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DYNAMICALLY CONSISTENT NONSTANDARD FINITE DIFFERENCE SCHEMES FOR CONTINUOUS DYNAMICAL SYSTEMS

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Abstract. This work deals with the relationship between a continuous dynamical system and numerical methods for its computer simulations, viewed as discrete dynamical systems. The term 'dynamic consistency' of a numerical scheme with the associated continuous system is usually loosely defined, meaning that the numerical solutions replicate some of the properties of the solutions of the continuous system. Here, this concept is replaced with *topological dynamic consistency*, which is defined in precise terms through the topological equivalence of maps. This ensures that *all* the topological properties (e.g., fixed points and their stability, periodic solutions, invariant sets, etc.) are preserved. Two examples are provided which demonstrate that numerical schemes satisfying this strong notion of dynamic consistency can be constructed using the nonstandard finite difference method.

1. Introduction.

The solutions of continuous dynamical systems given by systems of ordinary differential equations are typically computed by various numerical procedures defined on discretized time meshes. There has been a considerable effort in the recent years to construct numerical procedures which correctly replicate the properties of the original dynamical system by using the non-standard finite difference method. The most common requirement placed on the numerical method is *elementary stability*. This concept refers to preserving the hyperbolic fixed points of the original system with their correct linear stability. It was introduced by Mickens,

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[12], and has since been widely used,[13, 14, 6]. Other dynamical system properties have also been considered, e.g., invariant sets, dissipativity, and non-hyperbolic fixed points, [15, 8, 2, 3]. The most general concept used in the literature is that of *dynamic consistency*. In [1] we find the definition: "A difference equation is called *dynamically consistent* with the differential equation it approximates if they both possess the same dynamics such as stability, bifurcation and chaos." Clearly, the content of this concept is determined by what we call dynamics and it can mean different things in different situations.

As an example, the finite difference scheme producing the numerical solutions depicted in Fig. 2 may be considered to be dynamically consistent with the differential equation with the solutions given in Fig. 1, since they are clearly both not chaotic, and the only sets that are both positively and negatively invariant are the fixed points, which are the same and have the same asymptotic stability.

However, the mismatch between the solutions depicted in Figs. 1 and 2 is obvious. The lack of monotonicity of the numerical solutions is apparently a symptom of a more serious discrepancy between the discrete and the continuous dynamical system which will be made clear in the sequel. Other concepts attempting to link the difference equations with the differential equations they approximate have also been used, e.g., asymptotic consistency, [9], as well as general purpose concepts such as qualitative stability, [5], all referring to the preservation of particular properties of the involved dynamical systems.

The aim of this paper is to expand the mathematical theory which connects the continuous dynamical systems defined by systems of ordinary differential equations and the discrete dynamical schemes defined by their numerical schemes. To that end, we consider the problem in the general setting of Topological Dynamics. The *topological dynamic consistency* of a discrete dynamical systems with a certain continuous dynamical system is considered in the sense of topological equivalence (also called 'conjugacy') between the evolution operator of the continuous dynamical system and the set of maps defining the respective discrete dynamical system for all positive time step sizes. In this setting, the concept of dynamic consistency acquires a richer and precise meaning, as presented in the next section. Section 3 is devoted to classical and non-standard schemes for two examples that motivate our study. In Section 4, the analysis of topological equivalence of maps is made more explicit in the one-dimensional case. This enables us to properly revisit, in Section 5, the two examples. Concluding remarks on our future plan are made in Section 6.

2. **Topological Equivalence of maps.**

Let $D \subseteq \mathbb{R}^d$, be a domain for $d \geq 1$, and consider an initial value problem for a system of differential equations

$$\frac{dy}{dt} = f(y), \tag{1}$$

$$y(0) = x, \tag{2}$$

where $x \in D$ and $f \in C^0(D, D)$. We assume that (1) defines a (positive) dynamical system on D. This means that for every $x \in D$ the problem (1)–(2) has a unique solution $y = y(x,t) \in D$ for all $t \in [0, \infty)$. For a given $t \in (0, \infty)$, the mapping $S(t) : D \to D$ given by $S(t)(x) \to y(x,t)$ is called the *evolution operator* and the set

$$\{S(t) : t \in (0, \infty)\} \tag{3}$$

is the well-known *evolution semi-group*. For every $x \in D$ the set $\{S(t)(x) : t \in (0, \infty)\}$ is called the (positive) orbit of x.

