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CHARACTERIZING THE EXISTENCE OF COEXISTENCE STATES IN A CLASS OF COOPERATIVE SYSTEMS

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Abstract. This paper characterizes the existence of coexistence states for a class of sublinear elliptic cooperative systems with a linear equation and the other non-linear, but yet linear on a subdomain of the underlying domain. The analysis of this problem is imperative for ascertaining the dynamics of wider general classes of cooperative systems with spatially heterogeneous nonlinearities, like those introduced in López-Gómez & Molina-Meyer [10] for weakly coupled systems. Our characterization relies upon a *spectral bound* associated to a certain non-local second order differential operator, which has not been previously documented in the literature.

1. **Introduction.** This paper characterizes the existence of *coexistence states* for the elliptic boundary value problem

$$\begin{cases} -\Delta u = \lambda u + \alpha v - a(x) f(x,u) u & \text{in} \quad \Omega, \\ -\Delta v = \beta u + \lambda v & \\ (u,v) = (0,0) & \text{on} \quad \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 1$, with boundary $\Gamma_1 := \partial\Omega$ of class $\mathcal{C}^{2+\nu}$ for some $\nu \in (0,1)$, Δ stands for the Laplace operator in $\mathbb{R}^N$, $\lambda \in \mathbb{R}$, $\beta > 0$ and $\alpha > 0$ are regarded as real *continuation parameters*, and $f \in \mathcal{C}^{\nu,1+\nu}(\bar{\Omega} \times [0,\infty))$ and $a \in \mathcal{C}^{\nu}(\bar{\Omega})$ are two functions satisfying

(A) $a(x) \geq 0$ for every $x \in \bar{\Omega}$, and the open set

$$\Omega_+ := \{ x \in \Omega \ : \ a(x) > 0 \} = a^{-1}((0,\infty))$$

satisfies $\bar{\Omega}_+ \subset \Omega$, with $\Gamma_2 := \partial\Omega_+$ of class $\mathcal{C}^{2+\nu}$,

and

(F) $f(x,0) = 0$ and $\partial_u f(x,u) > 0$, for all $(x,u) \in \bar{\Omega} \times (0,\infty)$, and

$$\lim_{u \uparrow \infty} f(x,u) = \infty \quad \text{uniformly in } \ x \in \bar{\Omega}_+. \tag{2}$$

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Figure 1 shows a typical situation where condition (A) is fulfilled. Throughout this paper, we are denoting

