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NONLOCAL PROBLEMS FOR PARABOLIC INCLUSIONS

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Abstract. In this paper we investigate a class of parabolic inclusions with a nonlocal condition of integral type. We provide sufficient conditions that guarantee the existence of at least one solution. Our technique is based on Green's function for linear parabolic partial differential equations and fixed point theorems for multivalued maps.

1. **Introduction.** Let Ω be a an open bounded domain in $\mathbb{R}^N$, $N \geq 2$, with a smooth boundary $\partial\Omega$. We denote the norm (usually the Euclidean norm) of $x \in \Omega$ by $\|x\|$. Let T be a positive real number. Set $D = \Omega \times (0, T)$ and $\Gamma = \partial\Omega \times [0, T]$. Our objective is to investigate the existence of solutions of the following parabolic problem with a multivalued right-hand side and nonlocal initial condition

$$D_t u + Lu \in F(x, t, u), \qquad (x, t) \in D, \tag{1}$$

$$u(x, t) = 0, \qquad (x, t) \in \Gamma, \tag{2}$$

$$u(x, 0) = \int_0^T g(x, t, u(x, t))dt, \ x \in \Omega, \tag{3}$$

where L is an elliptic operator given by

$$Lu = -\sum_{i,j=1}^{N} a_{ij}(x, t) D_i D_j u + c(x, t)u.$$

Parabolic problems with discontinuous nonlinearities arise as simplified models in the description of porous medium combustion, see for instance [14], chemical reactor theory [15]. Also, best response dynamics arising in game theory can be modeled by a parabolic equation with a discontinuous right hand side [10] and [17] for details and references. Parabolic problems with discontinuous nonlinearities have been also investigated in the papers [6], [5], [30], [31] and [32]. On the other hand parabolic problems with integral boundary conditions appear in the modeling of concrete problems, such as heat conduction [4] and [19], thermoelasticity [9]. Several papers have been devoted to the study of parabolic problems with integral conditions [8], [27] and [34]. Many authors have dealt with parabolic problems with continuous nonlinearities and nonlocal conditions of the form $u(x, 0) + \sum_{i=0}^{m} \beta_i(x) u(x, t_i) = \psi(x)$ for $x \in \Omega$. See for instance [3] , [13], [20], [24] and [25]. We refer to [7] for

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details and references concerning the linear problem with the above type of nonlocal conditions. A good account on numerical treatment of parabolic problems with integral conditions can be found in [11].

In this paper we consider a nonlocal problem for a class of nonlinear parabolic equations with a multivalued right hand side. We shall convert Problem (1), (2), (3), to an integral inclusion using the properties of Green's function corresponding to the linear problem. We, then, provide sufficient conditions on the data that will guarantee that the problem under consideration has at least one solution. Our approach is based on fixed point theorems for suitable multivalued operators.

The outline of the paper is as follows. In section 2 we introduce notations and preliminary results which will be used in the paper. In section 3, we shall recall the main properties of upper semicontinuous multivalued maps. We state and prove our main results in section 4.

2. **Preliminaries.** In this section we introduce some notations and preliminary results which will be used in the paper. Let Ω be a an open bounded domain in $\mathbb{R}^N$, $N \geq 2$, with a smooth boundary $\partial\Omega$. Let T be a positive real number. Define $D = \Omega \times (0,T)$ and $\Gamma = \partial\Omega \times [0,T]$. For $u : D \to \mathbb{R}$ we denote its partial derivatives (when they exists) by $D_t u = \partial u/\partial t$, $D_i u = \partial u/\partial x_i$, $D_i D_j u = \partial^2 u/\partial x_i \partial x_j$, $i,j = 1,...,N$.

Let $X = C(\overline{D})$ denote the Banach space of continuous functions $u : \overline{D} \to \mathbb{R}$, endowed with the norm

$$|u|_0 = \sup\{|u(x,t)|\,;\ (x,t) \in \overline{D}\}.$$

$u \in C^{2,1}(D)$ if $u(.,t) \in C^2(\Omega), t \in (0,T),\ u(x,.) \in C^1(0,T), x \in \Omega$. $u \in C(D)$ is called Hö lder continuous of order $\alpha \in (0,1]$ if

$$H_\alpha(u) = \sup\{\frac{|u(x,t) - u(\xi,\tau)|}{\left(\|x-\xi\|^2 + |t-\tau|^2\right)^{\alpha/2}}; (x,t), (\xi,\tau) \in D\} < +\infty.$$

