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ANOSOV AUTOMORPHISMS OF NILPOTENT LIE ALGEBRAS

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ABSTRACT. Each matrix A in $GL_n(\mathbb{Z})$ naturally defines an automorphism f of the free r-step nilpotent Lie algebra $\mathfrak{f}_{n,r}$. We study the relationship between the matrix A and the eigenvalues and rational invariant subspaces for f. We give applications to the study of Anosov automorphisms.

1. INTRODUCTION

1.1. **Anosov maps.** Anosov maps are fundamental objects in the field of dynamical systems. A C^1 diffeomorphism f of a compact Riemannian manifold M is called an *Anosov diffeomorphism* if there exist constants λ in $(0,1)$ and $c > 0$ along with a df-invariant splitting $TM = E^s \oplus E^u$ of the tangent bundle of M such that for all $n \geq 0$,

$$\|df_x^n \boldsymbol{v}\| \leq c\lambda^n \|\boldsymbol{v}\| \quad \text{for all } \boldsymbol{v} \text{ in } E^s(x), \text{ and}$$
$$\|df_x^{-n} \boldsymbol{v}\| \leq c\lambda^n \|\boldsymbol{v}\| \quad \text{for all } \boldsymbol{v} \text{ in } E^u(x).$$

The standard example of an Anosov map is a toral map defined by a unimodular hyperbolic automorphism of $\mathbb{R}^n$ that preserves an integer lattice.

All of the other known examples of Anosov maps arise from automorphisms of nilpotent groups. A hyperbolic automorphism of a simply connected nilpotent Lie group N that fixes a torsion-free lattice $\Gamma < N$ descends to an Anosov diffeomorphism of the compact nilmanifold N/Γ. It is also possible that such an N/Γ has a finite quotient, called an *infranilmanifold*, and that the Anosov map on N/Γ finitely covers an Anosov map of the infranilmanifold.

An automorphism of a Lie algebra that descends to an Anosov map of a compact quotient of the corresponding Lie group is called an *Anosov automorphism*. Nilpotent Lie algebras are the only Lie algebras that admit Anosov automorphisms. A Lie algebra $\mathfrak{n}$ is called *Anosov* if there exists a basis $\mathscr{B}$ for $\mathfrak{n}$ with rational structure constants and there exists a hyperbolic automorphism f of $\mathfrak{n}$ with respect to which f is represented relative to $\mathscr{B}$ by a matrix in $GL_n(\mathbb{Z})$. A simply connected Lie group admits a hyperbolic automorphism preserving a lattice if and only if its Lie algebra is Anosov [1].

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In this paper we study the properties of Anosov automorphisms and Anosov Lie algebras. There has already been some progress in this area. S. G. Dani showed that the free r-step nilpotent Lie algebra $\mathfrak{f}_{n,r}$ on n generators admits an Anosov automorphism when $r < n$ [4]. Dani and Mainkar considered when two-step nilpotent Lie algebras defined by graphs admit Anosov automorphisms [7]. Real Anosov Lie algebras and all their rational forms have been classified in dimension eight and less [14]. Lauret observed that the classification problem for Anosov Lie algebras contains within it the problem of classifying all Lie algebras admitting $\mathbb{Z}^+$ derivations [13, 12]. See also [6, 5, 16, 21, 15] and [20] for other recent results.

In [1], Auslander and Scheuneman established the correspondence between Anosov automorphisms of nilpotent Lie algebras and semisimple hyperbolic automorphisms of free nilpotent Lie algebras preserving ideals of a certain type. A matrix A in $GL_n(\mathbb{Z})$, together with a rational basis $\mathscr{B}$ of $\mathfrak{f}_{n,r}$, induces an automorphism f^A of $\mathfrak{f}_{n,r}$. Suppose that $\mathfrak{i}$ is an ideal of $\mathfrak{f}_{n,r}$ such that

1. $\mathfrak{i}$ is invariant under f^A,
2. the restriction of f^A to $\mathfrak{i}$ is unimodular,
3. $\mathfrak{i}$ has a basis that consists of $\mathbb{Z}$-linear combinations of elements of $\mathscr{B}$, and
4. all eigenspaces for f^A for eigenvalues with modulus one are contained in $\mathfrak{i}$.