$$\Omega_0 := \Omega \setminus \bar{\Omega}_+ = \operatorname{int} a^{-1}(0).$$

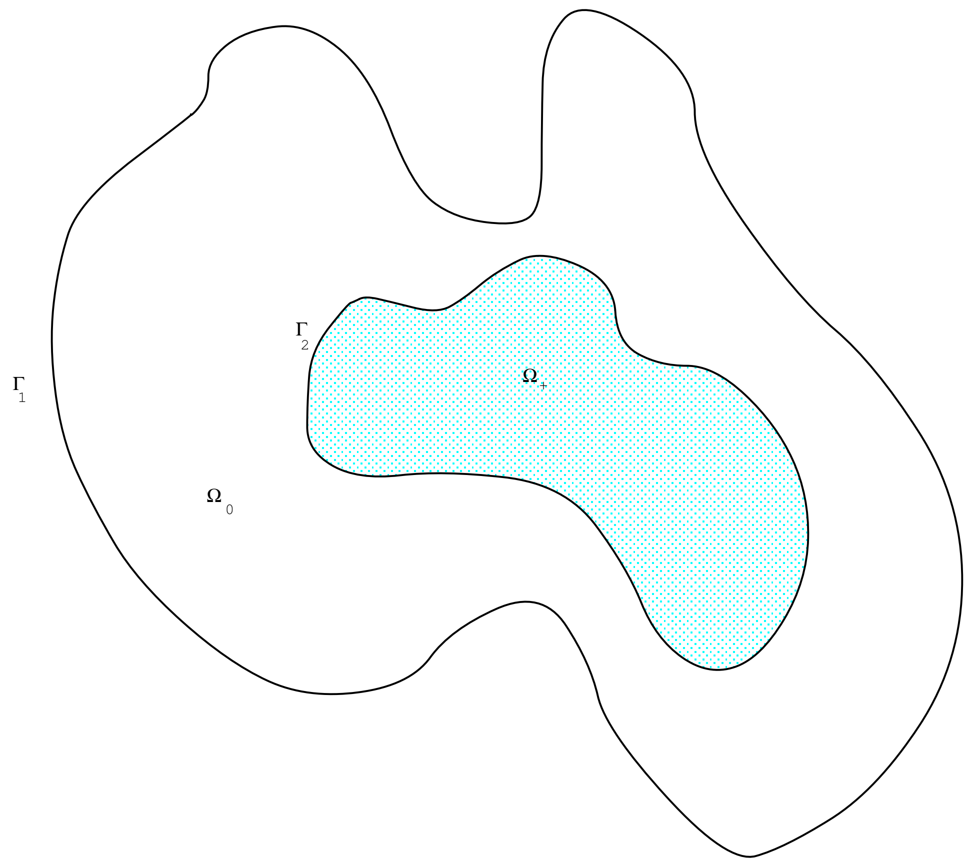

FIGURE 1. Nodal configuration of $a(x)$

The model (1) lies outside the general scope of the authors in [1], and of Molina-Meyer [11], [12], [13], for as the nonlinearities of each of the two equations do vanish in different subdomains of Ω. Indeed, although the nonlinearity of the v-equation of (1) equals zero in Ω, i.e., the v-equation is linear, the nonlinearity of the u-equation only vanishes in Ω_0, and, consequently, the assumption (A1) of [1], which was pivotal for most of the mathematical analysis carried out there in, is not longer satisfied in this paper. Here relies the main novelty of this work.

Our interest in analyzing the existence of coexistence states for (1) comes from the fact that this analysis is imperative as a first necessary step towards ascertaining the dynamics of more general classes of cooperative parabolic problems with general non-negative coefficients in front of the nonlinearities.

Now, we will introduce some useful notations used throughout this paper. For a given $h \in \mathcal{C}(\bar{\Omega})$, it is said that $h > 0$ if $h \geq 0$ but $h \neq 0$. Similarly, given

$h_1, h_2 \in \mathcal{C}(\bar{\Omega})$, it is said that $(h_1, h_2) > (0,0)$ if $h_1 \geq 0$, $h_2 \geq 0$, and $(h_1, h_2) \neq (0,0)$. Also, for every $V_1, V_2 \in \mathcal{C}^\nu(\bar{\Omega})$, we will denote

$$\mathfrak{L}(V_1, V_2) := \begin{pmatrix} -\Delta + V_1 & -\alpha \\ -\beta & -\Delta + V_2 \end{pmatrix}. \tag{3}$$

Since $\alpha > 0$ and $\beta > 0$, $\mathfrak{L}(V_1, V_2)$ is *cooperative* (as discussed by López-Gómez and Molina-Meyer [9], and Amann [5]). Therefore, for any sufficiently smooth $D \subset \Omega$, there is a unique value τ for which the linear eigenvalue problem

$$\begin{cases} \mathfrak{L}(V_1, V_2)\binom{\varphi}{\psi} = \tau\binom{\varphi}{\psi} & \text{in } \quad D, \\ (\varphi, \psi) = (0,0) & \text{on } \quad \partial D, \end{cases} \tag{4}$$

possesses a solution (φ, ψ) with $\varphi > 0$ and $\psi > 0$. Such value of τ is usually referred to as the *principal eigenvalue* of $\mathfrak{L}(V_1, V_2)$ in D (under homogeneous Dirichlet boundary conditions); in this paper, it will be denoted by

$$\sigma[\mathfrak{L}(V_1, V_2), D].$$

It is well known that $\sigma[\mathfrak{L}(V_1, V_2), D]$ is *simple* and *dominant*, in the sense that

$$\operatorname{Re}\tau > \sigma[\mathfrak{L}(V_1, V_2), D]$$

for any other eigenvalue τ of (4). Moreover, the *principal eigenfunction* (φ, ψ) is unique, up to a positive multiplicative constant, and

$$\varphi \gg 0, \qquad \psi \gg 0.$$

A function $w \in \mathcal{C}^1(\bar{D})$ is said to satisfy $w \gg 0$ (in D) if it lies in the interior of the cone of non-negative functions of $\mathcal{C}^1(\bar{D})$, i.e., if $w(x) > 0$ for all $x \in D$ and $\partial w / \partial n(x) < 0$ for all $x \in w^{-1}(0) \cap \partial D$, where $n = n(x)$ stands for the outward unit normal to D at $x \in \partial D$.