In this case we write $u \in C^\alpha(D)$ and we define its norm by

$$|u|_\alpha = |u|_0 + H_\alpha(u).$$

If $\alpha = 1$, u is called Lipschitz continuous, and we write $u \in Lip(D)$. Note that the natural injection $i : C^\alpha(D) \to C(D)$ is continuous. For $1 \leq p < +\infty$, we say that $u : D \to \mathbb{R}$ is in $L^p(D)$ if u is measurable and $\int_D |u(x,t)|^p\, dxdt < +\infty$, in which case we define its norm by

$$|u|_{L^p} = \left(\int_D |u(x,t)|^p\, dxdt\right)^{1/p}.$$

Consider the linear nonhomogeneous problem

$$D_t u + Lu = f(x,t), \qquad (x,t) \in D, \tag{4}$$

$$u(x,t) = 0, \qquad (x,t) \in \Gamma, \tag{5}$$

with the following nonlocal boundary condition

$$u(x,0) = \int_0^T g(x,t,u(x,t))dt \quad x \in \Omega. \tag{6}$$

We shall assume throughout this paper that the functions $a_{ij}, c : D \to \mathbb{R}$ are Hölder continuous, $a_{ij} = a_{ji}$ and moreover, there exist positive numbers λ_0, λ_1 such that

$$\lambda_0 \|\xi\|^2 \leq \sum_{i,j=1}^{N} a_{ij}(x,t)\xi_i\xi_j \leq \lambda_1 \ \|\xi\|^2, \ \forall \xi \in \mathbb{R}^N \text{ and } \forall (x,t) \in D.$$

For the problem (4), (5) together with initial condition

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{7}$$

where $u_0 \in C(\Omega;\mathbb{R})$, we have the following classical result

Lemma 2.1. *([16], [22], [23], [29]) Assume that the functions f and u_0 are Hölder continuous. Then problem (4), (5), (7) has a unique solution $u \in C^{2,1}(D) \cap C(\overline{D})$, given by*

$$u(x,t) = \int_\Omega G(x,t;y,0)\ u_0(y)\,dy + \int_0^t \int_\Omega G(x,t;y,s)\ f(y,s)\,dyds, \quad (x,t) \in D, \tag{8}$$

where $G(x,t;y,s)$, is Green's function corresponding to the linear homogeneous problem.

It follows from the lemma that the functions ϕ and ψ, defined respectively by $\phi(x,t) = \int_0^t \int_\Omega G(x,t;y,s)dyds$ and $\psi(x,t) = \int_\Omega G(x,t;y,0)\ dy$, are continuous on D. Let $\phi_0 = \sup_{(x,t)\in D} \int_0^t \int_\Omega G(x,t;y,s)dyds$.

3. **Multivalued Functions.**

We, now, introduce some useful definitions and properties from set-valued analysis. For complete details on multivalued maps we refer the interested reader to the books [1], [2], [12] and [18].

Let $(X, |.|_X)$ and $(Y, |\cdot|_Y)$ be Banach spaces. We shall denote the set of all subsets of X having property ℓ by $P_\ell(X)$. For instance, $U \in P_{cl}(X)$ means U closed in X; when $\ell = b$ we have the bounded subsets of X, $\ell = cv$ for convex subsets, $\ell = cp$ for compact subsets and $\ell = cp, cv$ for compact and convex subsets. The domain of a multivalued map $F : X \to 2^Y$ is the set $domF = \{z \in X;\ R(z) \neq \emptyset\}$. F is convex (closed) valued if $F(z)$ is convex (closed) for each $z \in X$. F is bounded on bounded sets if $F(A) = \cup_{z\in A} F(z)$ is bounded in Y for all $A \in P_b(X)$ (i.e. $\sup_{z\in A}\{\sup\{|y|; y \in F(z)\}\} < \infty$). F is called upper semicontinuous (usc) on X if for each $z \in X$ the set $F(z) \in P_{cl}(Y)$ is nonempty, and for each open subset Y_0 of Y containing $F(z)$, there exists an open neighborhood Π of z such that $F(\Pi) \subset Y_0$. In terms of sequences, F is usc if for each sequence $(z_n) \subset X$, $z_n \to z_0$, and B a closed subset of Y such that $F(z_n) \cap B \neq \emptyset$ then $F(z_0) \cap B \neq \emptyset$. The set-valued map F is called completely continuous if $F(A)$ is relatively compact in Y for every $A \in P_b(X)$. If F is completely continuous with nonempty compact values, then F is usc if and only if F has a closed graph (i.e. $z_n \to z$, $w_n \to w$, $w_n \in F(z_n)$ $\Rightarrow w \in F(z)$). When $X \subset Y$ then F has a fixed point if there exists $z \in X$ such $z \in F(z)$. Now, when $X = \mathbb{R}^{N+1}$ and $Y = \mathbb{R}$, we call a multivalued map $F : \overline{D} \to P_{cl}(\mathbb{R})$ measurable if for every $\theta \in \mathbb{R}$, the function $\varsigma : \overline{D} \to \mathbb{R}$ defined by $\varsigma(v) = dist(\theta, F(v)) = \inf\{|\theta - z|; z \in F(v)\}$ is measurable. Finally, if $F : D \times \mathbb{R} \to 2^{\mathbb{R}}$, we let $|F(x,t,u)| := \sup\{|v|; v(x,t) \in F(x,t,u)\}$.

