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EXISTENCE OF NONCONTINUABLE SOLUTIONS OF A SYSTEM OF DIFFERENTIAL EQUATIONS

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Abstract. In the paper a system of differential equations $y_i' = f_i(t, y_1, \dots, y_{n-1})g_i(y_n)$, $i = 1, \dots, n$ is studied. Sufficient (necessary) conditions for the existence of a solution y fulfilling $\lim_{t\to\tau_-} y_i(t) = C_i$, $i = 1, 2, \dots, n-1$, $\lim_{t\to\tau_-} |y_n(t)| = \infty$ are derived where $\tau < \infty$ and $C_i \in \mathbb{R}$ are given.

1. Introduction.

Consider the system of differential equations

$$y_i' = f_i(t, y_1, \dots, y_{n-1})g_i(y_n) \tag{1}$$

where $n \geq 2$, $i = 1, 2, \dots, n$, $f_i \in C^0(\mathbb{R}_+ \times \mathbb{R}^{n-1})$, $g_i \in C^0(\mathbb{R})$, $i = 1, \dots, n$, $\mathbb{R}_+ = [0, \infty)$, $\mathbb{R} = (-\infty, \infty)$ and $M_0 > 0$ exists such that

$$g_n(x) > 0 \quad \text{for} \quad |x| \geq M_0\,. \tag{2}$$

We will study nonextendable solutions $y(t) = \big(y_1(t), \dots, y_n(t)\big)$, $t \in [T, \tau) \subset \mathbb{R}_+$, i.e. solutions that can not be defined at $t = \tau$ if $\tau < \infty$. A solution y of (1) defined on $[T, \tau) \subset \mathbb{R}_+$ is called noncontinuable if $\tau < \infty$. Sometimes, such solutions are called singular of the second kind (see [1, 6, 7, 4]).

Special case of (1) is the n-th order differential equation

$$y^{(n)} = f(t, y, y', \dots, y^{(n-2)})g(y^{(n-1)}) \tag{3}$$

with $f \in C^0(\mathbb{R}_+ \times \mathbb{R}^{n-1})$, $g \in C^0(\mathbb{R})$, $f_i(t, x_1, \dots, x_{n-1}) = x_{i+1}$, $g_i \equiv 1$, $i = 1, 2, \dots, n-2$, $f_{n-1} \equiv 1$, $f_n = f$, $g_{n-1}(x) = x$ and $g_n(x) = g(x)$.
Sufficient conditions for the existence (the nonexistence) of noncontinuable solutions of (3) are given e.g. in [1, 5, 7, 8, 9, 10, 11, 12, 4]. In these papers the Cauchy initial problem is studied and, in case of existence results, it is proved that the right-hand side point τ of the definition interval exists. Similarly, see [6] for (1).

Jaroš and Kusano [4] investigated the differential equation

$$y'' = r(t)|y|^\sigma |y'|^\lambda \operatorname{sgn} y \tag{4}$$

with $\sigma > 0$ and $r < 0$ on $\mathbb{R}_+$. If $\tau \in (0, \infty)$ is given, they proved the existence of a noncontinuable solution y of (4) such that

$$\lim_{t\to\tau_-} y(t) \in [0, \infty) \quad \text{and} \quad \lim_{t\to\tau_-} y'(t) = -\infty\,;$$

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they call it a black hole solution. In papers [1] and [2] this boundary value problem is generalized into the following one.

To find conditions under which (3) has a noncontinuable solution y fulfilling

$$\begin{gathered} \tau \in (0,\infty), \quad C_i \in \mathbb{R} \quad \text{for} \quad i = 1,2,\dots,n-1\,, \\ \lim_{t\to\tau_-} y^{(i)}(t) = C_{i+1}, \quad i = 0,1,\dots,n-2\,, \quad \lim_{t\to\tau_-} |y^{(n-1)}(t)| = \infty \end{gathered} \tag{5}$$

and y is defined in a left neighbourhood of τ.

Some results are summed up into the following theorem.