If we let $\mathfrak{n} = \mathfrak{f}_{n,r}/\mathfrak{i}$ and let $p\colon \mathfrak{f}_{n,r} \to \mathfrak{n}$ be the projection map, there is an Anosov automorphism $\bar{f}\colon \mathfrak{n} \to \mathfrak{n}$ such that $\bar{f}p = pf^A$. We will call the four conditions the *Auslander–Scheuneman conditions*. Auslander and Scheuneman showed that any semisimple Anosov automorphism f of an r-step nilpotent Lie algebra $\mathfrak{n}$ may be represented in the manner just described, relative to a rational basis $\mathscr{B}$ of a free nilpotent Lie algebra $\mathfrak{f}_{n,r}$, a semisimple matrix A in $GL_n(\mathbb{Z})$, and an ideal $\mathfrak{i}$ in $\mathfrak{f}_{n,r}$ satisfying the four conditions. We will always assume without loss of generality that $\mathfrak{i} < [\mathfrak{f}_{n,r}, \mathfrak{f}_{n,r}]$.

In order to understand general properties of Anosov Lie algebras, one must first understand the kinds of ideals of free nilpotent Lie algebras that satisfy the Auslander–Scheuneman conditions for some automorphism f^A defined by a matrix $A \in GL_n(\mathbb{Z})$. The dynamical properties of a toral Anosov automorphism of $\mathbb{R}^n/\mathbb{Z}^n$ are closely related to the algebraic properties of the characteristic polynomial p of the matrix A in $GL_n(\mathbb{Z})$ used to define the automorphism (see [8]). We show in this work that, similarly, the algebraic properties of the characteristic polynomial p of the matrix A defining the automorphism f^A of a free nilpotent Lie algebra determine the structure of the ideals that satisfy the Auslander–Scheuneman conditions for f^A.

1.2. **Summary of results.**

Now we summarize the main ideas of the paper. We associate to any automorphism $f^A, A \in GL_n(\mathbb{Z})$, of a free r-step nilpotent Lie algebra $\mathfrak{f}_{n,r}$ an r-tuple of polynomials $(p_1, p_2, \ldots, p_r)$. We decompose $\mathfrak{f}_{n,r}$ into the direct sum of invariant subspaces $V_1, V_2, \ldots, V_r$, where, for $i = 1, \ldots, r$, the subspace V_i is spanned by i-fold brackets of elements from a set of n generators for $\mathfrak{f}_{n,r}$. For $i = 1, \ldots, r$, the polynomial p_i is the characteristic polynomial of a matrix representing the restriction of the automorphism f^A to the subspace V_i of $\mathfrak{f}_{n,r}$. Let K denote the splitting field of p_1 over $\mathbb{Q}$, and let G denote the Galois group for K over $\mathbb{Q}$. We associate to the automorphism the action of the finite group G on $\mathfrak{f}_{n,r}(K)$, the free nilpotent Lie algebra over K. We show in Theorem 5.1 that G-orbits in $\mathfrak{f}_{n,r}(K)$ correspond to rational invariant subspaces for f^A, and the characteristic polynomial for the restriction of f^A to such an invariant subspace is a power of an irreducible polynomial.

We analyze Anosov Lie algebras using the following general approach. We fix a free r-step nilpotent Lie algebra $\mathfrak{f}_{n,r}$. We consider the class of automorphisms of $\mathfrak{f}_{n,r}$ whose associated polynomial p_1 has Galois group G, where G is isomorphic to a subgroup of the symmetric group S_n. We let $(p_1, p_2, \ldots, p_r)$ be the r-tuple of polynomials associated to such a polynomial p_1. Our goal is to determine the factorizations of the polynomials $p_1, \ldots, p_r$; this will tell us what rational invariant subspaces for f are. Such subspaces generate any ideal satisfying the Auslander–Scheuneman conditions. First, we analyze the factorizations of $p_2, \ldots, p_r$ into powers of irreducibles by understanding orbits of the action of the Galois group of p_1 on $\mathfrak{f}_{n,r}(K)$. Then we determine whether the corresponding rational invariant subspaces are minimal and whether there are eigenvalues of modulus one using ideas from number theory (see Proposition 3.6 and Lemma 3.8).

We extend the classification of Anosov Lie algebras to some new classes of two-step Lie algebras.

