:
"if a $(CPS)_a$–sequence exists, then the level a is critical."

Firstly, let us point out more information about $(wC)_a$. To this aim, let us introduce another definition whose statement is closer to the initial Condition (C).

Definition 2.1. The functional J satisfies (wC^*) in the interval $[k_1, k_2]$ ($k_1, k_2 \in \mathbb{R}$, $k_1 \le k_2$) if, whenever $a \in [k_1, k_2]$, $\sigma > 0$ exist such that

$$J^{-1}([a-\sigma, a+\sigma]) \cap K_J = \emptyset,$$

then there exists $\alpha > 0$ such that

$$\|dJ(u)\|_{X'}(1 + \|u\|_X) \ge \alpha \qquad \text{for all } u \in J^{-1}([a-\sigma, a+\sigma]).$$

Lemma 2.2. *If $\sigma_0 > 0$ exists such that the functional J satisfies $(wC)_a$ for all $a \in [k_1 - \sigma_0, k_2 + \sigma_0]$, then J satisfies (wC^*) in the interval $[k_1, k_2]$.*

Proof. Arguing by contradiction, let $a \in [k_1, k_2]$ and assume that $\sigma > 0$ exists so that

$$J^{-1}([a-\sigma, a+\sigma]) \cap K_J = \emptyset \tag{2.1}$$

and $(u_n)_n \subset J^{-1}([a-\sigma, a+\sigma])$ can be found so that $\|dJ(u_n)\|_{X'}(1 + \|u_n\|_X) \to 0$. But from $a - \sigma \le J(u_n) \le a + \sigma$ for all $n \in \mathbb{N}$ it follows that, up to subsequences, $(J(u_n))_n$ converges to some $a^* \in [a-\sigma, a+\sigma]$. As we can choose $\sigma \le \sigma_0$, $(wC)_{a^*}$ holds and a^* is a critical level in contradiction with (2.1). ◻

Now, let us prove that, even if $(wC)_a$ is weaker that $(CPS)_a$, a deformation lemma still holds (see, e.g., [12, Lemma 3.2.3] for the $(CPS)_a$ case).

Lemma 2.3 (Deformation Lemma)**.** *Let $[k_1, k_2]$, with $k_1 \le k_2$, be an interval such that*

$$J^{-1}([k_1, k_2]) \cap K_J = \emptyset \tag{2.2}$$

and J satisfies $(wC)_a$ at each level $a \in [k_1, k_2]$.
Then, fixed any $\bar{\varepsilon} > 0$, there exists $\varepsilon > 0$ such that $2\varepsilon < \bar{\varepsilon}$ and $J^{k_1-\varepsilon}$ is a deformation retract of $J^{k_2+\varepsilon}$, i.e. $h \in C(X, X)$ exists such that:
(h_1) $h(J^{k_2+\varepsilon}) \subset J^{k_1-\varepsilon}$,
(h_2) $h(u) = u$ for all $u \notin J^{-1}([k_1 - 2\varepsilon, k_2 + 2\varepsilon])$,
where for each $k \in \mathbb{R}$ it is $J^k = \{u \in X : J(u) \leq k\}$.
In particular, if J is even then h can be chosen odd.

Proof. Firstly, we claim that, as J satisfies $(wC)_a$ in $[k_1, k_2]$, then there exist $\varepsilon > 0$, $2\varepsilon < \bar{\varepsilon}$, and $\alpha > 0$ so that

$$J^{-1}([k_1 - 2\varepsilon, k_2 + 2\varepsilon]) \cap K_J = \emptyset, \tag{2.3}$$

$$\|dJ(u)\|_{X'}(1 + \|u\|_X) \geq \alpha \qquad \text{for all } u \in J^{-1}([k_1 - 2\varepsilon, k_2 + 2\varepsilon]). \tag{2.4}$$

In fact, if (2.3) or (2.4) do not hold for all ε small enough, a sequence $(u_n)_n$ can be found so that

$$k_1 - \frac{1}{n} \leq J(u_n) \leq k_2 + \frac{1}{n} \quad \text{for all } n \in \mathbb{N}, \quad \lim_{n\to+\infty} \|dJ(u_n)\|_{X'}(1 + \|u_n\|_X) = 0.$$