Suppose that the solution of (1)–(2) is approximated on the time grid $\{t_k = kh : k = 0, 1, ...\}$ by a difference equation of the form

$$y_{k+1} = F(h)(y_k), \tag{4}$$
$$y_0 = x, \tag{5}$$

where the maps $F(h) : D \to D$ are defined for every $h > 0$. Hence, for every given $h > 0$, the equation (4) defines a discrete dynamical system with an evolution semi-group

$$\{(F(h))^k : k = 1, 2, ...\}. \tag{6}$$

The orbit of a point $x \in D$ is a the sequence

$$\{(F(h))^k(x) : k = 0, 1, 2, ...\}.$$

Our approach is to relate the properties of the maps in (3) and the maps in the set

$$\{F(h) : h > 0\}. \tag{7}$$

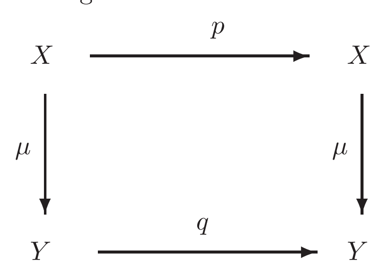
More precisely, we compare the set of continuous dynamical systems defined by the maps in (3) and the set of discrete dynamical systems defined by the maps in (7). In the spirit of the philosophy of the non-standard finite difference method, it is essential that the properties of the exact solution be preserved for all step sizes (as opposed to only for sufficiently small step sizes).

The strongest connection between dynamical systems, from topological point of view, is their topological equivalence.

Definition 2.1. Let X and Y be two topological spaces. The maps $p : X \to X$ and $q : Y \to Y$ are called *topologically equivalent* if there exists a homeomorphism $\mu : X \to Y$ such that

$$p \circ \mu = \mu \circ q \; . \tag{8}$$

In the standard literature on Topological Dynamical Systems topological equivalence is also called 'topological conjugacy' and equation (8) is referred to as the *conjugacy equation*, [10], reflecting the fact that the following diagram is commutative



We prefer to use the term topological equivalence since the respective maps are topologically indistinguishable.

Definition 2.2. The difference scheme (4) is called *topologically dynamically consistent* with the dynamical system (1), whenever all the maps in the set (3) are topologically equivalent to each other and every map in the set (7) is topologically equivalent to them. (Thus, the maps $S(t)$ and $F(h)$ are topologically the same for every $t > 0$ and $h > 0$).

An important property of the operator $S(t)$, for a given $t > 0$, is that it is homeomorphisms of D onto its image $S(t)(D)$. Equivalently, this means that $S(t)$ is continuous and injective. Certainly, if an operator $F(h)$ is required to be topologically equivalent to $S(t)$ then it should be also continuous and injective. This basic requirement for $F(h)$ is of fundamental importance as it ensures that we are comparing homeomorphisms in (3) with homeomorphisms in (7). Thus, we make the following assumptions about $F(h)$ which imply this property:

$$F \in C^1([0,\infty], C^1(D,D)), \tag{9}$$

$$\frac{dF(h)}{dx}(x) \text{ is a nonsingular matrix for each } \ h > 0,\ x \in D\ . \tag{10}$$

Naturally, F should satisfy also the usual consistency requirements

$$F(0)(x) = x, \quad \frac{dF(0)}{dh}(x) = f(x). \tag{11}$$

3. **Some Examples.** For a one-dimensional dynamical system the requirement on $F(h)$ to be continuous and injective implies that $F(h)$ is strictly increasing. More precisely, condition (10) is equivalent to

$$\frac{dF(h)}{dx}(x) > 0,\ x \in D. \tag{12}$$

This condition on $F(h)$ was discussed in [2], where it was shown that it ensures the monotonicity of the scheme with respect to initial value. It was also proved there that it is a relevant substitute for elementary stability in the case of non-hyperbolic fixed points.

We will demonstrate in the sequel that (12) is an essential ingredient of the topological dynamic consistency of one-dimensional numerical schemes. Here, we consider some well-known examples of equations and numerical schemes, where we observe that the schemes which do not replicate well the dynamical properties of the original equation also violate (12).