THEOREM 1.1. *Suppose that $\mathfrak{n}$ is a two-step Anosov Lie algebra of type (n_1, n_2) with associated polynomials (p_1, p_2). Let G denote the Galois group of p_1.*

1. *If $n_1 = 3, 4$ or 5, then $\mathfrak{n}$ is one of the Anosov Lie algebras listed in Table 3.*
2. *If p_1 is irreducible and the action of G on the roots of p_1 is doubly transitive, then $\mathfrak{n}$ is isomorphic to the free nilpotent Lie algebra $\mathfrak{f}_{n,2}$.*

We can also classify Anosov Lie algebras admitting automorphisms whose polynomials p_1 have certain specified Galois groups.

THEOREM 1.2. *Let $\bar{f}$ be a semisimple Anosov automorphism of an r-step Anosov Lie algebra. Let $(p_1, \ldots, p_r)$ be the r-tuple of polynomials associated to the automorphism f of the free nilpotent Lie algebra $\mathfrak{f}_{n,r}$ induced by $\bar{f}$. Suppose that p_1 is irreducible.*

1. *If the polynomial p_1 is of prime degree with cyclic Galois group, then $\mathfrak{n}$ is one of the Lie algebras of type C_n defined over $\mathbb{R}$ as in Definition 4.5. Conversely, if n is prime, and $\mathfrak{i}$ is an ideal of $\mathfrak{f}_{n,r}$ of cyclic type defined over $\mathbb{R}$ containing the ideal $\mathfrak{j}_{n,r}$ in Definition 5.3, then the Lie algebra $\mathfrak{n} = \mathfrak{f}_{n,r}/\mathfrak{i}$ is Anosov.*
2. *If the Galois group of p_1 is symmetric, then*
   - (a) *If $r = 2$, then $\mathfrak{n}$ is isomorphic to $\mathfrak{f}_{n,2}$,*
   - (b) *If $r = 3$, then $\mathfrak{n}$ is isomorphic to one of the following five Lie algebras: $\mathfrak{f}_{n,3}$, $\mathfrak{f}_{n,3}/F_1$, $\mathfrak{f}_{n,3}/F_2$, $\mathfrak{f}_{n,3}/(F_1 \oplus F_{2a})$, or $\mathfrak{f}_{n,3}/F_{2a}$, where the ideals F_1 and F_2 are as defined in Equation (1) of Section 3.2 and F_{2a} is as in Proposition 5.5.*

Matrices in $GL_n(\mathbb{Z})$ having characteristic polynomial with symmetric Galois group are dense in the sense of thick and thin sets [24]; hence, the second part of the previous theorem describes Anosov automorphisms of two- and three-step Lie algebras that are generic in this sense.

We investigate some general properties of Anosov automorphisms. We can describe the dimensions of minimal nontrivial rational invariant subspaces. Out of such analyses we obtain the following special case.

THEOREM 1.3. *Suppose that* $\mathfrak{n}$ *is an Anosov Lie algebra of type* $(n_1,\ldots,n_r)$. *If* $n_1 = 3$, *then* n_i *is a multiple of* 3 *for all* $i = 2,\ldots,r$, *and if* $n_1 = 4$, *then* n_i *is even for all* $i = 2,\ldots,r$. *If* n_1 *is prime and the polynomial* p_1 *is irreducible, then* n_1 *divides* n_i *for all* $i = 2,\ldots,r < n$.

The results of [15] give an alternate proof of this theorem.

One way to approach the classification problem is to fix the field in which the spectrum of an Anosov automorphism lies. The following theorem describes all Anosov automorphisms whose spectrum lies in a quadratic extension of $\mathbb{Q}$.

THEOREM 1.4. *Let* f *be a semisimple Anosov automorphism of a two-step nilpotent Lie algebra* $\mathfrak{n}$. *Let* $\Lambda \subset \mathbb{R}$ *denote the spectrum of* f, *and let* K *denote the finite extension* $\mathbb{Q}(\Lambda)$ *of* $\mathbb{Q}$. *If* K *is a quadratic extension of* $\mathbb{Q}$, *then* $\mathfrak{n}$ *is one of the Anosov Lie algebras defined in Definition 7.1.*