Definition 3.1. A multivalued map $F : D \times \mathbb{R} \to 2^{\mathbb{R}}$ is called an L^2- Carathéodory multifunction if

(i) $(x,t) \mapsto F(x,t,u)$ is measurable for each $u \in \mathbb{R}$,
(ii) $u \mapsto F(x,t,u)$ is upper semicontinuous for almost all $(x,t) \in D$,
(iii) for each $r > 0$ there exists $\omega_r \in L^2(D)$ such that $|F(x,t,u)| \leq \omega_r(x,t)$ a.e. on D whenever $|u| \leq r$.

Definition 3.2. The set of L^2-selections of F is the set $S^2_{F,u} = \{ v \in L^2(D); v\,(x,t) \in F(x,t,u)\}$ with the following properties
(i) $S^2_{F,u} \neq \emptyset$ if $\inf\{|v|\,;\ v \in F(.,.,u\}$ is in $L^2(D)$. See [33, Theorem 5.10].
(ii) It is closed (convex) if and only if $F(x,t,u)$ has closed (convex) values.
(iii) It is bounded if $\sup\{|v|\,; v\,(x,t) \in F(x,t,u)\} \in L^2(D)$.

Using the properties of Green's function we get the following results.

Lemma 3.3. *([28, Lemma 3.1])Assume the single valued map $h \in C(D)$. Then the operator $\gamma : C(D) \to C(D)$ defined by*
$\gamma(f)(x,t) = h(x,t) + \int_0^t \int_\Omega G(x,t;y,s)f(y,s)dyds$ is continuous.

Lemma 3.4. *([28, Theorem 3.2])Assume $F : D \times \mathbb{R} \to 2^{\mathbb{R}}$ is an L^2- Carathéodory multifunction with nonempty, compact, convex values and $h \in C(D)$. Then the operator $\gamma \circ F$ is of usc type; i.e. is usc, completely continuous and has nonempty, compact, convex values.*

Theorem 3.5. *(Nonlinear alternative for multivalued maps [28, Theorem 2.5]). Let K be a convex subset of a Banach space E, $U \subseteq K$ be relatively open, and $p \in U$. Suppose F:$\overline{U} \to K$ is an usc compact multivalued operator with nonempty, compact, convex values. Then either*
(i) there is $u \in \overline{U}$ such that $u \in F\,(u)$; or
(ii) there is $u \in \partial U$ and a $\lambda \in (0,1)$ such that $u \in \lambda F\,(u) + (1-\lambda)\,p$.

Definition 3.6. Let (Z,d) be a metric space and let A, B be two nonempty subsets of Z. The Hausdorff distance between A and B is defined by

$$d_H(A,B) = \max_{a\in A}(\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b))$$

where $d(a,B) = \inf\{d(a,b); b \in B\}$ and $d(A,b) = \inf\{d(a,b); a \in A\}$.

Then one can show that $(P_{b,cl}(Z),\ d_H)$ is a metric space.