3.1. **The logistic equation.**

$$\frac{dy}{dt} = f(y) = y(1-y). \tag{13}$$

The exact solution of this equation is well-known and is given here in Fig. 1. This equation defines a dynamical system on $D = [0, +\infty)$ with an asymptotically stable fixed point at $y = 1$ and an unstable fixed point at $y = 0$. We consider the Forward Euler method

$$y_{k+1} = F(h, y_k) = y_k + hf(y_k), \tag{14}$$

and the second order Runge-Kutta method

$$y_{k+1} = F(h, y_k) = y_k + \frac{h}{2}(f(y_k) + f(y_k + hf(y_k))). \tag{15}$$

The numerical solutions by these methods for $h = 1.8$ are presented in Figs. 2 and 3. It is easy to see that the function F in each case violates condition (12). Hence the fact that the numerical solutions in Figs. 2 and 3 do not replicate the behavior of the exact solution in Fig. 1 is not surprising. The dynamical inconsistency of these methods is considered in [12] and other publications. However, the discussion typically relates only to the fixed points and their stability properties. Here, we can observe that for the selected value of the step size the numerical solutions correctly preserve the fixed points and their stability properties. Nevertheless, these numerical solutions can hardly be considered a qualitatively correct replica of the exact solutions (Fig. 1). In particular, one can mention that the equations (14) and (15) do not define discrete dynamical systems on $[0, +\infty)$ but only on a closed finite interval of the form $[0, M]$. Furthermore, the solutions do not have the correct monotonicity and display varying levels of oscillations. We claim that the condition (12) plays an important role in preserving the topological properties of the solutions of dynamical systems. Indeed, the non-standard schemes which are known to replicate well the properties of the exact solutions satisfy (12). In [12, Section 2.4] Mickens considered the non-standard scheme

$$\frac{y_{k+1} - y_k}{h} = y_k(1 - y_{k+1}),$$

or, equivalently,

$$y_{k+1} = F(h, y_k) = \frac{(1+h)y_k}{1 + hy_k}. \tag{16}$$

A set of numerical solutions by this scheme using the same step size $h = 1.8$ is depicted in Fig. 4, where the preservation of the topological properties is apparent. The function F associated with the scheme satisfies (12). Indeed,