Theorem A ([2]). *Let $\tau \in (0,\infty)$ and $C_i \in \mathbb{R}$, $i = 1,2,\dots,n-1$ be such that $f(\tau, C_1,\dots,C_{n-1}) \neq 0$.*

(i) *If $\lambda > 2$ and $g(x) \geq |x|^\lambda$ for $|x| \geq M > 0$, then (3) has a solution y fulfilling (5).*

(ii) *Let $g(x) = |x|^\lambda$ for $|x| \geq M > 0$ and $\lambda \in \mathbb{R}$. Then (3) has a noncontinuable solution y fulfilling (5) if and only if $\lambda > 2$.*

In the present paper we generalize the results for (1).

Problem. To find conditions for the existence (the nonexistence) of noncontinuable solution $y = (y_1,\dots,y_n)$ of (1) such that

$$\begin{gathered} \tau \in (0,\infty)\,, \quad C_i \in \mathbb{R}\,, \qquad i = 1,2,\dots,n-1\,, \\ \lim_{t\to\tau_-} y_i(t) = C_i\,, \quad i = 1,2,\dots,n-1\,, \quad \lim_{t\to\tau_-} |y_n(t)| = \infty \end{gathered} \tag{6}$$

holds and y is defined in a left neighbourhood of τ.

Note that noncontinuable solutions appear when e.g. water flow problems in one space dimension are studied. Recall only the problems of the front of flood wave (the solution has the form (5), $n = 2$) or of a water spring (the solution is unbounded in a neighbourhood of τ); see [3]. In these cases the mathematical models are either very rough or are wrong and the behaviour of a solution y in a neighbourhood of τ has to be to be computed in another way.

2. Main results.

The first theorem gives sufficient conditions for the nonexistence of a solution fulfilling (6).

Theorem 1. *Let $\tau > 0$, $C_i \in \mathbb{R}_+$, $i = 1,2,\dots,n-1$, and let either*

(i) $\int_{M_0}^\infty \frac{ds}{g_n(s)} = \infty$

or

(ii) *$i \in \{1,2,\dots,n-1\}$, $M > 0$, $f_i(\tau, C_1,\dots,C_{n-1}) \neq 0$, and*

$$g_i(x) \neq 0 \quad \textit{for} \quad |x| \geq M\,, \qquad \int_{M_0}^\infty \frac{|g_i(s)|}{g_n(s)}\,ds = \infty\,.$$

Then (1) has no noncontinuable solution y fulfilling (6).

Proof. Suppose, contrarily, that y is a noncontinuable solution of (1) fulfilling (6) and it is defined on $[\tau_1, \tau) \subset \mathbb{R}_+$. Furthermore, suppose that $\lim_{t\to\tau_-} y_n(t) = \infty$; the opposite case, if $\lim_{t\to\tau_-} y_n(t) = -\infty$, can be studied similarly. Without loss of generality, we can suppose that $M \geq M_0$ and $M \geq 1$. Then $T \in [\tau_1, \tau)$, $M_1 > 0$

and $M_2 > 0$ exist such that

$$\left.\begin{aligned} &\bigl|f_j\bigl(t, y_1(t), \dots, y_{n-1}(t)\bigr)\bigr| \le M_1\,, \qquad j = 1, 2, \dots, n\,,\\ &y_n(t) \ge M\,, \qquad t \in [T, \tau)\\ &\text{and in case (ii)}\\ &\bigl|f_i\bigl(t, y_1(t), \dots, y_{n-1}(t)\bigr)\bigr| \ge M_2\,, \quad t \in [T, \tau)\,. \end{aligned}\right\} \tag{7}$$

(i) The assumptions of this case, (7) and the integration of the last equation of (1) on $[T, \tau)$ yield

$$\begin{aligned} \infty = \int_{y_n(T)}^{\infty} \frac{ds}{g_n(s)} &= \int_T^\tau \frac{y_n'(s)\,ds}{g_n(y_n(s))}\\ &= \int_T^\tau f_n\bigl(s, y_1(s), \dots, y_{n-1}(s)\bigr)\,ds \le M_1(\tau - T)\,. \end{aligned}$$

The contradiction proves the statement in this case.