Definition 3.7. A multivalued operator $F : Z \to P_{cl}(Z)$ is called (i) $\delta-$Lipschitz if and only if there exists $\delta > 0$ such that $d_H\,(F(u), F(v)) \leq \delta\; d\,(u,v)$ for all $u, v \in Z$,
(ii) a contraction if and only if it is $\delta-$Lipschitz with $\delta < 1$.

Theorem 3.8. *([26]). Let (Z,d)be a complete metric space and $\Theta : Z \to P_{b,cl}(Z)$ be a multivalued contraction. Then Θ has a fixed point.*

Theorem 3.9. *(Bohnenblust-Karlin [12, Cor.11.3(e)]) Let X be a Banach space, V a nonempty subset of X, which is bounded, closed and convex. Suppose $\digamma : V \to 2^X\backslash\{\emptyset\}$ is upper semicontinuous, with closed, convex values, such that $\digamma\,(V) \subset V$ and $\overline{\digamma\,(V)}$ is compact. Then $\digamma$ has a fixed point.*

4. **Existence Results.** Before stating and proving our main results we introduce the notion of strong solutions of problem (1), (2), (3).

Definition 4.1. $u \in C^{2,1}(D) \cap C(\overline{D})$ is called a strong solution of (1), (2), (3) if there exists a single-valued function $f \in Lip(D)$, i.e. $|f(x,t) - f(y,s)| \leq \ell_f\,(\|x-y\| + |t-s|)$ such that $f(x,t) \in F(x,t,u(x,t))$ and (4), (5), (6) hold.

Moreover, the integral representation (8) shows that u is a solution of problem (4), (5), (6) if and only if u satisfies

$$\begin{aligned} u(x,t) = &\int_{\Omega} G(x,t;y,0) \ \int_0^T g(y,s,u(y,s))dsdy \\ &+ \int_0^t \int_{\Omega} G(x,t;y,s) \ f\left(y,s\right) dyds, (x,t) \in D. \end{aligned} \tag{9}$$

Theorem 4.2. *Let F:D $\times\mathbb{R} \to 2^{\mathbb{R}}$ be a multifunction with compact values. Suppose that the following conditions are satisfied,*

(H0) there exists $f \in Lip(D)$ such that $f(x,t) \in F(x,t,u(x,t))$,
(H1) there exists $\ell_0 \in C(D)$ such that $d_H\left(F(x,t,u), F(x,t,z)\right) \le \ell_0\left(x,t\right)\left|u-z\right|$, $a.e. \, (x,t) \in D$, $u, z \in \mathbb{R}$,
(H2) $g : D \times \mathbb{R} \to \mathbb{R}$ is continuous, $g(x,t,0) = 0$ for all $(x,t) \in D$ and there exist $\eta \in Lip(\Omega)$, $\omega \in L^1(0,T)$, $\sigma_0 \in C(\mathbb{R}_+)$ nondecreasing with $\sigma_0(r) < r$ such that $|g(x,t,u) - g(x,t,v)| \le \eta\left(x\right)\omega(t)\sigma_0\left(|u-v|\right)$,
(H3) $\phi_0 \left|\ell_0\right|_0 + (max_{(x,t)\in D} \int_{\Omega} G(x,t;y,0)\eta\left(y\right)dy)\left|\omega\right|_{L^1} < 1$.
Then problem (1), (2), (3) has a strong solution.

Remark 1. Condition (H0) is not very restrictive as shown by the following result.

Theorem 4.3. *([21]) Let F: $\mathbb{R} \to 2^{\mathbb{R}}$ be a Lipschitz multivalued map with nonempty, closed values. Then F has a Lipschitz selection.*

Proof. It follows from the representation (9) that u is a solution of problem (1), (2), (3) if and only if u is a fixed point of the multivalued operator

$$\Lambda : X \to 2^X,$$

defined by

$$\Lambda u = h_g + GN_F(u) \tag{10}$$

where

$$h_g(x,t) = \int_{\Omega} G(x,t;y,0) \ \int_0^T g(y,s,u(y,s))dsdy, \quad (x,t) \in D, \tag{10.a}$$

and

$$GN_F(u)(x,t) = \int_0^t \int_{\Omega} G(x,t;y,s) \ N_F(u)\left(y,s\right))dyds, \quad (x,t) \in D. \tag{10.b}$$

Notice that Λ is the sum of the single-valued operator $h_g \in C(D)$ and a multivalued operator GN_F, where N_F is the Nemitskii operator associated with the multifunction F.