Also, throughout this paper, we set

$$\mathfrak{L}_0 := \mathfrak{L}(0,0) \qquad \sigma_1 := \sigma[-\Delta, \Omega], \tag{5}$$

and denote by $\phi_1 \gg 0$ the principal eigenfunction associated with σ_1, normalized so that, for example,

$$\max_{\bar{\Omega}} \phi_1 = 1.$$

Then, by a direct calculation, it becomes apparent that

$$\sigma[\mathfrak{L}_0, \Omega] = \sigma_1 - \sqrt{\alpha\beta} \qquad \text{and} \qquad (\varphi, \psi) = \left(\sqrt{\alpha}\, \phi_1, \sqrt{\beta}\, \phi_1\right). \tag{6}$$

But, except in this special case, the principal eigenvalue $\sigma[\mathfrak{L}(V_1, V_2), D]$ is a geometrical magnitude depending on a rather hidden way of the potentials V_1 and V_2, and of the domain D. According to (6), we also have that

$$\sigma[\mathfrak{L}_0, \Omega_0] = \sigma_1^0 - \sqrt{\alpha\beta},$$

where

$$\sigma_1^0 := \sigma[-\Delta, \Omega_0].$$

Now, we go back to (1). By the u-equation, $v = 0$ if $u = 0$, because $\alpha > 0$. Similarly, since $\beta > 0$, it follows from the v-equation that $u = 0$ if $v = 0$. Thus, (1) admits two types of non-negative solutions: the *trivial state* $(0,0)$, and the *coexistence states*; those of the form (u, v) with $u \gg 0$ and $v \gg 0$. Moreover, according to the v-equation, if (u, v) is a coexistence state of (1), then

$$(-\Delta - \lambda)v = \beta u > 0 \qquad \text{in } \ \Omega$$

and, hence, v provides us with a positive strict supersolution of the differential operator $-\Delta - \lambda$ in Ω under homogeneous Dirichlet boundary conditions. Thus, owing to López-Gómez and Molina-Meyer [9, Theorem 2.1] (cf. [8, Theorem 2.5]), the following estimate holds

$$\lambda < \sigma_1. \tag{7}$$

Consequently, $\lambda < \sigma_1$ is necessary for the existence of a coexistence state. The main result of this paper can be stated as follows.

Theorem 1.1. *Suppose $\lambda < \sigma_1$ and set*

$$\Sigma(\lambda) := \sup_{w\in\mathcal{P}} \inf_{\Omega_0} \frac{(-\Delta-\lambda)w}{(-\Delta-\lambda)^{-1}w}, \qquad \mathcal{P} := \{w \in \mathcal{C}_0^2(\bar{\Omega}) \ : \ w \gg 0\}. \tag{8}$$

Then, (1) possesses a coexistence state if and only if

$$(\sigma_1 - \lambda)^2 < \alpha\beta < \Sigma(\lambda), \tag{9}$$

and it is unique, if it exists.

The problem (1) can be formulated adopting another perspective. Indeed, suppose $\lambda < \sigma_1$. Then, we find from the v-equation of (1) that

$$v = \beta(-\Delta - \lambda)^{-1}u$$

and substituting it into the u-equation, it becomes apparent that the component u satisfies the non-local problem

$$\begin{cases} -\Delta u = \lambda u + \alpha\beta(-\Delta-\lambda)^{-1}u - a(x)f(x,u)u & \text{in } \ \Omega, \\ u = 0 & \text{on } \ \partial\Omega, \end{cases} \tag{10}$$

which can be regarded as a *non-local perturbation* of the generalized logistic boundary value problem

$$\begin{cases} -\Delta u = \lambda u - a(x)f(x,u)u & \text{in } \ \Omega, \\ u = 0 & \text{on } \ \partial\Omega, \end{cases} \tag{11}$$

by switching off to 0 the product $\alpha\beta$. As (10) admits a positive solution if and only if (1) possesses a coexistence state, invoking to Theorem 1.1, we find that (10) possesses a positive solution if and only if (9) is satisfied.

On the other hand, according to J. M. Fraile et al. [7], the unperturbed problem (11) possesses a positive solution if and only if

$$\sigma_1 < \lambda < \sigma_1^0.$$

As for such range of values of λ, (10) cannot admit a positive solution, it becomes apparent that the classical results of Brézis and Oswald [6], T. Ouyang [14], and J. M. Fraile et al. [7] are of a rather different nature than those derived from Theorem 1.1 for the nonlocal problem (10).