Hence, up to subsequences, $(J(u_n))_n$ converges to some $a \in [k_1, k_2]$; thus, being $(u_n)_n$ a $(CPS)_a$–sequence, from $(wC)_a$ it follows that a is a critical level in contradiction with (2.2).
On the other hand, as J is C^1 in X, there exists $V : X \setminus K_J \to X$ pseudogradient vector field of J, odd if J is even (see [16, Chapter II]), and we can define $\tilde{V} : X \to X$ such that

$$\tilde{V}(u) = \begin{cases} -\chi(u)\dfrac{V(u)}{\|V(u)\|^2} & \text{if } u \in J^{-1}([k_1 - 2\varepsilon, k_2 + 2\varepsilon]), \\ 0 & \text{otherwise,} \end{cases}$$

where $\chi : X \to [0, 1]$ is a locally Lipschitz continuous function (even if so is J) such that

$$\chi(u) = \begin{cases} 0 & \text{if } u \notin J^{-1}([k_1 - 2\varepsilon, k_2 + 2\varepsilon]), \\ 1 & \text{if } u \in J^{-1}([k_1 - \varepsilon, k_2 + \varepsilon]). \end{cases}$$

From the properties of pseudogradient vector fields, we have that

$$\min\{1, \|dJ(u)\|_{X'}\} \leq \|V(u)\|_X < 2\min\{1, \|dJ(u)\|_{X'}\}; \tag{2.5}$$

hence, from (2.3) it follows that for all $u \in J^{-1}([k_1 - 2\varepsilon, k_2 + 2\varepsilon])$ $V(u)$ is well defined and $\|V(u)\|_X \neq 0$.
Furthermore, (2.4) implies that for all $R > 0$ a positive constant $b_R > 0$ exists so that

$$u \in J^{-1}([k_1 - 2\varepsilon, k_2 + 2\varepsilon]),\ \|u\|_X \leq R \qquad \Longrightarrow \qquad \|dJ(u)\|_{X'} \geq b_R,$$

whence, (2.5) implies $\|V(u)\|_X \geq b_R^*$ for some $b_R^* > 0$.
Now, essentially applying classical arguments as developed in [3, Theorem 1.3], we prove that $\tilde{V}$ is locally Lipschitz continuous and sublinear, i.e. γ_1, $\gamma_2 > 0$ exist so that $\|\tilde{V}(u)\|_X \leq \gamma_1 + \gamma_2\|u\|_X$ for all $u \in X$.
Whence, there exists a unique continuous function $\eta : \mathbb{R} \times X \to X$ which solves the Cauchy problem

$$\begin{cases} \frac{\partial\eta}{\partial t}(t; u) = \tilde{V}(\eta(t; u)) & \text{if } t \in \mathbb{R}, \\ \eta(0; u) = u, \end{cases} \tag{2.6}$$

so that $\eta(t;u) \equiv u$ for all $u \notin J^{-1}([k_1 - 2\varepsilon, k_2 + 2\varepsilon])$ and $\eta(t;\cdot)$ is odd if J is even. From the properties of pseudogradient vector fields and some previous estimates, we have

$$dJ(u)[\tilde{V}(u)] \ \le\ -\ \frac{1}{4}\ \chi(u) \quad \text{for all } u \in X. \tag{2.7}$$

Thus, for all $(t,u) \in \mathbb{R} \times X$, (2.6) and (2.7) imply

$$\frac{d}{dt} J(\eta(t;u)) \ =\ dJ(u)[\tilde{V}(\eta(t;u))] \ \le\ -\ \frac{1}{4}\ \chi(\eta(t;u)). \tag{2.8}$$

Whence,

$$t \in \mathbb{R} \longmapsto J(\eta(t;u)) \quad \text{is decreasing for each } u \in X. \tag{2.9}$$

Now, let $u \in J^{-1}([k_1 - \varepsilon, k_2 + \varepsilon])$ be fixed. We claim that a constant $T(u) > 0$ exists such that

$$\eta(T(u);u) \in J^{k_1 - \varepsilon}.$$

In fact, if $\tau > 0$ is such that $\eta(t;u) \in J^{-1}([k_1 - \varepsilon, k_2 + \varepsilon])$ for all $t \in [0,\tau]$, being $\chi(\eta(t;u)) = 1$ in $[0,\tau]$ from (2.8) it follows

$$k_1 - \varepsilon \ \le\ J(\eta(\tau;u)) \ \le\ J(u) - \frac{\tau}{4},$$

which implies

$$\sup\{\tau > 0 :\ \eta(t;u) \in J^{-1}([k_1 - \varepsilon, k_2 + \varepsilon]) \text{ for all } t \in [0,\tau]\} \ \le\ 4(k_2 - k_1 + 2\varepsilon).$$