We show that $\Lambda u \in P_{cl}(X)$ for any $u \in X$. For, let $(z_n)_{n\in\mathbb{N}} \subset X$, $z_n \in \Lambda u$, $z_n \to z$ in X. Then, $z \in X$ and there exists $f_n \in Lip(D)$ such that $f_n(x,t) \in F(x,t,u(x,t))$ and

$$z_n(x,t) = h_g(x,t) + \int_0^t \int_{\Omega} G(x,t;y,s) \ f_n(y,s)dyds, \quad (x,t) \in D.$$

Since F has closed and bounded values it follows that $(\ f_n)_{n\in\mathbb{N}}$ is bounded, and passing to subsequences if necessary, converges to some $f \in Lip(D)$, and $f(x,t) \in F(x,t,u(x,t))$. By the Lebesgue dominated convergence we get

$$z(x,t) = h_g(x,t) + \int_0^t \int_{\Omega} G(x,t;y,s) \ f(y,s)dyds, \quad (x,t) \in D,$$

which shows that $z \in \Lambda u$. Hence Λu is nonempty and closed. Also, we can easily show that Λ is bounded. Next, we show that Λ is a contraction. For, let $u_1, u_2 \in X$ and consider $z_i \in \Lambda u_i$, $i = 1, 2$. Then, there exist $h_i \in Lip(D)$, $i = 1, 2$ such that for every $(x,t) \in D$ and $i = 1, 2$

$$z_i(x,t) = \int_\Omega G(x,t;y,0) \int_0^T g(y,s,u_i(y,s))dsdy + \int_0^t \int_\Omega G(x,t;y,s)\, h_i(y,s)dyds, \quad .$$

Then

$$\begin{aligned}&z_1(x,t) - z_2(x,t) = \\ &\int_\Omega G(x,t;y,0) \int_0^T [g(y,s,u_1(y,s)) - g(y,s,u_2(y,s))]dsdy \\ &+ \int_0^t \int_\Omega G(x,t;y,s)\ [h_1(y,s) - h_2(y,s)]dyds, \quad (x,t) \in D.\end{aligned}$$

(H1) and (H2) yield

$$\begin{aligned}&|z_1(x,t) - z_2(x,t)| \leq \\ &\int_\Omega G(x,t;y,0)\eta(y) \int_0^T \omega(s)\sigma_0(|u_1(y,s) - u_2(y,s)|)\, dsdy \\ &+ \int_0^t \int_\Omega G(x,t;y,s)\ell_0(y,s)\ |u_1(y,s) - u_2(y,s)|\, dyds \leq \\ &\int_\Omega G(x,t;y,0)\eta(y)\, dy\, |\omega|_{L^1}\, \sigma_0(|u_1 - u_2|_0) \\ &+ \int_0^T \int_\Omega G(x,t;y,s)\ \ell_0(y,s)dyds\, |u_1 - u_2|_0\end{aligned}$$

Hence

$$\begin{aligned}&|z_1(x,t) - z_2(x,t)| \leq \\ &\int_\Omega G(x,t;y,0)\eta(y)\, dy\, |\omega|_{L^1}\, \sigma_0(|u_1 - u_2|_0) + \phi_0\, |\ell_0|_0\, |u_1 - u_2|_0 \leq \\ &(\int_\Omega G(x,t;y,0)\eta(y)\, dy\, |\omega|_{L^1} + \phi_0\, |\ell_0|_0)\, |u_1 - u_2|_0\, .\end{aligned}$$

Interchanging the role of z_1 and z_2 we see that

$$d_H(\Lambda u_1, \Lambda u_2) \leq \delta\, |u_1 - u_2|_0\, ,$$

where

$$\delta := \max_{(x,t)\in D} (\int_\Omega G(x,t;y,0)\eta(y)\, dy)\, |\omega|_{L^1} + \phi_0\, |\ell_0|_0\, .$$

It follows from (H3) that Λ is a contraction. Theorem 3.8 implies that Λ has a fixed point u_0, which is a solution of problem (1), (2), (3). □

Remark. In what follows $C(\eta)$ denotes the generic constant $\max_{(x,t)\in D}$ $(\int_\Omega G(x,t;y,0)\eta(y)\, dy)$, which depends on a function $\eta \in Lip(\Omega)$.