This paper is distributed as follows. Section 2 collects some existing results that are necessary to prove Theorem 1.1. Section 3 provides us with a proof of Theorem 1.1, and Section 4 consists of some final comments.

2. **Some preliminary and auxiliary results.** This section collects some pivotal results for the proof of Theorem 1.1. The following characterization of the strong maximum principle goes back to López-Gómez and Molina-Meyer [9] (see the discussion of Amann [5] and the references there in). Complete details of the proof for the single equation were given in [8, Section 2].

Theorem 2.1. *Suppose D is an open smooth subdomain of $\mathbb{R}^N$, $N \geq 1$, and $V_1, V_2 \in \mathcal{C}^\nu(\bar{D})$. Then, the following assertions are equivalent.*

(a) $\sigma\left[\mathfrak{L}(V_1,V_2), D\right] > 0$.

(b) *There exist $\bar{u}, \bar{v} \in \mathcal{C}^2(D) \cap \mathcal{C}(\bar{D})$ such that $\bar{u} > 0$, $\bar{v} > 0$ in D,*

$$\mathfrak{L}(V_1,V_2)\begin{pmatrix}\bar{u}\\ \bar{v}\end{pmatrix} \geq \begin{pmatrix}0\\ 0\end{pmatrix} \qquad \text{in} \quad D,$$

and either $(\bar{u},\bar{v}) > (0,0)$ on ∂D, or else

$$\mathfrak{L}(V_1,V_2)\begin{pmatrix}\bar{u}\\ \bar{v}\end{pmatrix} > \begin{pmatrix}0\\ 0\end{pmatrix} \qquad \text{in} \quad D;$$

$(\bar{u},\bar{v})$ is said to be a positive strict supersolution of $\mathfrak{L}(V_1,V_2)$ in D.

(c) *The operator $\mathfrak{L}(V_1,V_2)$ satisfies the strong maximum principle in D, in the sense that for every $F, G \in \mathcal{C}^\nu(\bar{D})$, $h_1, h_2 \in \mathcal{C}^{2+\nu}(\partial D)$, and $u, v \in \mathcal{C}^{2+\nu}(\bar{D})$ satisfying*

$$\begin{cases} \mathfrak{L}(V_1,V_2)\binom{u}{v} = \binom{F}{G} \geq \binom{0}{0} & \text{in } D,\\ (u,v) = (h_1,h_2) \geq (0,0) & \text{on } \partial D, \end{cases}$$

with some of these inequalities strict, one has that $u \gg 0$ and $v \gg 0$ in D.

From Theorem 2.1 one can easily obtain all the continuity and monotonicity properties of the principal eigenvalues with respect to the support domains and the potentials (see, if necessary, [9], [8], [2] and [5]), as well as most of the subsequent results.

Throughout the rest of this section we simply suppose that

$$f \in \mathcal{C}^{\nu,\nu}(\bar{\Omega} \times [0,\infty)) \;\text{ and }\; D \subset \Omega \;\text{ is smooth with }\; D \cap \Omega_+ \neq \emptyset. \tag{12}$$

Adapting the abstract theory of Amann [3], [4], the next consequence from Theorem 2.1 (going back to Molina-Meyer [11], [12]) is obtained.

Theorem 2.2. *Suppose f satisfies (12), $h_1, h_2 \in \mathcal{C}^{2+\nu}(\partial D)$ and*

$$\begin{cases} -\Delta u = \lambda u + \alpha v - a f(\cdot,u)u & \\ -\Delta v = \beta u + \lambda v & \text{in} \quad D,\\ (u,v) = (h_1,h_2) & \text{on} \quad \partial D, \end{cases} \tag{13}$$

has a subsolution $(\underline{u},\underline{v})$, $\underline{u},\underline{v} \in \mathcal{C}^{2+\nu}(\bar{D})$, and a supersolution $(\bar{u},\bar{v})$, $\bar{u},\bar{v} \in \mathcal{C}^{2+\nu}(\bar{D})$, such that $(\underline{u},\underline{v}) \leq (\bar{u},\bar{v})$. Then, (13) possesses a solution (u,v), $u,v \in \mathcal{C}^{2+\nu}(\bar{D})$, such that

$$(\underline{u},\underline{v}) \leq (u,v) \leq (\bar{u},\bar{v}).$$

Actually, (13) possesses a minimal and a maximal solution in $[(\underline{u},\underline{v}),(\bar{u},\bar{v})]$.