The paper is organized as follows. In Section 2, we review background material on nilpotent Lie algebras, Anosov automorphisms and algebraic numbers, and we define the r-tuple of polynomials associated to an Anosov automorphism of a Lie algebra. In Section 3, we describe properties of the r-tuple of polynomials, such as their reducibility, and their Galois groups. In Proposition 3.6, we consider the set of roots of an Anosov polynomial and describe multiplicative relationships among them; this number-theoretic result may be interesting in its own right. In Section 4, we associate to an automorphism f of a free nilpotent Lie algebra $\mathfrak{f}_{n,r}$ the action of a Galois group G, and in Theorem 5.1 we relate rational invariant subspaces of $\mathfrak{f}_{n,r}$ to the orbits of G. In Section 6, we consider Anosov Lie algebras for which the associated Galois group is symmetric or cyclic. Finally, in Section 7, we apply the results from previous sections to the problem of classification of Anosov Lie algebras whose associated polynomial p_1 has small degree. Although the theorems we have stated above follow from various results distributed throughout this work, for the sake of clarity, in Section 8 we provide self-contained proofs of the theorems.

2. PRELIMINARIES

In this section, we describe the structure of free nilpotent Lie algebras and their automorphisms, and we review some concepts from number theory that we will use later. We conclude with some examples to illustrate the concepts presented.

2.1. **Nilpotent Lie algebras.** Let $\mathfrak{n}$ be a Lie algebra defined over field K. The *central descending series* for $\mathfrak{n}$ is defined by $\mathfrak{n}^0 = \mathfrak{n}$, and $\mathfrak{n}^i = [\mathfrak{n}, \mathfrak{n}^{i-1}]$ for $i \geq 1$. If $\mathfrak{n}^r = 0$ and $\mathfrak{n}^{r-1} \neq 0$, then $\mathfrak{n}$ is said to be *r-step nilpotent*. When $\mathfrak{n}$ is a nilpotent Lie algebra defined over field K and n_i is the dimension of the vector space $\mathfrak{n}^i/\mathfrak{n}^{i-1}$ over K, then $(n_1, n_2, \dots, n_r)$ is called the *type* of $\mathfrak{n}$.

The *free r-step nilpotent Lie algebra on n generators over the field K*, denoted $\mathfrak{f}_{n,r}(K)$, is defined to be the quotient algebra $\mathfrak{f}_n(K)/\mathfrak{f}_n^{r+1}(K)$, where $\mathfrak{f}_n(K)$ is the free nilpotent Lie algebra on n generators over K. Given a set $\mathscr{B}_1$ of n generators, the free nilpotent Lie algebra $\mathfrak{f}_{n,r}(K)$ can be written as the direct sum $V_1(K) \oplus \cdots \oplus V_r(K)$, where $V_1(K)$ is defined to be the span of $\mathscr{B}_1$ over K and for $i = 2, \dots, r$, the subspace $V_i(K)$ is defined to be the span over K of i-fold brackets of the generators. We will call the space $V_i(K)$ the *ith step* of $\mathfrak{f}_{n,r}(K)$ without always explicitly mentioning the dependence on $\mathscr{B}_1$. When the field K has characteristic zero, we identify the prime subfield of K with $\mathbb{Q}$. For our purposes, fields that we consider will be intermediate to $\mathbb{Q}$ and $\mathbb{C}$: one of $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, or the splitting field for a polynomial in $\mathbb{Z}[x]$. We will always assume that a generating set $\mathscr{B}_1$ for a free nilpotent Lie algebra $\mathfrak{f}_{n,r}(K)$ has cardinality n.

The most natural basis to use for a free nilpotent Lie algebra is a Hall basis. Let $\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_n$ be n generators for $\mathfrak{f}_{n,r}(K)$. We call these the *standard monomials of degree one*. *Standard monomials of degree n* are defined inductively: after the monomials of degree $n-1$ and less have been defined, we define an order relation $<$ on them: if degree $u <$ degree v, then $u < v$. Any linear combination of monomials of degree i will be said to be of degree i. If u has degree i and v has degree j, and $i + j = k$, we define $[u, v]$ to be a standard monomial of degree k if u and v are standard monomials and $u > v$, and if $u = [x, y]$ is the form of the standard monomial u, then $v \geq y$. The standard monomials of degree r or less form a basis for $\mathfrak{f}_{n,r}(K)$, called the *Hall basis* [10]. For $i = 1, \dots, r$, the subset $\mathscr{B}_i = \mathscr{B} \cap V_i(K)$ of the basis $\mathscr{B}$ is a basis for the ith step $V_i(K)$ of $\mathfrak{f}_{n,r}(K)$ consisting of elements of the Hall basis of degree i. To each monomial of degree i, we can also associate a *Hall word* of length i from a given alphabet $\alpha_1, \dots, \alpha_n$ of n letters; for example, $[[\boldsymbol{x}_3, \boldsymbol{x}_1], \boldsymbol{x}_2]$ becomes the word $\alpha_3\alpha_1\alpha_2$.

