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INFINITELY MANY SOLUTIONS TO SUPERQUADRATIC PLANAR DIRAC-TYPE SYSTEMS

Alberto Boscaggin

SISSA-ISAS International School for Advanced Studies
Via Beirut, 2-4 - 34014 Trieste - Italy

Anna Capietto

Department of Mathematics - University of Torino
Via Carlo Alberto, 10 - 10123 Torino - Italy

Abstract. It is proved the existence of infinitely many solutions to a superquadratic Dirac-type boundary value problem of the form $\tau z = \nabla_z F(t, z)$, $y(0) = y(\pi) = 0$ ($z = (x, y) \in \mathbb{R}^2$). Solutions are distinguished by using the concept of rotation number. The proof is performed by a global bifurcation technique.

1. **Introduction.** In this paper we prove a multiplicity result for a nonlinear boundary value problem of the form

$$\begin{cases} \tau z = \nabla_z F(t, z) \\ y(0) = y(\pi) = 0 \end{cases} \qquad t \in [0, \pi], \quad z = (x, y) \in \mathbb{R}^2, \tag{1}$$

where, being

$$J = \begin{pmatrix} O & 1 \\ -1 & O \end{pmatrix}$$

the standard symplectic matrix,

$$\tau z = 2q(t)Jz' + q'(t)Jz + P(t)z. \tag{2}$$

The following assumptions on the differential operator τ will be assumed throughout the paper:

- $q \in C^1([0, \pi], \mathbb{R})$, with $q(t) > 0$ for every $t \in [0, \pi]$;
- $P \in C([0, \pi], \mathcal{L}(\mathbb{R}^2))$, with $P(t)$ symmetric for every $t \in [0, \pi]$.

Moreover, we suppose that

- $F \in C^2([0, \pi] \times \mathbb{R}^2, \mathbb{R})$ with $\nabla_z F(t, 0) = 0$.

Problems associated with the differential operator (2) are called of Dirac-type; they originate from the separation of the physically relevant Dirac operator and they have many applications. For complete references we refer to the book of J. Weidmann

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[14]; in particular, in [14] it is given a detailed study of the oscillation theory for linear systems associated to (2).
In our main results (Theorems 3.2 and 3.5) we shall prove the existence of infinitely many solutions, with prescribed nodal properties, of a superquadratic problem (cf. the convexity assumption $(H2)$).
The literature on boundary value problems for nonlinear Dirac-type systems is not wide. In [12],[13] J. Ward has given multiplicity results for some systems of first order equations based on bifurcation techniques. In [12], the author considered the system $z' = \lambda Jz$; in [13] the more general cases $z' = \lambda A(t)z$ and $z' = \lambda Jz + A(t)z$, with

$$A(t) = \begin{pmatrix} O & a_1(t) \\ -a_2(t) & O \end{pmatrix} \tag{3}$$

are investigated. It is easy to check that these are systems of the form $\tau z = \lambda B(t)z$. For related results, we refer to the work of C. Bereanu [1]. For asymptotically linear Dirac-type systems, we recall the recent contributions [2] and [4] (by a shooting-type technique). Superquadratic Hamiltonian problems of the form

$$Jz' = \nabla_z F(t, z) \tag{4}$$

have been treated both with variational and topological methods: we refer, among others, to the works by Y. Long [7], W. Dambrosio and the second author [3] and references therein. However, it has to be remarked that Dirac-type systems cannot be compared in a straightforward way with standard Hamiltonian systems of the form (4); we refer to Remark 2.3 for more comments on this subject.
The proof of our main results is developed by a bifurcation technique, on the lines of the seminal paper by P. Rabinowitz [10] (who applied his abstract bifurcation result to scalar second order elliptic equations) and of the more recent contribution in [3]. To this end, one has to go through two main steps: the knowledge of the linear eigenvalue problem associated to τ, and the possibility to define a topological invariant which enables to distinguish the nodal properties of the solutions. In case of Dirac-type systems, there is no well-established linear theory: we thus develop this first task in Subsection 2.1 (Theorem 2.2). We quote again the papers of J. Ward [12], [13] for a similar result in the particular case described above. Theorem 2.2 might be considered a result of independent interest. As for the topological invariant, we use the classical rotation number.
A global bifurcation result for the Dirichlet boundary value problem associated to systems of second order equations involving p-laplacian like differential operators has been recently given by M. García Huidobro, R. Manásevich and J. Ward in [6].