Thus, from (2.9), $T(u) \le 4(k_2 - k_1 + 2\varepsilon)$ exists such that $J(\eta(T(u);u)) \le k_1 - \varepsilon$. At last, if $T = 4(k_2 - k_1 + 2\varepsilon)$, from (2.9) and the previous arguments the map $h : u \in X \mapsto h(u) := \eta(T;u) \in X$ satisfies the required properties. □

In particular, the previous lemma applies if the interval $[k_1, k_2]$ is reduced to a single point.

Corollary 2.4. *Let $\beta \in \mathbb{R}$ be a regular value of J. Then, if J satisfies $(wC)_\beta$, fixed any $\bar{\varepsilon} > 0$, there exists $\varepsilon > 0$ such that $2\varepsilon < \bar{\varepsilon}$ and $J^{\beta-\varepsilon}$ is a deformation retract of $J^{\beta+\varepsilon}$. In particular, if J is even then the deformation retract can be chosen odd.*

Remark 2.5. In the hypotheses of Lemma 2.3, taking $\eta : \mathbb{R} \times X \to X$ and $T > 0$ defined as in its proof, we have that the continuous function

$$\tilde{\eta} : (s;u) \in [0,1] \times X \mapsto \tilde{\eta}(s;u) := \eta(sT;u) \in X$$

satisfies the following properties:

(*i*) $\tilde{\eta}(0;u) = u$ for all $u \in X$;
(*ii*) $\tilde{\eta}(s;u) \equiv u$ for all $s \in [0,1]$ if $u \notin J^{-1}([k_1 - 2\varepsilon, k_2 + 2\varepsilon])$;
(*iii*) $\tilde{\eta}(1; J^{k_2+\varepsilon}) \subset J^{k_1-\varepsilon}$;
(*iv*) $J(\tilde{\eta}(s;u)) \le J(u)$ for all $s \in [0,1]$, $u \in X$;
(*v*) $\tilde{\eta}$ is odd if J is even.

Now, we can apply the previous Deformation Lemma in the proofs of our main abstract results.

Proof of Theorem 1.6. If β is not attained, from Corollary 2.4 it follows that $\varepsilon > 0$ and $h \in C(X,X)$ exist such that $h(J^{\beta+\varepsilon}) \subset J^{\beta-\varepsilon}$. But $J^{\beta+\varepsilon} \neq \emptyset$ while $J^{\beta-\varepsilon} = \emptyset$: a contradiction. □

Proof of Theorem 1.7. Define

$$\beta = \inf_{\xi\in\Gamma} \sup_{s\in[0,1]} J(\xi(s)),$$

with $\Gamma = \{\xi \in C([0,1], X) : \xi(0) = 0,\ \xi(1) = e\}$.
Firtly, we claim that

$$\beta \geq \varrho.$$

In fact, taking any $\xi \in \Gamma$, from $X \hookrightarrow W$ it follows $\xi \in C([0,1], W)$ (W equipped with $\|\cdot\|_W$). Whence,

$$g : s \in [0,1] \longmapsto \|\xi(s)\|_W \in \mathbb{R}$$

is continuous and such that $g(0) = 0 < r$, $g(1) = \|e\|_W > r$; thus, $\bar{s} \in]0,1[$ exists such that $\|\xi(\bar{s})\|_W = r$, and $J(\xi(\bar{s})) \geq \varrho$ implies $\sup_{s\in[0,1]} J(\xi(s)) \geq \varrho$.
Now, we claim that β is a critical level.
In fact, arguing by contradiction, if β is a regular value, as $(wC)_\beta$ holds, taking $\bar{\varepsilon} = \min\{\varrho, \varrho - J(e)\} > 0$ from Corollary 2.4 there exist $\varepsilon > 0$ and $h \in C(X, X)$ so that $2\varepsilon < \bar{\varepsilon}$ and (h_1), (h_2) hold. Thus, $J(0) = 0 < \beta - 2\varepsilon$ and $J(e) < \beta - 2\varepsilon$ imply $h(0) = 0$, $h(e) = e$.
On the other hand, $\xi_\varepsilon \in \Gamma$ exists such that