(ii) The integration of the i-th equation of (1) on $[T, \tau)$, (7) and the assumptions of (ii) yield y_i' does not change its sign on $[T, \tau)$ and

$$\begin{aligned} \infty > M_1\bigl|C_i - y_i(T)\bigr| &= M_1 \int_T^\tau \bigl|y_i'(s)\bigr|\,ds\\ &= M_1 \int_T^\tau \bigl|f_i(s, y_1(s), \dots, y_{n-1}(s))\bigr|\bigl|g_i(y_n(s))\bigr|\,ds \ge M_1 M_2 \int_T^\tau \bigl|g_i(y_n(s))\bigr|\,ds\\ &\ge M_2 \int_T^\tau \frac{|g_i(y_n(s))|}{g_n(y_n(s))} f_n\bigl(s, y_1(s), \dots, y_{n-1}(s)\bigr) g_n(y_n(s))\,ds\\ &\ge M_2 \int_T^\tau \frac{|g_i(y_n(s))|}{g_n(y_n(s))} y_n'(s)\,ds = M_2 \int_{y_n(T)}^\infty \frac{|g_i(\sigma)|}{g_n(\sigma)}\,d\sigma = \infty\,. \end{aligned}$$

The contradiction proves the statement. □

The following theorem gives a sufficient condition for the existence of a solution fulfilling (6).

Theorem 2. *Let $\tau > 0$, and $C_i \in \mathbb{R}_+$, $i = 1, 2, \dots, n-1$ be such that*

$$f_n(\tau, C_1, \dots, C_{n-1}) \ne 0\,. \tag{8}$$

Let $M > 0$, λ_i, $i = 1, 2, \dots, n$ be such that $\lambda_n > 1$,

$$\lambda_i < \lambda_n - 1 \quad \text{for} \quad i = 1, 2, \dots, n-1\,, \tag{9}$$

$$\bigl|g_i(x)\bigr| \le |x|^{\lambda_i} \quad \text{for} \quad |x| \ge M\,, \quad i = 1, 2, \dots, n-1 \tag{10}$$

and

$$g_n(x) \ge |x|^{\lambda_n} \quad \text{for} \quad |x| \ge M\,. \tag{11}$$

Then (1) has a noncontinuable solution y fulfilling (6).

Proof. Suppose for the simplicity that $M \ge M_0$, $M > 1$ and $f_n(\tau, C_1, \dots, C_{n-1}) > 0$; the opposite case can be studied similarly. Then $N > 0$, M_1, M_2 and $\bar\tau \in [0, \tau)$ exist such that

$$0 < M_1 \le f_n(t, x_1, \dots, x_{n-1}) \le M_2 \tag{12}$$

for $t \in [\bar\tau, \tau]$, $|x_i - C_i| \le N$, $i = 1, 2, \dots, n-1$. Let M_3 be such that

$$\bigl|f_j(t, x_1, \dots, x_{n-1})\bigr| \le M_3 \tag{13}$$

for $t \in [\bar{\tau}, \tau]$, $|x_i - C_i| \le N$, $i = 1, 2, \dots, n-1$, $j = 1, 2, \dots, n-1$. Consider the auxiliary Cauchy problem

$$\begin{aligned} y_i' &= f_i\big(t, \chi_1(y_1), \dots, \chi_{n-1}(y_{n-1})\big) g_i(y_n)\,, \\ y_i(\tau) &= C_i\,, \quad i = 1, \dots, n-1\,, \quad y_n(\tau) = k \end{aligned} \tag{14}$$

$i = 1, \dots, n$ where $k \in \{k_0, k_0 + 1, \dots\}$, $k_0 \ge 2M$,

$$\begin{aligned} \chi_j(s) &= s && \text{for} \quad |s - C_j| \le N \\ &= C_j + N && \text{for} \quad s > C_j + N \\ &= C_j - N && \text{for} \quad s < C_j - N\,, \\ j &= 1, 2, \dots, n-1\,. \end{aligned} \tag{15}$$

Furthermore, let $J = [T, \tau) \subset [\bar{\tau}, \tau)$ be such that

$$M_2(\tau - T) < \int_M^{2M} \frac{ds}{g_n(s)}\,, \tag{16}$$

$$M_3(\lambda_n - 1) \max_{1 \le i < n} \big\{ [(\lambda_n - 1) M_1]^{-\lambda_i/(\lambda_n - 1)} (\tau - T)^{1 - \lambda_i/(\lambda_n - 1)} \\ /(\lambda_n - \lambda_i - 1) \big\} < N\,. \tag{17}$$

Note that right-hand sides in (14) are continuous functions of $(t, y_1, \dots, y_n)$ on $\mathbb{R}_+ \times R^n$. Denote by $y^k = (y_{1k}, \dots, y_{nk})$ a solution of (14) and by J_k the penetration of its maximal definition interval and J. We prove that $J_k = J$ for $k \ge k_0$ and we estimate y_{nk}.