In Section 2 we introduce the concept of rotation number and develop the linear theory; then we state a global bifurcation result.
In Section 3 we state and prove our main results.

Notation. Given two symmetric matrices $A(t), B(t)$, by $A(t) \leq B(t)$ (respectively, $A(t) < B(t)$) we mean that the matrix $B(t) - A(t)$ is semi-positive definite (resp., positive definite). We denote with $\mu_{min}(A)$ (respectively, $\mu_{max}(A)$) the minimum (resp., maximum) of the (real) eigenvalues of a symmetric matrix A. For brevity, the gradient $\nabla_z F(t, z)$ of a function F w.r.t. the variable z will be denoted by $\nabla F(t, z)$.

2. **Some useful preliminaries.**

2.1. **The linear eigenvalue problem.** In this subsection we focus on the linear eigenvalue problem associated to the differential operator τ. Based on the classical notion of winding number, we introduce an index associated to the linear system under consideration and list its properties. Then, we prove the existence of an unbounded double sequence of simple eigenvalues and describe, by means of the index, the nodal properties of the associated eigenfunctions.

Let $B \in C([0,\pi], \mathcal{L}(\mathbb{R}^2))$, with $B(t)$ symmetric for every $t \in [0,\pi]$, and consider the linear boundary value problem

$$\begin{cases} \tau z = B(t)z \\ y(0) = y(\pi) = 0 \end{cases} \qquad t \in [0,\pi], \quad z = (x,y) \in \mathbb{R}^2. \tag{5}$$

Consider the (unique) solution $z : [0,\pi] \to \mathbb{R}$ of the Cauchy problem

$$\begin{cases} \tau z = B(t)z \\ z(0) = (1,0) \end{cases} \qquad t \in [0,\pi], \quad z = (x,y) \in \mathbb{R}^2;$$

as $z(t) \neq 0$ for every $t \in [0,\pi]$, by the path lifting theorem there exists a unique continuous function $\theta : [0,\pi] \to \mathbb{R}$ such that

$$\begin{cases} z(t) = |z(t)|(\cos\theta(t), \sin\theta(t)), \\ \theta(0) = 0. \end{cases}$$

We now use the classical rotation number rot_z of the path $z(\cdot)$. Precisely, we write

$$\mathrm{rot}_z = \frac{\theta(\pi)}{\pi}. \tag{6}$$

Due to the linearity of the equation, the rotation number is the same for all non-trivial solutions of $\tau z = B(t)z$ satisfying $y(0) = 0$. Thus, in what follows we shall emphasize only the dependence on the matrix B and denote this rotation number by $j(B)$. As proved in [14], we have

$$j(B) = \frac{1}{\pi}\int_0^\pi G(t)\begin{pmatrix} \cos\theta(t) \\ \sin\theta(t) \end{pmatrix} \cdot \begin{pmatrix} \cos\theta(t) \\ \sin\theta(t) \end{pmatrix} dt, \tag{7}$$

where, for all $t \in [0,\pi]$, $G(t)$ is the symmetric matrix defined by $G(t) = \frac{1}{2q(t)}(B(t) - P(t))$.