Suppose $\mathfrak{g}$ is a Lie algebra defined over a field K of characteristic zero. Suppose that $\mathscr{B}$ is a basis of $\mathfrak{g}$ having rational structure constants. The basis $\mathscr{B}$ determines a *rational structure* on $\mathfrak{g}$. A subspace E of $\mathfrak{g}$ spanned by $\mathbb{Q}$-linear combinations of elements of $\mathscr{B}$ is called a *rational subspace* for this rational structure. Since the structure constants for the free nilpotent Lie algebra $\mathfrak{f}_{n,r}(K)$ relative to a Hall basis $\mathscr{B}$ are rational, a Hall basis $\mathscr{B}$ for $\mathfrak{f}_{n,r}(K)$ defines a rational structure on $\mathfrak{f}_{n,r}(K)$.

EXAMPLE 2.1. Let $\mathscr{C}_1 = \{\boldsymbol{z}_i\}_{i=1}^n$ be a set of n generators for the free r-step nilpotent Lie algebra $\mathfrak{f}_{n,r}(K)$ on n generators over a field K and let $\mathscr{C} = \bigcup_{i=1}^r \mathscr{C}_i$ be the Hall basis determined by $\mathscr{C}_1$. Elements of $\mathscr{C}_2$, where $r \geq 2$, are of the form $[\boldsymbol{z}_i, \boldsymbol{z}_j]$ with $i > j$, hence the dimension of $V_2(K)$ over K is $\binom{n}{2}$. When $r \geq 3$, from the definition of Hall monomials, elements in the set $\mathscr{C}_3$ for $\mathfrak{f}_{n,r}(K)$ are of the form $[[\boldsymbol{z}_i, \boldsymbol{z}_j], \boldsymbol{z}_i]$ or $[[\boldsymbol{z}_i, \boldsymbol{z}_j], \boldsymbol{z}_j]$ with $i > j$ or, if $n \geq 3$, of the form $[[\boldsymbol{z}_i, \boldsymbol{z}_j], \boldsymbol{z}_k]$ with

i, j, k distinct and i and k greater than j. There are $n(n-1)$ standard Hall monomials of the first type, and when $n \geq 3$, there are $2\binom{n}{3}$ standard Hall monomials of the second type, for a dimensional total of $\frac{1}{3}(n+1)n(n-1)$ for the third step $V_3(K)$ of $\mathfrak{f}_{n,r}(K)$. We let $\mathscr{C}_3'$ denote the set of standard Hall monomials of the first type, and let $\mathscr{C}_3''$ denote the set of standard Hall monomials of the second type:

$$\begin{aligned} \mathscr{C}_3' &= \bigcup_{1 \leq j < i \leq n} \{[[\boldsymbol{z}_i, \boldsymbol{z}_j], \boldsymbol{z}_i], [[\boldsymbol{z}_i, \boldsymbol{z}_j], \boldsymbol{z}_j]\}, \quad \text{and} \\ \mathscr{C}_3'' &= \bigcup_{1 \leq j < i < k \leq n} \{[[\boldsymbol{z}_i, \boldsymbol{z}_j], \boldsymbol{z}_k], [[\boldsymbol{z}_k, \boldsymbol{z}_j], \boldsymbol{z}_i]\}. \end{aligned}$$

Define subspaces $F_1(K)$ and $F_2(K)$ of $\mathfrak{f}_{n,r}(K)$ by

$$F_1(K) = \operatorname{span}_K \mathscr{C}_3', \qquad \text{and} \quad F_2(K) = \operatorname{span}_K \mathscr{C}_3''. \tag{1}$$

The subspace $V_3(K)$ is the direct sum of $F_1(K)$ and $F_2(K)$, since $\mathscr{C}_3$ spans $V_3(K)$ and is the disjoint union of $\mathscr{C}_3'$ and $\mathscr{C}_3''$.