Remark 1. By Theorem 2.1, if $(\underline{u},\underline{v})$ (resp. $(\bar{u},\bar{v})$) is a strict subsolution (resp. supersolution) of (13), then any solution $(u,v) \in [(\underline{u},\underline{v}),(\bar{u},\bar{v})]$ must satisfy $\underline{u} < u \leq \bar{u}$ and $\underline{v} < v \leq \bar{v}$ (resp. $\underline{u} \leq u < \bar{u}$ and $\underline{v} \leq v < \bar{v}$).

The next result goes back to [1, Lemma 3.5]. By the discussion already carried out in Section 1, we already know that in case $h_1 = h_2 = 0$ the problem (13) admits two types of non-negative solution couples: $(0,0)$ and the coexistence states.

Lemma 2.3. *Suppose f satisfies (12),*

$$f(x,0) = 0, \quad \partial_u f(x,u) > 0, \qquad \forall\ (x,u) \in \bar{\Omega} \times (0,\infty), \tag{14}$$

$h_1, h_2 \in \mathcal{C}^{2+\nu}(\partial D)$, $(h_1,h_2) \geq (0,0)$, and $(\bar u, \bar v) \in \mathcal{C}^{2+\nu}(\bar D) \times \mathcal{C}^{2+\nu}(\bar D)$ is a supersolution of (13) with $\bar u > 0$ and $\bar v > 0$. Then, $\bar u \gg 0$ and $\bar v \gg 0$ in D; in particular, any non-negative solution $(u,v) \neq (0,0)$ of (13) satisfies $u \gg 0$ and $v \gg 0$. Moreover,

$$\lambda \leq \sigma\left[\mathfrak{L}(af(\cdot,\bar u),0)), D\right],$$

and, for every $\kappa > 1$, the pair $(\kappa \bar u, \kappa \bar v)$ also provides us with a supersolution of (13). If, in addition, $(h_1,h_2) = (0,0)$ and $(\bar u, \bar v)$ solves (13), then

$$\lambda = \sigma\left[\mathfrak{L}(af(\cdot,\bar u),0), D\right].$$

The next result goes back to [1, Theorem 3.7].

Theorem 2.4. *Suppose:*

(a) *f satisfies (12) and (14),*
(b) *(13) possesses a supersolution $(\bar u, \bar v)$ with $\bar u > 0$ and $\bar v > 0$,*
(c) *$h_1, h_2 \in \mathcal{C}^{2+\nu}(\partial D)$ satisfy $h_1 \geq 0$, $h_2 \geq 0$, and $\lambda > \sigma[\mathfrak{L}_0, D]$ if $h_1 = h_2 = 0$.*

Then, (13) has a unique coexistence state. Moreover, if we denote it by Θ, then, for any positive subsolution (resp. supersolution) $(\underline{u}, \underline{v})$ (resp. $(\bar u, \bar v)$) of (13) such that $(\underline{u}, \underline{v}) > (0,0)$ (resp. $\bar u > 0$ and $\bar v > 0$), one has that $(\underline{u}, \underline{v}) \leq \Theta$ (resp. $\Theta \leq (\bar u, \bar v)$).

Remark 2. According to [1, Lemma 3.9], under the general assumptions of Theorem 2.4, it is apparent that for any positive strict subsolution (resp. supersolution) $(\underline{u}, \underline{v})$ (resp. $(\bar u, \bar v)$) of (13) one has that $(\underline{u}, \underline{v}) \ll \Theta$ (resp. $\Theta \ll (\bar u, \bar v)$).

3. **Proof of Theorem 1.1.** The following result provides us with some necessary conditions for the existence of a coexistence state.

Proposition 1. *Suppose $a > 0$, f satisfies (14), and the problem (1) possesses a solution $(u,v) > (0,0)$. Then, $u \gg 0$, $v \gg 0$, and*

$$0 < \sigma_1 - \lambda < \sqrt{\alpha\beta}. \tag{15}$$

If, in addition, $a(x)$ satisfies (A) (see the beginning of Section 1), then

$$0 < \sigma_1 - \lambda < \sqrt{\alpha\beta} \leq \sqrt{\Sigma(\lambda)}, \tag{16}$$

where $\Sigma(\lambda)$ is the spectral bound defined in (8).