Remark 2.1. In case of a linear Hamiltonian system (obtained from τ by putting $q \equiv \mathrm{const}$), the number $j(B)$ is strictly related to the Maslov index [11] of the symplectic path given by the fundamental matrix solution of $Jz' = \frac{1}{2q}(B(t) - P(t))z$ used in [3]. For linear Dirac-type systems, the fundamental matrix solution is not, in general, symplectic. The possibility of defining a Maslov index for (5) is the object of a current research.

By (7), we can easily deduce some properties of the index, which we summarize in the following

Lemma 2.1. *The following facts hold true:*

- *Monotonicity: if $A, B \in C([0,\pi], \mathcal{L}(\mathbb{R}^2))$, with $A(t), B(t)$ symmetric for every $t \in [0,\pi]$, then*

  $$A(t) \leq B(t) \textit{ for every } t \in [0,\pi] \Longrightarrow j(A) \leq j(B);$$

  moreover

  $$A(t) < B(t) \textit{ for every } t \in [0,\pi] \Longrightarrow j(A) < j(B));$$

- *Continuity: if $B_n, B \in C([0,\pi], \mathcal{L}(\mathbb{R}^2))$, with $B_n(t), B(t)$ symmetric for every n and $t \in [0,\pi]$, then*
$$B_n(t) \to B(t) \text{ uniformly in } t \in [0,\pi] \Longrightarrow j(B_n) \to j(B);$$
- *Divergence: if $(B_n) \subset C([0,\pi], \mathcal{L}(\mathbb{R}^2))$, with $B_n(t)$ symmetric for every n and $t \in [0,\pi]$, then*
$$\mu_{min}(B_n(t)) \to +\infty \text{ uniformly in } t \in [0,\pi] \Longrightarrow j(B_n) \to +\infty.$$

Proof. The monotonicity property is proved in [14]; the continuity property is easily deduced by a standard continuous dependence argument for solutions of Cauchy problems. The divergence property follows from the estimate

$$\begin{aligned} j(B_n) &= \frac{1}{\pi}\int_0^\pi \frac{1}{2q(t)}(B_n(t) - P(t)) \begin{pmatrix} \cos\theta_n(t) \\ \sin\theta_n(t) \end{pmatrix} \cdot \begin{pmatrix} \cos\theta_n(t) \\ \sin\theta_n(t) \end{pmatrix} dt = \\ &= \frac{1}{\pi}\int_0^\pi \frac{1}{2q(t)} B_n(t) \begin{pmatrix} \cos\theta_n(t) \\ \sin\theta_n(t) \end{pmatrix} \cdot \begin{pmatrix} \cos\theta_n(t) \\ \sin\theta_n(t) \end{pmatrix} dt + \\ &+ \frac{1}{\pi}\int_0^\pi -\frac{1}{2q(t)} P(t) \begin{pmatrix} \cos\theta_n(t) \\ \sin\theta_n(t) \end{pmatrix} \cdot \begin{pmatrix} \cos\theta_n(t) \\ \sin\theta_n(t) \end{pmatrix} dt \geq \\ &\geq \frac{1}{\pi}\int_0^\pi \frac{1}{2q(t)} \mu_{min}(B_n(t))dt + \frac{1}{\pi}\int_0^\pi \frac{1}{2q(t)} \mu_{min}(-P(t))dt \to +\infty. \end{aligned}$$

□

We are now ready to state

Theorem 2.2. *Suppose that $B(t)$ is positive definite for every $t \in [0,\pi]$. Then the linear boundary value problem*

$$\begin{cases} \tau z = \lambda B(t) z \\ y(0) = y(\pi) = 0 \end{cases} \qquad t \in [0,\pi], \quad z = (x,y) \in \mathbb{R}^2 \tag{8}$$

has a countable set of (simple) eigenvalues, with no accumulation points. Moreover, the eigenvalues can be ordered in a double sequence $(\lambda_k)_{k\in\mathbb{Z}}$, such that

i) $\cdots < \lambda_{-1} < \lambda_0 < \lambda_1 < \cdots$;
ii) $\lambda_k \to \pm\infty$ *for* $k \to \pm\infty$;
iii) $j(\lambda_k B) = k$.