$$\sup_{s\in[0,1]} J(\xi_\varepsilon(s)) \ < \ \beta + \varepsilon. \tag{2.10}$$

Thus, if we define $\xi^*(s) = h(\xi_\varepsilon(s))$, it is easy to prove that $\xi^* \in \Gamma$, too. Furthermore, (h_1) and (2.10) imply

$$\sup_{s\in[0,1]} J(\xi^*(s)) \ < \ \beta - \varepsilon$$

in contradiction with the definition of β. □

Now, we want to prove the multiplicity results in the symmetric case. To this aim, some topological information needs.

Lemma 2.6 (Intersection Lemma)**.** *Assume that a Banach space $(W, \|\cdot\|_W)$ exists such that $X \hookrightarrow W$ continuously. Moreover, let E and Z be two closed subspaces of X such that $E + Z = X$ and* $\mathrm{codim} Z < \dim E < +\infty$.
Fixed any r, $R > 0$ and defined $S_r^W = \{u \in X : \|u\|_W = r\}$,

$$\Gamma^* = \{h \in C(X, X) : \ h \text{ is odd}, \quad h(u) = u \text{ for all } u \in E \text{ and } \|u\|_X \geq R\}, \tag{2.11}$$

then

$$h(E) \cap S_r^W \cap Z \neq \emptyset \qquad \text{for all } h \in \Gamma^*. \tag{2.12}$$

Proof. Fixing $h \in \Gamma^*$, for simplicity denote $Q_r = h(E) \cap S_r^W \cap Z$. We claim that

$$\gamma(Q_r) \geq \dim E - \mathrm{codim} Z \geq 1, \tag{2.13}$$

where γ is the Krasnoselskii genus (see [16, Section 5 in Chapter II]; hence (2.12) holds.
In order to prove (2.13), firstly let us point out that Q_r is symmetric with respect to the origin but $0 \notin Q_r$ (from the hypotheses). Moreover, Q_r is compact in X with $\|\cdot\|_X$. In fact, if $B_R^X = \{u \in X : \|u\|_X \leq R\}$, we have $E = (E \cap B_R^X) \cup (E \setminus B_R^X)$, with $E \cap B_R^X$ compact (as $\dim E < +\infty$ and B_R^X is closed and bounded in X) and $h(E \setminus B_R^X) = E \setminus B_R^X$ (by the definition of Γ^*). Hence, Q_r is compact as $Q_r = (h(E \cap B_R^X) \cap S_r^W \cap Z) \cup ((E \setminus B_R^X) \cap S_r^W \cap Z)$ with $h(E \cap B_R^X) \cap S_r^W \cap Z$ compact (as closed subset of the compact set $h(E \cap B_R^X)$) and $(E \setminus B_R^X) \cap S_r^W \cap Z$

compact (as subset of E where $\|\cdot\|_X$ and $\|\cdot\|_W$ are equivalent).
Whence, the estimate in (2.13) holds working as in the proof of [16, Lemma 6.4 in Chapter II]. □

Proof of Theorem 1.8. Now, define

$$\beta = \inf_{h\in\Gamma^*} \sup_{u\in E_\varrho} J(h(u)),$$

with Γ^* defined in (2.11) with R as in $(A_\varrho)(ii)$. From Lemma 2.6 it follows $\beta \geq \varrho$ while, being the identity map in Γ^*, it is also $\beta \leq \varrho_1$.
We claim that β is a critical level.
In fact, arguing by contradiction, if β is a regular value, as $(wC)_\beta$ holds, taking $\bar\varepsilon = \varrho > 0$ from Corollary 2.4 there exist $\varepsilon > 0$ and $h \in C(X,X)$ so that $2\varepsilon < \bar\varepsilon$ and (h_1), (h_2) hold and, as J is even, h can be chosen odd. Hence, from $(A_\varrho)(ii)$ it follows $h \in \Gamma^*$.
On the other hand, $h_\varepsilon \in \Gamma^*$ exists such that

$$\sup_{u\in E_\varrho} J(h_\varepsilon(u)) \ < \ \beta + \varepsilon. \tag{2.14}$$