$$\frac{dF(h)}{dy}(x) = \frac{1+h}{(1+hx)^2} > 0, \quad h > 0, \ x \in D.$$

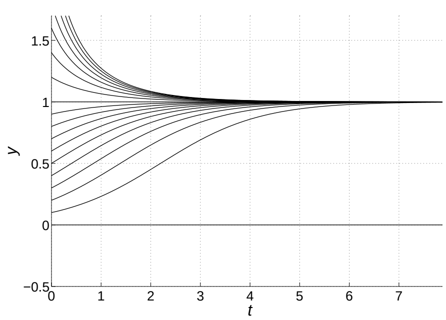


Figure 1. Exact solution

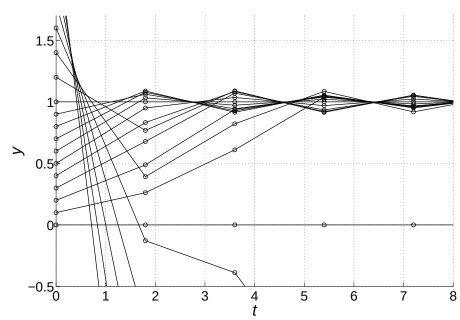


Figure 2. Forward Euler method

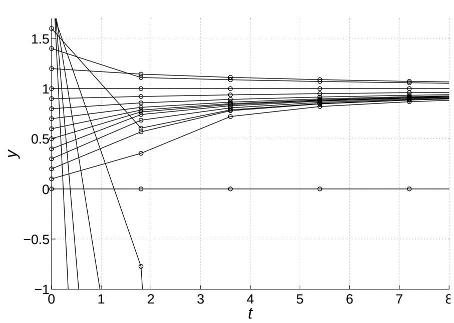


Figure 3. Runge-Kutta method

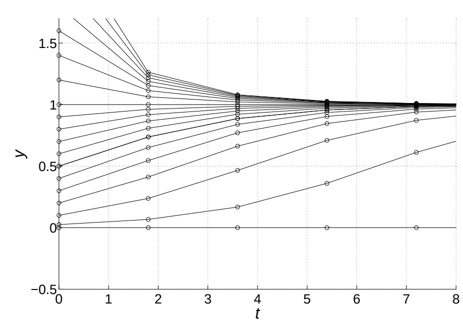


Figure 4. Mickens' non-standard method

3.2. The combustion equation.

$$\frac{dy}{dt} = f(y) = y^2(1-y). \tag{17}$$

The exact solution of this equation is given in Fig. 5. This equation defines a dynamical system on $D = (-\infty, +\infty)$ with an asymptotically stable fixed point at $y = 1$ and an unstable fixed point at $y = 0$. Note that the fixed point at $y = 0$ is not hyperbolic since $\frac{df}{dy}(0) = 0$. Therefore, the behavior of the solutions around this point cannot be described in terms of linear stability. Here $y = 0$ is attractive for solutions above it and repelling for solutions below it.

We consider the Forward Euler method (14) and the Runge-Kutta method (15). A set of numerical solutions produced by these schemes with $h = 1.7$ is given in Figs. 6 and 7. The discrepancy with the exact solutions is obvious. However, it should be also observed that the numerical solutions in both cases preserve the fixed points $y = 0$ and $y = 1$ as well as their stability. This, once again, suggests that preserving only the fixed points and their stability is not sufficient to adequately replicate the qualitative behavior of the dynamical system. It is easy to see that (12) does not hold for any of these schemes. It is seen that non-standard schemes which preserve the fixed points and their stability do not automatically satisfy (12). The numerical solutions of the combustion equation was studied in some detail in [13] and the following non-standard scheme was proposed,

$$\frac{y_{k+1} - y_k}{h} = y_k^2(1 - y_{k+1}),$$

or, equivalently,

$$y_{k+1} = F(h)(y_k) = \frac{y_k + hy_k^2}{1 + hy_k^2}. \tag{18}$$

It was proved that this scheme preserves the positivity and the boundedness of the solution as well as the asymptotic stability of the fixed point $y = 1$. It is easy to see that this scheme violates (12). As a results there are numerical solutions which merge or intersect. For example, see Fig. 8, where a set of solutions produced by this scheme with $h = 1.7$ are presented. Furthermore, while the fixed points $y = 0$, $y = 1$ and their stability are preserved, their basins of attraction are quite different from the respective basins of attraction for the solutions of the original equation. Hence, $F(h)$ in (18) is not topologically equivalent to the evolution operator $S(t)$ of (17).