Proof. It is easily seen that λ is an eigenvalue of (8) if and only if $\theta(\lambda) = j(\lambda B) \in \mathbb{Z}$. From Lemma 2.1, it follows that θ is strictly increasing, continuous and such that $\lim_{\lambda\to\pm\infty} \theta(\lambda) = \pm\infty$. Hence, it is sufficient to define λ_k as the unique real number such that $\theta(\lambda_k) = j(\lambda_k B) = k$. □

2.2. **Global bifurcation.** In this subsection we turn to the study of the nonlinear boundary value problem (1). More precisely, we describe how it can be handled with bifurcation techniques. Finally, we state a global bifurcation theorem which we will use in the next section for the proof of our main result.

Define $S(t,z) = \int_0^1 \partial_z^2 F(t,sz)ds$, so that problem (1) can be written in the form

$$\begin{cases} \tau z = S(t,z)z \\ y(0) = y(\pi) = 0, \end{cases} \tag{9}$$

where $t \in [0,\pi]$, $z = (x,y) \in \mathbb{R}^2$; consider the nonlinear eigenvalue problem

$$\tau z = \lambda S_0(t)z + G(t,z)z \tag{10}$$

$$y(0) = y(\pi) = 0 \tag{11}$$

with $S_0(t) = S(t,0)$ and $G(t,z) = S(t,z) - S_0(t)$. In what follows, we shall assume

(H1) The Hessian matrix $\partial_z^2 F(t,z)$ is positive definite for every $(t,z) \in [0,\pi] \times \mathbb{R}^2$.

By $(H1)$, the matrix $S_0(t)$ is positive definite. Thus, by Theorem 2.2, the linear problem

$$\begin{cases} \tau z = \lambda S_0(t) z \\ y(0) = y(\pi) = 0 \end{cases} \qquad t \in [0,\pi], \quad z = (x,y) \in \mathbb{R}^2 \tag{12}$$

has a double sequence $(\lambda_k)_{k\in\mathbb{Z}}$ of eigenvalues. Replacing if necessary $P(t)$ with $P(t) + cS_0(t)$ we can suppose that $\lambda_k \neq 0$ for every $k \in \mathbb{Z}$, so that problem (10)-(11) can be written in a standard way in the form

$$z = \lambda K z + H(z) \qquad H(z) = o(\|z\|_\infty) \quad z \to 0, \tag{13}$$

with $K \in \mathcal{L}(C([0,\pi],\mathbb{R}^2))$ compact and $H : C([0,\pi],\mathbb{R}^2) \to C([0,\pi],\mathbb{R}^2)$ (nonlinear and) compact. By Rabinowitz theorem [10] we then obtain, for every $k \in \mathbb{Z}$, the existence of a bifurcating branch of solutions $C_k \subset \mathbb{R} \times C([0,\pi],\mathbb{R}^2)$.

Remark 2.2. The equality between the multiplicity of an eigenvalue of the linear problem (12) and the algebraic multiplicity of the corresponding characteristic value of K is due to the fact that the operator τ is formally self-adjoint in the sense that, for every $z, w \in C^1([0,\pi],\mathbb{R}^2)$ satisfying the boundary condition:

$$\int_0^\pi \tau z(t) \cdot w(t) dt = \int_0^\pi z(t) \cdot \tau w(t) dt.$$

Now we are going to define, on the closure $\Sigma \subset \mathbb{R} \times C([0,\pi],\mathbb{R}^2)$ of the set of non-trivial solutions of (10)-(11), a topological invariant which enables us to distinguish the solutions that belong to the different branches C_k. Precisely, we define

$$\Phi(\lambda, z) = j(\lambda S_0(\cdot) + G(\cdot, z(\cdot))).$$

Then we have

Lemma 2.3. *Assume $(H1)$. Then the map $\Phi : \Sigma \to \mathbb{Z}$ is well defined and continuous.*