Theorem 4.4. *Let $F : D \times \mathbb{R} \to 2^{\mathbb{R}}$ be an L^2-Carathéodory multifunction with nonempty, compact, convex values. Assume that, in addition to (H0), the following conditions are satisfied*

- (H4) there exist $M_g > 0$ and $\eta \in Lip(\Omega)$ such that $\int_0^T |g(y,s,u(y,s))|\, ds \leq M_g \eta(y)$ for all $y \in \Omega$,
- (H5) there exist $q \in C(D)$ and $\Psi : [0,\infty) \to (0,\infty)$ continuous and nondecreasing such that $|F(x,t,u)| \leq q(x,t)\Psi(|u|)$ for almost all $(x,t) \in D$ and $u \in \mathbb{R}$,
- (H6) $sup_{\rho\in[0,\infty)} \dfrac{\rho}{M_g C(\eta) + \phi_0\, |q|_0\, \Psi(\rho)} > 1.$

Then problem (1), (2), (3) has a strong solution.

Proof. Recall that u is a strong solution of (1), (2), (3) if and only if u is a fixed point of the multivalued operator Λ given by (10). Lemma 3.2 implies that Λ is of upper semi-continuous type. By (H6) there exists $M_0 > 0$ such that

$$\frac{M_0}{M_g C(\eta) + \phi_0\, |q|_0\, \Psi(M_0)} > 1.$$

Consider $U := \{u \in X;\ |u|_0 < M_0\}$. Then U is relatively open in $K = X = C(\overline{D})$. We shall apply Theorem 3.5 to the operator Λ, and show that the second alternative does not hold. Let $u \in X$ be a solution of

$$u(x,t) \in \lambda(h_g(x,t) + \int_0^t \int_\Omega G(x,t;y,s)\ F(y,s,u\,(y,s))dyds), \quad (x,t) \in D. \quad (11)$$

with $\lambda \in (0,1)$. From (11) and (H5) we obtain for each $(x,t) \in D$

$$\begin{aligned} |u(x,t)| &\leq |h_g(x,t)| + \int_0^t \int_\Omega G(x,t;y,s)\ q(y,s)\Psi\left(|u(y,s)|\right) dyds) \\ &\leq |h_g|_0 + \int_0^t \int_\Omega G(x,t;y,s) q(y,s) dyds) \Psi\left(|u|_0\right) \end{aligned}$$

Since $h_g(x,t) = \int_\Omega G(x,t;y,0)\ \int_0^T g(y,s,u(y,s))dsdy$, for each $(x,t) \in D$, (H4) implies

$$|h_g|_0 \leq M_g C(\eta). \quad (12)$$

Hence

$$|u|_0 \leq M_g C(\eta) + \phi_0 |q|_0 \Psi\left(|u|_0\right). \quad (13)$$

Suppose now that there exist $u \in \partial U$ and $\lambda \in (0,1)$ such that $u \in \lambda \Lambda u$. Then u satisfies (11) and $|u|_0 = M_0$. It follows from the condition on Ψ and (13) that

$$M_0 \leq M_g C(\eta) + \phi_0 |q|_0 \Psi\left(M_0\right).$$

This, obviously, contradicts the definition of M_0. Consequently, the first alternative in Theorem 3.5 holds; i.e. the multivalued operator Λ has a fixed point u. Therefore u is a solution of (1), (2), (3). □

Theorem 4.5. *Let $F : D \times \mathbb{R} \to 2^{\mathbb{R}}$ be an L^2-Carathéodory multifunction with nonempty, compact, convex values. Assume that, in addition to (H0) and (H4), the following conditions are satisfied*

(H7) there exists $\Phi : D \times \mathbb{R} \to \mathbb{R}_+$ a continuous function, nondecreasing with respect to its third argument such that
  (i) $limsup_{\rho \to \infty} \frac{1}{\rho}\left[M_g C(\eta) + \phi_0 \max_{(x,t)\in D} |\Phi(x,t,\rho)|\right] < 1$,
  (ii) $|F(x,t,u| \leq \Phi\left(x,t,|u|\right)$ for a.e. $(x,t) \in D$, $u \in \mathbb{R}$.

Then problem (1), (2), (3) has at least one solution.

Proof. First, we show that all possible solutions of our problem are a priori bounded, i.e. there exists $\rho^* > 0$ such that any solution u of the problem satisfies $|u|_0 \leq \rho^*$.