Proof. Suppose $a > 0$ and f satisfies (14). Let $(u,v) > (0,0)$ a solution of (1). Then, according to Lemma 2.3, we have that $u \gg 0$ and $v \gg 0$. Moreover,

$$\begin{pmatrix} -\Delta + af(\cdot,u) & -\alpha \\ -\beta & -\Delta \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} u \\ v \end{pmatrix}$$

and, hence, by the uniqueness of the principal eigenvalue,

$$\lambda = \sigma[\mathfrak{L}(af(\cdot,u),0),\Omega],$$

as it has been already established by Lemma 2.3. As $af(\cdot,u) > 0$, we find from (6) and the monotonicity of the principal eigenvalue with respect to the potential that

$$\lambda > \sigma[\mathfrak{L}(0,0),\Omega] = \sigma[\mathfrak{L}_0,\Omega] = \sigma_1 - \sqrt{\alpha\beta}.$$

Moreover, since $v = 0$ on $\partial\Omega$ and

$$(-\Delta - \lambda)v = \beta u > 0 \qquad \text{in } \Omega,$$

it follows from Theorem 2.1 that

$$0 < \sigma[-\Delta - \lambda, \Omega] = \sigma_1 - \lambda,$$

which completes the proof of (15).

Once we know that $\lambda < \sigma_1$, it can be inferred from the v-equation of (1) that

$$v = \beta(-\Delta - \lambda)^{-1} u$$

and, hence, substituting it into the u-equation, we are driven to

$$(-\Delta - \lambda)u = \alpha\beta(-\Delta - \lambda)^{-1} u - af(\cdot, u)u.$$

Therefore,

$$(-\Delta - \lambda)u = \alpha\beta(-\Delta - \lambda)^{-1} u \qquad \text{in } \Omega_0,$$

because $a = 0$ in Ω_0, and, hence,

$$\mathfrak{J}(u) = \inf_{\Omega_0} \frac{(-\Delta - \lambda)u}{(-\Delta - \lambda)^{-1} u} = \alpha\beta.$$

Consequently,

$$\alpha\beta \leq \Sigma(\lambda). \tag{17}$$

Clearly, (16) is a consequence from (15), (17). ☐

The next lemma will show that, actually, Proposition 1 can be refined up to substitute condition (16) by the next one

$$0 < \sigma_1 - \lambda < \sqrt{\alpha\beta} < \sqrt{\Sigma(\lambda)} \tag{18}$$

in its statement.

Lemma 3.1. *Suppose $a > 0$ in Ω, $f \in \mathcal{C}^{\nu,1+\nu}(\bar{\Omega} \times [0, \infty))$ satisfies (14), and (1) possesses a coexistence state. Then, there exists $\varepsilon > 0$ such that the perturbed problem*

$$\begin{cases} -\Delta u = \lambda u + (\alpha + t)v - af(\cdot, u)u & \\ -\Delta v = \beta u + \lambda v & \text{in } \Omega, \\ (u, v) = (0, 0) & \text{on } \partial\Omega. \end{cases} \tag{19}$$

has a coexistence state for every $t \in [0, \varepsilon)$.

Proof. It consists of a simple application of the implicit function theorem based on the fact that any coexistence state of (1) is *non-degenerate*. Let (u_0, v_0) be a coexistence state of (1) and consider the operator

$$\mathfrak{F} : E := \mathcal{C}_0^{2+\nu}(\bar{\Omega}) \times \mathcal{C}_0^{2+\nu}(\bar{\Omega}) \times \mathbb{R} \longrightarrow F := \mathcal{C}^{\nu}(\bar{\Omega}) \times \mathcal{C}^{\nu}(\bar{\Omega})$$

defined by

$$\mathfrak{F}(u, v, t) := \begin{pmatrix} -\Delta u - \lambda u - (\alpha + t)v + af(\cdot, u)u \\ -\Delta v - \lambda v - \beta u \end{pmatrix}, \qquad (u, v, t) \in E. \tag{20}$$