Proof. If $(\lambda, z) \in \Sigma$ and $z \neq 0$, then (λ, z) is a solution of the linear boundary value problem

$$\begin{cases} \tau w = [\lambda S_0(t) + G(t, z(t))] w \\ v(0) = v(\pi) = 0 \end{cases} \qquad t \in [0,\pi], \quad w = (u,v) \in \mathbb{R}^2,$$

so that $\Phi(\lambda, z) = j(\lambda S_0(\cdot) + G(\cdot, z(\cdot))) = \mathrm{rot}_z \in \mathbb{Z}$.
If $(\lambda, z) \in \Sigma$ and $z = 0$, then $\lambda = \lambda_k$ for some $k \in \mathbb{Z}$, so we still have $\Phi(\lambda, z) = j(\lambda_k S_0) = k \in \mathbb{Z}$.
Finally, the continuity of Φ follows from the continuity property of the rotation number. □

We are now ready to state (on the lines of [10], [3]) a global bifurcation result.

Theorem 2.4. *Assume $(H1)$. Then, for every $k \in \mathbb{Z}$, problem (10)-(11) has an unbounded branch of solutions C_k, and $C_k \neq C_j$ for $k \neq j$. Moreover $\Phi(\lambda, z) = k$ for every $(\lambda, z) \in C_k$.*

Proof. Consider the bifurcating branches C_k obtained by Rabinowitz theorem [10]. By the connectedness of C_k and the continuity (cf. Lemma 2.3) of Φ, we have that $\Phi(\lambda, z) = \Phi(\lambda_k, 0) = k$ for every $(\lambda, z) \in C_k$. It clearly follows that $C_k \neq C_j$ for $k \neq j$. □

In the next section we will show that there are infinitely many branches C_k that intersect the hyperplane $\lambda = 1$. From (10) and the definition of G, this means we can find infinitely many solutions to the given problem (10)-(11).

We end this section with the following important

Remark 2.3. It has to be observed that any system of the form $\tau z = S(t, z)z$, with $S(t, z)$ a symmetric matrix, can be transformed in a system of the (standard) form $Jw' = \tilde{S}(t, w)w$, with $\tilde{S}(t, w)$ symmetric. However, a definiteness condition on $S(t, z)$ (or, according to Remark 2.9 in [3], on $S_0(t)$)) is not in general transferred to $\tilde{S}$; hence, our main result (Theorem 3.2 below) cannot be deduced from Theorem 2.2 in [3]. However, it is possible to give a sufficient condition on the differential operator τ which provides a definiteness property for $\tilde{S}$; we develop this possibility in Theorem 3.5.

3. **The main result.**

In this section we state and prove our multiplicity result on the nonlinear problem (1). Before doing that, we introduce and comment the assumptions which, together with $(H1)$, will be considered in what follows. Indeed, let us suppose

(H2) $\lim_{|z|\to\infty} \mu_{min}(\partial_z^2 F(t, z)) = +\infty$ uniformly in $t \in [0, \pi]$;

(H3) There exist $C, R > 0$ such that

$$|\partial_t F(t, z)| \leq C|F(t, z)|,$$

$$|\nabla F(t, z)| \leq C\frac{|F(t, z)|}{|z|}$$

for every $t \in [0, \pi]$ and for every $z \in \mathbb{R}^2$ with $|z| \geq R$.

Observe first that $(H2)$ implies that $\lim_{|z|\to\infty} \mu_{min}(S(t, z)) = +\infty$ uniformly in $t \in [0, \pi]$.