Let $k \ge k_0$. We prove that

$$y_{nk}(t) > M \quad \text{for} \quad t \in J_k\,. \tag{18}$$

As $k \ge k_0 > M$, (14) yields (18) is valid in a left neighbourhood of τ. Suppose, contrarily, that $T_1 \in J_k$ exists such that $y_{nk}(T_1) = M$ and $y_{nk}(t) > M$ on $(T_1, \tau]$. Then (12), (14) and (15) yield

$$y_{nk}'(t) \le M_2 g_n\big(y_{nk}(t)\big)\,, \quad t \in [T_1, \tau]$$

or

$$\begin{aligned} \int_M^{2M} \frac{ds}{g_n(s)} \le \int_M^k \frac{ds}{g_n(s)} &= \int_{T_1}^{\tau} \frac{y_{nk}'(s)}{g_n(y_{nk}(s))}\, ds \\ &= \int_{T_1}^{\tau} f_n\big(s, y_{1k}(s), \dots, y_{n-1,k}(s)\big)\, ds \le M_2(\tau - T_1) \le M_2(\tau - T)\,. \end{aligned}$$

The contradiction with (16) proves that (18) is valid.

Furthermore, the integration of the n-th equation in (14), (12), (15) and (18) yield

$$\begin{aligned} &y_{nk}' \ge \ M_1 g_n\big(y_{nk}(t)\big) \ge M_1 y_{nk}^{\lambda_n}(t)\,, \quad t \in J_k\,, \\ &y_{nk}^{-\lambda_n + 1}(t) - y_{nk}^{-\lambda_n + 1}(\tau) \ge (\lambda_n - 1) M_1 (\tau - t) \end{aligned}$$

or

$$y_{nk}(t) \le \big[(\lambda_n - 1) M_1 (\tau - t) + k^{1 - \lambda_n}\big]^{-1/(\lambda_n - 1)}\,, \quad t \in J_k\,. \tag{19}$$

From (18) and (19), y_{nk} is bounded on J_k and hence $J_k = J$ for $k \ge k_0$. Moreover, it follows from (19) that

$$y_{nk}(t) \le M_4(\tau - t)^{-1/(\lambda_n - 1)} \tag{20}$$

on J with $M_4 = \left[(\lambda_n - 1)M_1\right]^{-1/(\lambda_n-1)}$.

Furthermore, we estimate y_{ik} for $i = 1, 2, \ldots, n-1$. It follows from (10), (13), (15), (17), (18) and (20) that

$$\begin{aligned} \left|y_{ik}(t) - C_i\right| &= \left| \int_t^\tau y'_{ik}(s)\, ds \right| \le M_3 \int_t^\tau \left|g_i(y_{nk}(s))\right| ds \\ &\le M_3 M_4^{\lambda_i} \int_t^\tau (\tau - s)^{-\lambda_i/(\lambda_n-1)} ds = M_5(\tau - t)^{1-\lambda_i/(\lambda_n-1)} \\ &\le M_5(\tau - T)^{1-\lambda_i/(\lambda_n-1)} \le N\,, \end{aligned} \tag{21}$$

with $M_5 = M_3 M_4^{\lambda_i} \frac{\lambda_n - 1}{\lambda_n - \lambda_i - 1}$, $t \in J$, $i = 1, 2, \ldots, n-1$.