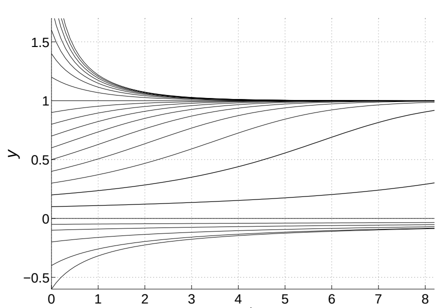


Figure 5. Exact solution

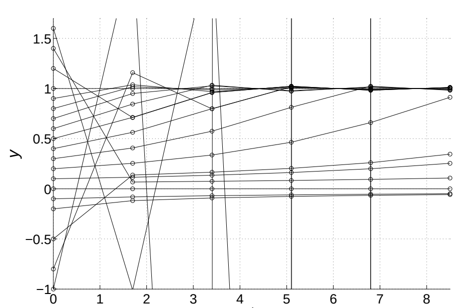


Figure 6. Forward Euler method

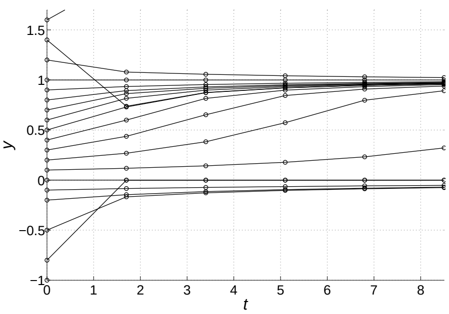


Figure 7. Runge-Kutta method

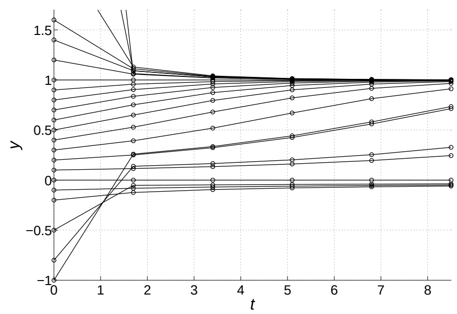


Figure 8. Mickens' non-standard method

These simple examples indicate that the difference schemes, both standard and nonstandard, which preserve the fixed points and their stability, are not necessarily topologically dynamically consistent. On the other hand, the topological dynamical consistency, that is the topological equivalence of $F(h)$, $h > 0$, and $S(t)$, $t > 0$, implies preservation (up to a homeomorphism) of *all* topological properties of the given continuous dynamical system, in particular, the fixed points and their stability.

4. **One-dimensional maps.** Here we consider the continuous dynamical system given by (1) and the discrete dynamical system given by (4), when D is an interval.

Dynamical systems defined by a single equation have attracted significant attention from the researchers working on the Nonstandard Finite Difference Method. The reason is two fold. On the one hand, since some important models in mathematical physics and in biomathematics belong to this class, there is a need for numerical schemes that replicate their essential properties. On the other hand, the one-dimensional case has relatively simple dynamics. For example, all orbits of (1) are monotone and all minimal invariant sets are fixed points; these are also the properties of the orbits of (4) whenever (9) and (10), i.e., (12) in this case, hold. Consequently, the one-dimensional case is a convenient setting for developing, demonstrating, and testing new methods and approaches, which then can be extended to multi-dimensional systems of ODEs and possibly PDEs, [12, 13, 14, 16].

The following theorem is a useful tool for proving the topological dynamic consistency of difference schemes. Its proof, omitted here for the sake of simplicity, uses techniques developed within the theory of Topological Dynamics, e.g., [10] and references therein. Indeed, the homeomorphism μ in the conjugacy equation (8), with $p = \varphi$ and $q = \psi$, is derived by a construction method used in [10, Proposition 2.1.7].

Theorem 4.1. *Let D be an interval and let $\varphi : D \to D$ and $\psi : D \to D$ be continuous injections. If*
(i) $(\varphi(x) - x)(\psi(x) - x) > 0$, $x \in \overset{o}{D}$,
(ii) $\varphi(D) = D \Longleftrightarrow \psi(D) = D$,
then φ and ψ are topologically equivalent.

5. **The examples revisited.** Let us consider again the logistic equation (13). Preserving the fixed points of the equation within a numerical scheme is typically not difficult. In fact, all the schemes considered in Section 3.1 have fixed points 0 and

1. Let us assume that we have a numerical scheme (4) for (13) which preserves its fixed points, and satisfies the conditions (9), (10), and (11). As was shown, Mickens' nonstandard scheme (16) is in this category. We apply Theorem 4.1 to prove that such a scheme is topologically dynamically consistent. The domain of the dynamical system defined by the logistic equation is $D = [0, +\infty)$ with fixed points at 0 and 1. First we apply Theorem 4.1 on $D = [0, 1]$. Since for each $t > 0$ and $h > 0$ neither $S(t)$ nor $F(h)$ have fixed points in $\overset{o}{D} = (0, 1)$ the product $(S(t)(x) - x)(F(h)(x) - x)$ has the same sign for all $x \in (0, 1)$, $t > 0$ and $h > 0$. Using the consistency (11) one can easily obtain that this sign is positive. Hence, condition (i) of Theorem 4.1 holds for the maps $S(t)$ and $F(h)$. Condition (ii) also holds, because both maps are continuous bijections on [0,1], that is $S(t)(D) = D$ and $F(h)(D) = D$. Therefore, $S(t)$ and $F(h)$ are topologically equivalent on the interval $[0, 1]$. Similarly, it is proved that $S(t)$ and $F(h)$ are topologically equivalent on the interval $[1, +\infty)$, by using the fact that in this case $S(t)(D) \subsetneq D$ and $F(h)(D) \subsetneq D$. This implies that $S(t)$ and $F(h)$ are topologically equivalent on $[0, +\infty)$, for all $t > 0$ and $h > 0$. Hence, this numerical scheme is topologically dynamically consistent. In particular, the Mickens' nonstandard scheme (16) is topologically dynamically consistent. This scheme was also considered in [2]. It was shown there that it is member of a larger family of numerical schemes given by