Remark 3.1. Hypothesis $(H1)$ and the fact that $\nabla_z F(t, 0) = 0$ for every $t \in [0, \pi]$ imply that there exist $\alpha > 0$, $\beta \in \mathbb{R}$ such that for every $t \in [0, \pi]$ and for every $z \in \mathbb{R}^2$ with $|z| \geq 1$, we have

$$F(t, z) \geq \alpha|z| + \beta. \tag{14}$$

In particular, $\lim_{|z|\to\infty} F(t, z) = +\infty$ uniformly in $t \in [0, \pi]$. The proof is a slight variant of Proposition 1.5 in [9].

The crucial estimate for the proof of the main result is a preliminary lemma (the so-called "elastic property") on the solutions of (10).

Lemma 3.1. *Assume $(H1), (H2), (H3)$ and suppose that (λ_n, z_n) is a sequence of solutions of (10), with $1 \leq \lambda_n \leq C$ for every n and $\|z_n\|_\infty \to \infty$. Then, up to a subsequence, we have $\min_{t\in[0,\pi]} |z_n(t)| \to \infty$.*

Proof. We can rewrite the differential equation (10) as

$$Jz' = \frac{1}{2q(t)}\nabla F(t, z) + \frac{1}{2q(t)}[(\lambda - 1)\partial_z^2 F(t, 0) - P(t) - q'(t)J]z.$$

Thus, arguing as in [5], it is sufficient to prove that there exists $V \in C([1,+\infty[\times[0,\pi]\times\mathbb{R}^2,\mathbb{R})$ such that:

i) for every $\lambda \in [1,+\infty[$, $V(\lambda,\cdot,\cdot) \in C^1([0,\pi]\times\mathbb{R}^2,\mathbb{R})$;
ii) $\lim_{|z|\to\infty} |V(\lambda,t,z)| = \infty$ uniformly in $t \in [0,\pi]$ and in bounded λ-intervals;
iii) for every $I \subset [1,+\infty[$ bounded λ-interval, there exist $M, K > 0$ such that, setting
$$R(\lambda,t,z) = -\frac{1}{2q(t)} J\nabla F(t,z) - \frac{1}{2q(t)}[(\lambda-1)J\partial_z^2 F(t,0) - JP(t) + q'(t)]z,$$
we can get
$$|\partial_t V(\lambda,t,z) + \nabla V(\lambda,t,z)\cdot R(\lambda,t,z)| \le M|V(\lambda,t,z)|$$
for every $\lambda \in I$, $t \in [0,\pi]$ and $z \in \mathbb{R}^2$ with $|z| \ge K$.

Define $V(\lambda,t,z) = F(t,z)$; conditions i) and ii) are clearly satisfied (recall (14)). Moreover,
$$\nabla F(t,z)\cdot R(\lambda,t,z) = \nabla F(t,z)\cdot A(\lambda,t)z,$$
where $A(\lambda,t)$ is the matrix
$$A(\lambda,t) = -\frac{1}{2q(t)}[(\lambda-1)J\partial_z^2 F(t,0) - JP(t) + q'(t)].$$
So, if $I \subset [1,+\infty[$ is a bounded interval, using $(H3)$ we have that
$$|\partial_t F(t,z)| \le C|F(t,z)|$$
and
$$\begin{aligned}|\nabla F(t,z)\cdot R(\lambda,t,z)| &\le \left(\max_{\lambda\in I, t\in[0,\pi]} \|A(\lambda,t)\|\right)|\nabla F(t,z)||z| \le \\ &\le C\left(\max_{\lambda\in I, t\in[0,\pi]} \|A(\lambda,t)\|\right)|F(t,z)|,\end{aligned}$$
for every $t \in [0,\pi]$ and every $z \in \mathbb{R}^2$ with $|z| \ge R$. We deduce that condition iii) holds true. □

We are now ready to state and prove the main result of this paper.

Theorem 3.2. *Assume (H1), (H2), (H3). Then there exists $m_0 \in \mathbb{R}$ such that for every $k \in \mathbb{Z}$ with $k > m_0$ there exists a (classical) solution of (1) whose rotation number is k.*

For the proof, we need the following two preliminary results.