2.2. **Anosov automorphisms.** As we discussed previously, every Anosov automorphism can be represented in terms of a matrix A in $GL_n(\mathbb{Z})$, an automorphism f^A of $\mathfrak{f}_{n,r}$ induced by A, and an ideal $\mathfrak{i} < \mathfrak{f}_{n,r}$ satisfying the four Auslander–Scheuneman conditions. In this section, we spell out some of the details involved in such a representation.

Let $\mathfrak{f}_{n,r}(K) = \bigoplus_{i=1}^r V_i(K)$ be the free r-step nilpotent Lie algebra over the field K with a set $\mathscr{B}_1$ of n generators. Let $\mathscr{B} = \bigcup_{i=1}^r \mathscr{B}_i$ be the Hall basis determined by $\mathscr{B}_1$. Let A be a matrix in $GL_n(\mathbb{Z})$ having no eigenvalues of modulus one. Together the matrix A and and the basis $\mathscr{B}_1$ define a linear map $f_1 \colon V_1(K) \to V_1(K)$. The map f_1 induces an automorphism f_K^A of $\mathfrak{f}_{n,r}(K)$, that when restricted to $V_1(K)$ equals f_1. For all $i = 1, \ldots, r$, the restriction f_i of f_K^A to $V_i(K)$ can be represented with respect to the basis $\mathscr{B}_i$ of $V_i(K)$ by a matrix A_i having integer entries that are independent of the field K.

For $i = 1, \ldots, r$, let p_i denote the characteristic polynomial of A_i. We define the *r-tuple of polynomials associated to f* to be $(p_1, \ldots, p_r)$. Note that all of the polynomials are monic with integer coefficients, and there is no dependence on K in defining the polynomials: either the matrix A or the polynomial p_1 alone is enough to uniquely define the r-tuple $(p_1, \ldots, p_r)$.

A Lie algebra admits an Anosov automorphism if and only if it admits a semisimple Anosov automorphism [1]. Assume that the linear map $f_1 \colon V_1(L) \to V_1(L)$ defined by $A \in GL_n(\mathbb{Z})$ and rational basis $\mathscr{B}$ is diagonalizable over the field L (where $\operatorname{char} L = 0$). The vector space $V_1(L)$ can be decomposed into the direct sum of minimal nontrivial rational f_1-invariant subspaces $E_1, \ldots, E_s$. For each rational invariant subspace E_j, $j = 1, \ldots, s$, the restriction of f_1 to E_j is diagonalizable over L. Hence, there is a basis $\mathscr{C}_1 = \{\boldsymbol{z}_1, \ldots, \boldsymbol{z}_n\}$ of $V_1(L)$ consisting of eigenvectors of f_1 with each eigenvector properly contained in one of the subspaces $E_1, \ldots, E_s$. Let $\mathscr{C}$ be the Hall basis of $\mathfrak{f}_{n,r}(L)$ determined by $\mathscr{C}_1$. We will call such an eigenvector basis for an automorphism f of $\mathfrak{f}_{n,r}(L)$ *compatible with the rational structure* and, in the future, when we use eigenvector bases for free nilpotent Lie algebras we will always choose them to be compatible with the rational structure determined by a fixed Hall basis.

NOTATION 2.2. We shall use $\mathscr{B}$ to denote the Hall basis of a free nilpotent Lie algebra $\mathfrak{f}_{n,r}(K)$ that determines the rational structure and that, with a matrix in $GL_n(\mathbb{Z})$, defines the Anosov automorphism, while we will use $\mathscr{C}$ to denote a Hall basis that diagonalizes the Anosov automorphism.

Suppose that K has characteristic zero and $\mathscr{B}$ is a fixed Hall basis of $\mathfrak{f}_{n,r}(K)$, and identify the prime subfield of K with $\mathbb{Q}$. We will use $\mathfrak{f}_{n,r}(\mathbb{Q})$ to denote the subset of $\mathfrak{f}_{n,r}(K)$ that is the $\mathbb{Q}$-span of $\mathscr{B}$ in $\mathfrak{f}_{n,r}(K)$.

At times we will move between free nilpotent Lie algebras $\mathfrak{f}_{n,r}(K)$ and $\mathfrak{f}_{n,r}(L)$ defined over different field extensions K and L of $\mathbb{Q}$. We define a correspondence between rational f_K^A-invariant subspaces of $\mathfrak{f}_{n,r}(K)$ and rational f_L^A-invariant subspaces of $\mathfrak{f}_{n,r}(L)$.