$$y_{k+1} = F(h)(y_k) = \frac{y_k + \phi(h)y_k - \phi(h)y_k^2}{1 + \phi(h)(1 - \alpha)y_k}, \tag{19}$$

where $\alpha \le 0$ and the real function ϕ is such that $\phi(h) = h + O(h^2)$. It was proved in [2] that under the assumptions made on α and ϕ the scheme satisfies (12), or equivalently, (10). Then, similarly to the Mickens' scheme, it follows from the above discussion that all the schemes given by (19) are topologically dynamically consistent.

Next, the combustion equation defines a dynamical system on $(-\infty, +\infty)$. Applying Theorem 4.1 on the three intervals $(-\infty, 0]$, $[0, 1]$ and $[1, +\infty)$ we obtain that every numerical scheme, which preserves the fixed points 0 and 1 and satisfies conditions (9), (10) and (11), is topologically dynamically consistent with the combustion equation. We noted in Section 3.2 that all schemes presented there satisfy these requirements, except condition (10). The following family of schemes for the combustion equation was derived in [2],

$$y_{k+1} = F(h)(y_k) = \frac{y_k + \phi(h)\alpha y_k^2 - \phi(h)\beta y_k^3}{1 + \phi(h)(\alpha - 1)y_k + \phi(h)y_k^2}, \tag{20}$$

where α and β are real parameters and $\phi(h) = h + O(h^2)$. It was proved that these schemes satisfy (12) for $\alpha \ge 1$, $\beta < -1/2$ and $0 < \phi(h) < c = -(2\beta + 1)/\alpha^2$. Therefore, under these conditions on the parameters of the family, the schemes are topologically dynamically consistent with the combustion equation. It should be noted that the Mickens' scheme (18) is obtained from (20) for $\alpha = 1$ and $\beta = 0$, that is, using values of the parameters which are outside the indicated range. Here, we propose a new scheme which does not require renormalization of the denominator:

$$\frac{y_{n+1} - y_n}{h} = y_n^2 + \frac{1 + h}{2}y_n^3 - \frac{h + 3}{2}y_n^2 y_{n+1},$$

or, equivalently,

$$y_{k+1} = F(h)(y_k) = \frac{y + hy_k^2 + \frac{1}{2}h(h+1)y_k^3}{1 + \frac{1}{2}h(h+3)y^2}. \tag{21}$$

It is easy to see that

$$\frac{dF(h,y)}{dy} = \frac{\frac{1}{4}h^2(h+1)(h+3)y^4 + (hy+1)^2}{(1+\frac{1}{2}h(h+3)y_k^2)^2} > 0.$$

Therefore, condition (10) holds, and this implies the topological dynamic consistency of the scheme. A set of numerical solutions obtained by (21) using $h = 1.7$ is given in Fig. 9, and the agreement with Fig. 5 is evident.

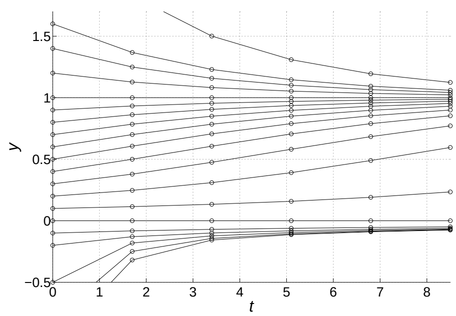


Figure 9. Nonstandard method (21)

6. **Conclusion.** The new concept of *topological dynamic consistency* was introduced in this paper and characterizes an alignment of the numerical schemes to the continuous dynamical systems they approximate, ensuring that all the topological properties of the continuous dynamical system are properly replicated. We demonstrated on two examples that topological dynamic consistency is a desirable property of the numerical schemes. Furthermore, we showed that by using the nonstandard finite difference method, one can construct schemes which are topologically dynamically consistent. In our future research, we plan to formulate and prove suitable sufficient conditions for topological dynamic consistency of the numerical schemes in the one-dimensional and the multi-dimensional cases, as well as developing a systematic approach to constructing such schemes.

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