Lemma 3.3. *There exists $\Lambda_k > 0$ such that $(\lambda, z) \in C_k$ implies $\lambda \le \Lambda_k$.*

Proof. By contradiction, suppose that there exist $(\lambda_n, z_n) \subset C_k$ such that $\lambda_n \to +\infty$. Up to a subsequence, we have $\lambda_n \ge 1$, so that by the monotonicity and divergence properties of the index (Lemma 2.1) we have
$$j(\lambda_n S_0(\cdot) + G(\cdot, z_n(\cdot))) = j((\lambda_n - 1)S_0(\cdot) + S(t, z_n(\cdot))) \ge j((\lambda_n-1)S_0(\cdot)) \to +\infty,$$
in contradiction with the constancy of Φ along bifurcation branches. □

Lemma 3.4. *There exists $M_k > 0$ such that $(\lambda, z) \in C_k$ and $\lambda \ge 1$ implies $\|z\|_\infty \le M_k$.*

Proof. By contradiction, suppose that there exist $(\lambda_n, z_n) \subset C_k$ with $\lambda_n \geq 1$ and $\|z_n\|_\infty \to +\infty$. We know that λ_n is bounded, so according to Lemma 3.1 we can suppose that $\min_{t\in[0,\pi]} |z_n(t)| \to \infty$. By the monotonicity and divergence properties of the index we have

$$j(\lambda_n S_0(\cdot) + G(\cdot, z_n(\cdot))) = j((\lambda_n - 1)S_0(\cdot) + S(\cdot, z_n(\cdot))) \geq j(S(\cdot, z_n(\cdot))) \to +\infty,$$

in contradiction with the fact that Φ is constant along bifurcation branches. □

Proof of Theorem 3.2. Define $m_0 = j(S_0)$ and let $k \in \mathbb{Z}$ with $k > m_0$. Observe that the monotonicity property of the index implies that $\lambda_k > 1$ and consider the branch C_k. By Lemma 3.3 and Lemma 3.4, it follows that the branch C_k contains a point $(1, z_k)$, i.e. this branch intersects the hyperplane $\lambda = 1$. Moreover, $k = \Phi(1, z_k) = \mathrm{rot}_{z_k}$. □

Finally, we shall give a second multiplicity result. Assumption $(H4)$ on F in Theorem 3.5 below is more general than $(H3)$ in Theorem 3.2; on the other hand, an extra condition on the matrix P in the differential operator τ has to be required (cf. Remark 2.3).

Theorem 3.5. *Assume (H1), (H2) and that $P(t)$ is C^1 and negative definite for all t. Moreover, suppose that $F \geq 0$, $\partial_z^2 F(t, 0)$ is C^1 and that the following assumption is satisfied:*

(H4) There exist $C, R > 0$ such that

$$|\partial_t F(t, z)| \leq C(|z|^2 + F(t, z)),$$

$$|\nabla F(t, z)| \leq C(|z|^2 + F(t, z))$$

for every $t \in [0, \pi]$ and for every $z \in \mathbb{R}^2$ with $|z| \geq R$.

Then there exists $m_0 \in \mathbb{R}$ such that for every $k \in \mathbb{Z}$ with $k > m_0$ there exists a (classical) solution of (1) whose rotation number is k.

Proof. With the change of variables $z(t) = \frac{1}{\sqrt{q(t)}} w(t)$, we have that z is a solution of the boundary value problem (1) with rotation number $k \in \mathbb{Z}$ if and only if w is a solution with rotation number k of the boundary value problem

$$\begin{cases} Jw' = \nabla_w \hat{F}(t, w) \\ v(0) = v(\pi) = 0 \end{cases} \qquad t \in [0, \pi], \quad w = (u, v) \in \mathbb{R}^2,$$

where

$$\hat{F}(t, w) = \frac{1}{2} F\Big(t, \frac{1}{\sqrt{q(t)}} w\Big) - \frac{P(t)}{2q(t)} w \cdot w.$$