The map $\mathfrak{F}$ is of class $\mathcal{C}^1$ and, by definition,

$$\mathfrak{F}(u_0, v_0, 0) = 0.$$

Moreover, the differential operator

$$D_0\mathfrak{F} := D_{(u,v)}\mathfrak{F}(u_0, v_0, 0) \in \mathcal{L}\left(\mathcal{C}_0^{2+\nu}(\bar{\Omega}) \times \mathcal{C}_0^{2+\nu}(\bar{\Omega}); \mathcal{C}^{\nu}(\bar{\Omega}) \times \mathcal{C}^{\nu}(\bar{\Omega})\right)$$

is given by

$$\begin{aligned} D_0\mathfrak{F}\binom{u}{v} &= \begin{pmatrix} -\Delta - \lambda + a\partial_u f(\cdot, u_0)u_0 + af(\cdot, u_0) & -\alpha \\ -\beta & -\Delta - \lambda \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \\ &= \mathfrak{L}\left(-\lambda + a\partial_u f(\cdot, u_0)u_0 + af(\cdot, u_0), -\lambda\right) \begin{pmatrix} u \\ v \end{pmatrix}, \end{aligned}$$

where $\mathfrak{L}(\cdot,\cdot)$ stands for the linear *cooperative operator* defined in (3). According to assumption (A), (14), and Lemma 2.3, we find from the monotonicity of the principal eigenvalue with respect to the potential that

$$\begin{aligned}\sigma[\mathfrak{L}\,(-\lambda\ &+a\partial_u f(\cdot,u_0)u_0+af(\cdot,u_0),-\lambda)\,,\Omega]\\ &=\sigma[\mathfrak{L}\,(a\partial_u f(\cdot,u_0)u_0+af(\cdot,u_0),0)\,,\Omega]-\lambda\\ &>\sigma[\mathfrak{L}\,(af(\cdot,u_0),0)\,,\Omega]-\lambda=0.\end{aligned}$$

Therefore, owing to Theorem 2.1, the linearized operator $D_0\mathfrak{F}$ is an isomorphism with strong positive inverse, and, consequently, thanks to the implicit function theorem, there exist $\varepsilon>0$ and two maps of class $\mathcal{C}^1$

$$U,V:(-\varepsilon,\varepsilon)\mapsto \mathcal{C}_0^{2+\nu}(\bar\Omega)\times\mathcal{C}_0^{2+\nu}(\bar\Omega)$$

such that

$$U(0)=u_0,\qquad V(0)=v_0,$$

and

$$\mathfrak{F}(U(t),V(t),t)=0\qquad\text{for every }\ t\in(-\varepsilon,\varepsilon).$$

As u_0 and v_0 lie in the interior of the cone of positive functions of the ordered Banach space $\mathcal{C}_0^1(\bar\Omega)$, it becomes apparent that $(U(t),V(t))$ is a coexistence state of (19) for sufficiently small $t>0$. This completes the proof. □

According to (16), if (18) fails, then

$$\alpha\beta=\Sigma(\lambda).$$

On the other hand, according to Proposition 1 and Lemma 3.1, we must have

$$(\alpha+t)\beta\le\Sigma(\lambda)\qquad\forall\ \ t\in[0,\varepsilon).$$

This contradiction shows that $\alpha\beta<\Sigma(\lambda)$ and, hence, (18) is indeed necessary for the existence of a coexistence state of (1).

According to Theorem 1.1, $\lambda<\sigma_1$ and (18) are not only necessary but also sufficient for the existence of a coexistence state of (1). Moreover, the coexistence state is unique if it exists. By Proposition 1 and Theorem 2.4, to complete the proof of Theorem 1.1 it suffices to show that under condition

$$0<\sigma_1-\lambda<\sqrt{\alpha\beta}<\sqrt{\Sigma(\lambda)},$$

(1) possesses a supersolution $(\bar u,\bar v)$ with $\bar u>0$ and $\bar v>0$. Note that the estimate $\lambda<\sigma_1$ entails $\Sigma(\lambda)$ to be well defined, and that $\sigma_1-\lambda<\sqrt{\alpha\beta}$ can be equivalently written as

$$\lambda>\sigma[\mathfrak{L}_0,\Omega]=\sigma_1-\sqrt{\alpha\beta};$$

consequently, it implies that $(0,0)$ is linearly unstable. In such case, (1) admits arbitrarily small positive subsolutions (cf. [1, Section 3] for further details). The existence of the supersolution is guaranteed provided

$$\alpha\beta<\Sigma(\lambda).\tag{21}$$

Indeed, due to (21), there exist $u\gg 0$ in Ω and $\varepsilon>0$ such that

$$\inf_{\Omega_0}\frac{(-\Delta-\lambda)u}{(-\Delta-\lambda)^{-1}u}>\alpha\beta+\varepsilon.\tag{22}$$