We have $u(x,t) \in h_g(x,t) + \int_0^t \int_\Omega G(x,t;y,s)\ F(y,s,u\,(y,s))dyds$, for $(x,t) \in D$, where h_g is given by (10.a).

Taking into account (12) we have by (H7) (ii)

$$|u(x,t)| \leq M_g C(\eta) + \int_0^t \int_\Omega G(x,t;y,s)\ \Phi(y,s,|u\,(y,s)|)dyds.$$

Hence

$$|u|_0 \leq M_g C(\eta) + \int_0^t \int_\Omega G(x,t;y,s)\ \Phi(y,s,|u|_0)dyds,$$

so that

$$|u|_0 \leq M_g C(\eta) + \phi_0 \max_{(x,t)\in D} |\Phi(x,t,|u|_0)|. \quad (14)$$

Let $\rho_0 := |u|_0$. Inequality (14) yields

$$1 \leq \frac{1}{\rho_0}\{M_g C(\eta) + \phi_0 \max_{(x,t)\in D} |\Phi(x,t,\rho_0)|\}. \tag{15}$$

It follows from (H7) (i) that there exists $\rho^* > 0$, independent of u, such that for all $\rho > \rho^*$

$$\frac{1}{\rho}\{M_g C(\eta) + \phi_0 \max_{(x,t)\in D} |\Phi(x,t,\rho)|\} < 1. \tag{16}$$

Comparing inequalities (15) and (16) we see that $\rho_0 \leq \rho^*$. Next, define a truncation function $\zeta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\zeta(\rho^*, \xi) = \begin{cases} 1 & 0 \leq \xi \leq \rho^*, \\ 2 - \frac{\xi}{\rho^*} & \rho^* \leq \xi \leq 2\rho^*, \\ 0 & \xi \geq 2\rho^*. \end{cases}$$

It is clear that ζ is continuous and $0 \leq \zeta(\rho^*, \xi) \leq 1$ for all $\xi \in \mathbb{R}_+$. Let

$$H(x,t,u) := \zeta(\rho^*, |u|)\, F(x,t,u).$$

Then H is a bounded L^2-Carathéodory multifunction and

$$|H(x,t,u)| \leq \zeta(\rho^*, |u|)\, |\Phi(x,t,|u|)| \text{ for all } (x,t) \in D, u \in \mathbb{R}.$$

Then there exists $\omega_H \in C(D)$ such that $|\varphi(x,t)| \leq \omega_H(x,t)$ for a.e. $(x,t) \in D$ and all φ such that $\varphi(x,t) \in H(x,t,u)$. In fact, we have $\omega_H(x,t) := \max\{|\Phi(x,t,|u|)|;\ (x,t) \in D, |u| \leq 2\rho^*\}$.

Consider the modified problem

$$\begin{cases} D_t u + Lu \in H(x,t,u), & (x,t) \in D, \\ u(x,t) = 0, & (x,t) \in \Gamma, \\ u(x,0) = \int_0^T g(x,s,u(x,s))ds, & x \in \Omega. \end{cases} \tag{17}$$

We show that (17) has at least one solution u satisfying the estimate $|u|_0 \leq \rho^*$.

The set $Y := \{u \in X; |u|_0 \leq M_g C(\eta) + \phi_0 |\omega_H|_0\}$ is nonempty, bounded, closed and convex. From the above results, we know that the solutions of (17) are fixed points of the multivalued operator $\Upsilon := h_g + GN_H$, where N_H is the Nemitskii operator of H. Moreover Υ is of u.s.c. type, maps Y into itself and $\overline{\Upsilon(Y)}$ is compact. It follows from Theorem 3.9 that Υ has a fixed point v, which is a solution of (17).

It remains to show that $|v|_0 \leq \rho^*$. Indeed, we have that $|H(x,t,u)| \leq \Phi_0(x,t,|u|) := \zeta(\rho^*, |u|)\, \Phi(x,t,|u|)$ for a.e. $(x,t) \in D$, $u \in \mathbb{R}$. It is easily seen that Φ_0 satisfies condition (H7). Hence, the first part of the proof shows that $|v|_0 \leq \rho^*$. But, for all u such that $|u|_0 \leq \rho^*$, the multivalued functions F and H coincide, and problem (17) reduces to our original problem. Therefore (1), (2), (3) has at least one solution. □

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