As the estimations (18), (20) and J do not depend on k, Arzèl-Ascoli Theorem (see [4] Lemma 10.2) yields the existence of a subsequence of $\{y^k\}_{k=k_0}^\infty$ that converges locally uniformly to a solution $y = (y_1, \ldots, y_n)$ of the differential equation in (14) on $[T, \tau)$; we denote it by $\{y^k\}_{k=k_0}^\infty$ for the simplicity. Moreover, (15) and (21) yield y^k, $k \ge k_0$ and y are solutions of (1), too. It follows from the second equality in (21) that

$$\lim_{t \to \tau_-} y_i(t) = C_i \quad \text{for} \quad i = 1, 2, \ldots, n-1\,,$$

and we prove that

$$\lim_{t \to \tau_-} y_n(t) = \infty\,. \tag{22}$$

According to (1), (2), (12) and (18), y_n is increasing on J and suppose, contrarily, that

$$\lim_{t \to \tau_-} y_n(t) = Q \in [M, \infty)\,.$$

Note, that $\int_M^\infty g_n^{-1}(s)\, ds < \infty$ according to the assumptions of the theorem.
Let $T_2 \in (T, \tau)$ be such that

$$M_2(\tau - T_2) < \int_{2Q}^\infty \frac{ds}{g_n(s)}\,. \tag{23}$$

As y_{nk} converges uniformly to y_n for $k \to \infty$ on $[T, T_2]$, $\bar{k} \ge \bar{k}_0$ exists such that

$$y_{nk}(T_2) \le 2Q \quad \text{for} \quad k \ge \bar{k}\,.$$

From this and from (12) we obtain by the integration of n-th equation in (1) on $[T_2, \tau)$

$$M_2(\tau - T_2) \ge \int_{T_2}^\tau \frac{y'_{nk}(s)}{g_n(y_{nk}(s))}\, ds = \int_{y_{nk}(T_2)}^k \frac{ds}{g_n(s)} \ge \int_{2Q}^k \frac{ds}{g_n(s)}$$

for $k \ge \bar{k}$ and, hence

$$M_2(\tau - T_2) \ge \int_{2Q}^\infty \frac{ds}{g_n(s)}\,.$$

The contradiction with (23) proves that (22) is valid and y fulfils (6). ☐

Remark 1. It is evident from the proof of Theorem 2 that a noncontinuable solution y of (1) fulfilling (6) exists if instead of (10) and (11) we suppose

$$\left|g_i(x)\right| \le x^{\lambda_i}\,, \quad i = 1, \ldots, n-1\,, \quad g_n(x) \ge x^{\lambda_n} \quad \text{for} \quad x \ge M$$

in case $f_n(\tau, C_1, \ldots, C_{n-1}) > 0$ and

$$\left|g_i(x)\right| \le |x|^{\lambda_i}\,, \quad i = 1, \ldots, n-1\,, \quad g_n(x) \ge |x|^{\lambda_n} \quad \text{for} \quad x \le -M$$

in case $f_n(\tau, C_1, \ldots, C_{n-1}) < 0$.

The following result shows that conditions in Theorem 2 are sharp, they cannot be weaken in case (8) is valid.

Corollary 1. *Let $\tau > 0$, $C_i \in \mathbb{R}$ for $i = 1, 2, \ldots, n-1$ and $\lambda_i \in \mathbb{R}$, $i = 1, \ldots, n$ be such that*

$$f_i(\tau, C_1, \ldots, C_{n-1}) \neq 0, \quad i = 1, 2, \ldots, n,$$
$$g_i(x) = |x|^{\lambda_i} \quad \text{for} \quad |x| \geq M > 0 \quad \text{and} \quad i = 1, 2, \ldots, n.$$

Then (1) has a noncontinuable solution y fulfilling (6) if and only if $\lambda_n > 1$ and

$$\lambda_i < \lambda_n - 1 \quad \text{for} \quad i = 1, 2, \ldots, n-1.$$

Proof. It follows from Theorems 1 and 2. □

Remark 2. It follows from Theorem 2 and Corollary 1 that Theorem A is a special case of them with $g_i \equiv 1$ for $i = 1, 2, \ldots, n-2$, $g_{n-1}(x) = x$, $g_n = g$, $\lambda_i = 0$ for $i = 1, 2, \ldots, n-2$, $\lambda_{n-1} = 1$, $\lambda_{n-1} = 1$ and $\lambda_n = \lambda$ in case (i); in case (ii) we obtain the result only if $C_i \neq 0$, $i = 2, 3, \ldots, n-1$.

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