Thus, if we define $h^*(u) = h(h_\varepsilon(u))$, it is easy to prove that $h^* \in \Gamma^*$, too. Furthermore, (h_1) and (2.14) imply

$$\sup_{u\in E_\varrho} J(h^*(u)) \ < \ \beta - \varepsilon$$

in contradiction with the definition of β. □

3. **Some applications.** Let $F : Y \to \mathbb{R}$ be defined as in (1.2), with Y as in (1.4), where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 3)$, $A = A(x,t,\xi)$ is a Carathéodory function on $\Omega\times\mathbb{R}\times\mathbb{R}^N$ such that the partial derivatives $A_t(x,t,\xi) = \frac{\partial A}{\partial t}(x,t,\xi)$ and $a(x,t,\xi) = (\frac{\partial A}{\partial \xi_1}(x,t,\xi),\dots,\frac{\partial A}{\partial \xi_N}(x,t,\xi))$ exist for almost all $x \in \Omega$ and all $(t,\xi_1,\dots,\xi_N) \in \mathbb{R}\times\mathbb{R}^N$ and are Carathéodory functions; furthermore, let $G(x,t) = \int_0^t g(x,s)ds$ with $g = g(x,t)$ a Carathéodory function on $\Omega\times\mathbb{R}$.

As already remarked in the introduction, here our aim is overcome both the lack of regularity of F in $W_0^{1,p}(\Omega)$ and some difficulties in the compactness of the Palais–Smale sequences by applying the abstract setting introduced in the first part of this paper to the functional F in Y. Thus, we obtain both existence and multiplicity results (see [5, 6, 7]).

In order to outline such statements, throughout this section we assume that there exist $1 < p < \theta$, some positive continuous functions Φ_1, Φ_2, ϕ_1, $\phi_2 : \mathbb{R}\to\mathbb{R}$, $R \geq 1$ and some positive constants λ, η_1, η_2, μ_1, $\mu_2 > 0$ such that the following estimates hold:

$$|A_t(x,t,\xi)| \ \leq \ \Phi_1(t) + \phi_1(t)\ |\xi|^p \quad \text{a.e. in } \Omega, \text{ for all } (t,\xi)\in\mathbb{R}\times\mathbb{R}^N;$$
$$|a(x,t,\xi)| \ \leq \ \Phi_2(t) + \phi_2(t)\ |\xi|^{p-1} \quad \text{a.e. in } \Omega, \text{ for all } (t,\xi)\in\mathbb{R}\times\mathbb{R}^N;$$
$$a(x,t,\xi)\cdot\xi \ \geq \ \lambda|\xi|^p \quad \text{a.e. in } \Omega, \text{ for all } (t,\xi)\in\mathbb{R}\times\mathbb{R}^N;$$
$$A(x,t,\xi) \ \leq \ \eta_1 a(x,t,\xi)\cdot\xi \quad \text{a.e. in } \Omega \text{ if } |(t,\xi)| \geq R;$$
$$\sup_{|(t,\xi)|\leq R} |A(x,t,\xi)| \ \leq \ \eta_2 \quad \text{a.e. in } \Omega;$$
$$a(x,t,\xi)\cdot\xi + A_t(x,t,\xi)t \ \geq \ \mu_1 a(x,t,\xi)\cdot\xi \quad \text{a.e. in } \Omega \text{ if } |(t,\xi)| \geq R;$$
$$\theta A(x,t,\xi) - a(x,t,\xi)\cdot\xi - A_t(x,t,\xi)t \geq \mu_2 a(x,t,\xi)\cdot\xi \text{ a.e. in } \Omega \text{ if } |(t,\xi)| \geq R;$$
$$[a(x,t,\xi) - a(x,t,\xi^*)]\cdot[\xi-\xi^*] > 0 \text{ a.e. in } \Omega, \text{ for all } t\in\mathbb{R},\ \xi,\ \xi^*\in\mathbb{R}^N,\ \xi\neq\xi^*.$$