Subsequently, for sufficiently small $\delta>0$, we set

$$\Omega_\delta:=\{\,x\in\Omega\ :\ \operatorname{dist}(x,\Omega_0)<\delta\,\}.$$

By continuity, it follows from (22) that there exists $\delta > 0$ such that

$$(-\Delta - \lambda)u \geq \alpha\beta(-\Delta - \lambda)^{-1}u \qquad \text{in } \Omega_\delta.$$

Thus, for every $\kappa > 0$, we have that

$$\begin{aligned}(-\Delta - \lambda)(\kappa u) &\geq \alpha\beta(-\Delta - \lambda)^{-1}(\kappa u)\\ &\geq \alpha\beta(-\Delta - \lambda)^{-1}(\kappa u) - af(\cdot, \kappa u)\kappa u \qquad \text{in } \Omega_\delta,\end{aligned}$$

because

$$af(\cdot, \kappa u)\kappa u \geq 0.$$

Moreover, according to (2) and (A), there exists $\kappa_0 > 0$ such that

$$(-\Delta - \lambda)u \geq \alpha\beta(-\Delta - \lambda)^{-1}u - af(\cdot, \kappa u)u \qquad \text{in } \bar{\Omega}_+ \setminus \Omega_\delta$$

for all $\kappa \geq \kappa_0$, because $a(x)$ is positive and bounded away from zero in $\bar{\Omega}_+ \setminus \Omega_\delta$. Therefore, the component-wise positive pair

$$(\bar{u}, \bar{v}) := \kappa_0 \left(u, \beta(-\Delta - \lambda)^{-1}u\right)$$

provides us with the required supersolution of (1) in Ω. This concludes the proof of Theorem 1.1.

4. **Final comments.**

As a direct consequence from (8), one has that

$$\Sigma(\lambda) = \sup_{u \in (-\Delta-\lambda)^{-1}(\mathcal{P})} \inf_{\Omega_0} \frac{(-\Delta - \lambda)^2 u}{u}.$$

On the other hand, it is well known that

$$(\sigma_1^0 - \lambda)^2 = \sup_{u \in \mathcal{P}} \inf_{\Omega_0} \frac{(-\Delta - \lambda)^2 u}{u} = \sigma[(-\Delta - \lambda)^2, \Omega_0]$$

(see [8, Theorem 3.1] and the references there in). As for obtaining $\Sigma(\lambda)$ one has to maximize

$$\inf_{\Omega_0} \frac{(-\Delta - \lambda)^2 u}{u}$$

among all the functions u of the form

$$u = (-\Delta - \lambda)^{-1}w,$$

for some $w \gg 0$, necessarily

$$-\Delta u = \lambda u + w > 0 \qquad \text{if } \lambda \geq 0,$$

and, therefore, the functions u_k, $k \geq 1$, of any maximizing sequence approximating $\Sigma(\lambda)$ must be concave. Consequently, they cannot approach any positive smooth extension approximating the principal eigenfunction associated to σ_1^0. So, the next estimate should hold

$$\Sigma(\lambda) < (\sigma_1^0 - \lambda)^2.$$

As the proof of Theorem 1.1 is based upon the method of sub and supersolutions, one can easily obtain the dynamics of the parabolic counterpart of (1),

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u = \lambda u + \alpha v - a(x)f(x,u)u & \text{in } \Omega \times (0,\infty), \\ \frac{\partial v}{\partial t} - \Delta v = \beta u + \lambda v & \\ (u, v) = (0, 0) & \text{on } \partial\Omega \times (0,\infty), \\ (u(\cdot,0), v(\cdot,0)) = (u_0, v_0) > (0,0) & \text{in } \Omega, \end{cases} \tag{23}$$

at least, in the special case when $\alpha\beta < \Sigma(\lambda)$. Indeed, under condition (9), the unique coexistence state of (1) must be a global attractor for (23), while if

$$\lambda \leq \sigma_1 - \sqrt{\alpha\beta},$$

then $(0,0)$ is a global attractor. When

$$\alpha\beta \geq \Sigma(\lambda),$$

the solution of (23) grows-up to infinity as $t \uparrow \infty$.

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