F Thus, we can use a slight variant of Theorem 2.2 in [3] (with the obvious changes due to the fact that in [3] it is considered the boundary condition $u(0) = 0 = u(\pi)$ instead of $v(0) = 0 = v(\pi)$) . In fact, $\hat{F} \geq 0$ and $\nabla \hat{F}(t, 0) = 0$; assumption $(H1)$ and the fact that P is negative definite guarantee that the Hessian matrix $\partial_w^2 \hat{F}(t, w)$ is positive definite for every $(t, w) \in [0, \pi] \times \mathbb{R}^2$, while assumption $(H2)$ implies that $\lim_{|w|\to\infty} \mu_{min}(\partial_w^2 \hat{F}(t, w)) = +\infty$ uniformly in $t \in [0, \pi]$. The regularity assumption in [3] can be weakened by requiring that $\hat{F}(t, \cdot)$ is of class C^2 for every $t \in [0, \pi]$, which is true, and finally $\partial_w^2 \hat{F}(t, 0)$ is of class C^1. Hence, it remains to verify that there exist $\tilde{C}, \tilde{R} > 0$ such that

$$|\partial_t \hat{F}(t, w)| \leq \tilde{C} \hat{F}(t, w) \tag{15}$$

for every $t \in [0,\pi]$ and every $w \in \mathbb{R}^2$ with $|w| \geq \tilde{R}$. Setting $\alpha(t) = \frac{1}{\sqrt{q(t)}}$ and $\tilde{P}(t) = \frac{P(t)}{2q(t)}$, we have

$$|\partial_t \hat{F}(t,w)| \leq \left|\frac{1}{2}\partial_t F(t,\alpha(t)w) + \frac{1}{2}\alpha'(t)\nabla F(t,\alpha(t)w)\right| + |\tilde{P}'(t)w \cdot w|;$$

then assumption (H4) implies that, for $|w|$ large enough,

$$\left|\frac{1}{2}\partial_t F(t,\alpha(t)w) + \frac{1}{2}\alpha'(t)\nabla F(t,\alpha(t)w)\right| \leq C_1(|w|^2 + F(t,\alpha(t)w)),$$

while the fact the $\tilde{P}$ is negative definite implies that

$$C_1(|w|^2 + F(t,\alpha(t)w)) \leq C_2\Big(-\tilde{P}(t)w\cdot w + \frac{1}{2}F(t,\alpha(t)w)\Big) = C_2\hat{F}(t,w).$$

Moreover,

$$|\tilde{P}'(t)w\cdot w| \leq C_3|w|^2 \leq -C_4\tilde{P}(t)w\cdot w \leq C_4\Big(-\tilde{P}(t)w\cdot w + \frac{1}{2}F(t,\alpha(t)w)\Big) = C_4\hat{F}(t,w).$$

So we have (15) and the proof is complete. □

Remark 3.2. The assumption $\hat{F}(t,w) \geq 0$ for every $(t,w) \in [0,\pi]\times\mathbb{R}^2$ in [3] is not necessary. In fact, the positivity of F is used only for w outside a sufficiently large ball, where it is clearly satisfied. As a consequence, the assumption $F \geq 0$ is not necessary also in Theorem 3.5.

Remark 3.3. We observe that the fact that in system (1) the nonlinearity has the form $\nabla F(t,z)$ is crucial only for the proof of Lemma 3.1; for a more general nonlinearity, one should require the existence of a "guiding-type" function [8].
We finally observe that a variant of our main results can be obtained by assuming, instead of $(H2)$, the condition
(H2)' $\lim_{|z|\to\infty} \mu_{max}(\partial_z^2 F(t,z)) = -\infty$ uniformly in $t \in [0,\pi]$.

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E-mail address: boscaggi@sissa.it
E-mail address: anna.capietto@unito.it