On the other hand, let us assume that there are δ_1, $\delta_2 > 0$ and $\theta \leq q < p^*$, where $p^* = Np/(N-p)$ if $p < N$ while $p^* = +\infty$ otherwise, such that the function G and its derivative g satisfy the following conditions:

$$|g(x,t)| \leq \delta_1 + \delta_2\ |t|^{q-1} \quad \text{a.e. in } \Omega, \text{ for all } t \in \mathbb{R};$$
$$0 < G(x,t) \leq \frac{1}{\theta}\ g(x,t)\ t \qquad \text{a.e. in } \Omega, \text{ if } |t| \geq R.$$

Remark 3.1. The previuos hypotheses imply some "good" growth estimates both on $A(x,t,\xi)$ and on $G(x,t)$. More precisely, there exist constants α_1^*, $\alpha_2^* > 0$ such that a.e. in Ω, for all $(t,\xi) \in \mathbb{R} \times \mathbb{R}^N$, we have

$$|A(x,t,\xi)| \leq\ \alpha_1^*(1+|t|^{\theta-\frac{\mu_2}{\eta_1}})\ +\alpha_2^*(1+|t|^{\theta-\frac{\mu_2}{\eta_1}-p})|\xi|^p,$$

where, without loss of generality, we can take $\theta - \frac{\mu_2}{\eta_1} - p > 0$ (for the proof, see [6, Lemma 6.5]). Furthermore, a function $\alpha_3^* \in L^\infty(\Omega)$, $\alpha_3^*(x) > 0$ a.e. in Ω, and a positive constant $\alpha_4^* > 0$ exist such that

$$G(x,t)\ \geq\ \alpha_3^*(x)\ |t|^\theta - \alpha_4^* \qquad \text{a.e. in } \Omega, \text{ for all } t \in \mathbb{R}.$$

Example 3.2. Particular examples of the function $A(x,t,\xi)$ which satisfy the previous conditions can be easily found. In fact, such conditions are trivially satisfied by $A(x,t,\xi) \equiv \frac{1}{p}|\xi|^p$, $1 < p < \theta$, i.e., if we consider the functional related to the classical p–Laplacian equation $-\Delta_p u\ =\ g(x,u)$.
Another example is $A(x,t,\xi) = (1+|t|^r)\ |\xi|^p$, when p, $r > 1$ are such that $p+r < \theta$.

Firstly, let us point out that in these hypotheses F is C^1 in Y with Euler–Lagrange equation

$$-\text{div}(a(x,u,\nabla u)) + A_t(x,u,\nabla u) = g(x,u),$$

(see [6, Corollary 3.2]).

Furthermore, it can be proved that F satisfies $(wC)_a$ in Y for all $a \in \mathbb{R}$ (see [6, Proposition 4.6]).

Then, by applying Theorem 1.7 the following result can be stated (for a complete proof, see [6, Theorem 6.1]):

Theorem 3.3. *Let λ_1 be the first eigenvalue of $-\Delta_p$ in $W_0^{1,p}(\Omega)$, i.e.*

$$\lambda_1 = \inf_{u \in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega |u|^p dx}.$$

If there exists $\mu_3 > 0$ such that

$$A(x,t,\xi) \geq \mu_3 |\xi|^p \quad \textit{a.e. in } \Omega, \textit{ for all } (t,\xi) \in \mathbb{R} \times \mathbb{R}^N,$$
$$\limsup_{t\to 0} \frac{g(x,t)}{|t|^{p-2}t}\ <\ p\ \mu_3\ \lambda_1 \quad \textit{uniformly with respect to } x \in \Omega,$$

then the functional F possesses at least one nontrivial critical point in Y.

On the other hand, as a good finite dimensional decomposition of Y can be obtained so that the "geometric" hypothesis (A_{ϱ_n}) holds for a sequence $(\varrho_n)_n$, $\varrho_n \nearrow +\infty$, and infinitely many subspaces $(E_n)_n$, $(Z_n)_n$ (see [6, Section 5]) we can apply the multiplicity result stated in Corollary 1.9 to prove the following theorem (for a complete proof, see [6, Theorem 6.3]):

Theorem 3.4. *If $A(x,\cdot,\cdot)$ is even and $g(x,\cdot)$ is odd for a.e. $x \in \Omega$, then the functional F possesses an unbounded sequence of critical values in Y.*

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