DEFINITION 2.3. Let K and L be fields with characteristic zero. Let $\mathscr{B}_1(K)$ and $\mathscr{B}_1(L)$ be generating sets, both of cardinality n, for free nilpotent Lie algebras $\mathfrak{f}_{n,r}(K)$ and $\mathfrak{f}_{n,r}(L)$, respectively, and let $\mathscr{B}(K)$ and $\mathscr{B}(L)$ be the Hall bases defined by $\mathscr{B}_1(K)$ and $\mathscr{B}_1(L)$, respectively. A bijection $i_1\colon \mathscr{B}_1(K) \to \mathscr{B}_1(L)$ of the generating sets naturally induces a bijection $i\colon \mathscr{B}(K) \to \mathscr{B}(L)$ of the Hall bases, and this in turn defines an isomorphism $\overline{i}$ from $\mathfrak{f}_{n,r}(\mathbb{Q}) < \mathfrak{f}_{n,r}(K)$ to $\mathfrak{f}_{n,r}(\mathbb{Q}) < \mathfrak{f}_{n,r}(L)$, where $\mathfrak{f}_{n,r}(\mathbb{Q})$ denotes the $\mathbb{Q}$-span of the fixed Hall basis.

Endow $\mathfrak{f}_{n,r}(K)$ and $\mathfrak{f}_{n,r}(L)$ with the rational structures defined by $\mathscr{B}(K)$ and $\mathscr{B}(L)$, respectively. Given a matrix $A \in GL_n(\mathbb{Z})$, let maps $f_K^A \in \mathrm{Aut}(\mathfrak{f}_{n,r}(K))$ and $f_L^A \in \mathrm{Aut}(\mathfrak{f}_{n,r}(L))$ be defined by A and $\mathscr{B}_1(K)$ and $\mathscr{B}_1(L)$, respectively. Observe that $[f_K^A]_{\mathscr{B}(K)} = [f_L^A]_{\mathscr{B}(L)} \in GL_N(\mathbb{Z})$, where $N = \dim \mathfrak{f}_{n,r}$.

Let E be a rational f_K^A-invariant subspace of $\mathfrak{f}_{n,r}(K)$ spanned by vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m$ in $\mathfrak{f}_{n,r}(K)$, *i.e.*, the coordinates of $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m$ with respect to $\mathscr{B}(K)$ are in $\mathbb{Q}$. Define the subspace E^L of $\mathfrak{f}_{n,r}(L)$ to be the L-span of the vectors $\overline{i}(\boldsymbol{v}_1), \ldots, \overline{i}(\boldsymbol{v}_m)$ in $\mathfrak{f}_{n,r}(L)$. Clearly, E^L is rational and f_L^A-invariant subspace of $\mathfrak{f}_{n,r}(K)$.

REMARK 2.4. Observe that if $\mathfrak{i}$ satisfies the Auslander–Scheuneman conditions for a semisimple automorphism f of a free nilpotent Lie algebra, then it satisfies the conditions for f^2. Therefore, when seeking ideals of a free nilpotent Lie algebra satisfying the four conditions for an automorphism f, by moving to f^2 if necessary, we may assume that the eigenvalues of the automorphism have product 1, and that all the real eigenvalues are positive.

The next example clarifies some of our definitions and notation.

EXAMPLE 2.5. Let $\mathfrak{f}_{3,2}(\mathbb{R}) = V_1(\mathbb{R}) \oplus V_2(\mathbb{R})$ be the free two-step nilpotent Lie algebra on three generators $\boldsymbol{x}_1, \boldsymbol{x}_2$, and $\boldsymbol{x}_3$. These three generators span the subspace $V_1(\mathbb{R})$. The Hall words of length two are $\boldsymbol{x}_1' = [\boldsymbol{x}_3, \boldsymbol{x}_2], \boldsymbol{x}_2' = [\boldsymbol{x}_3, \boldsymbol{x}_1]$, and $\boldsymbol{x}_3' = [\boldsymbol{x}_2, \boldsymbol{x}_1]$; they span $V_2(\mathbb{R})$. The union $\mathscr{B}$ of $\mathscr{B}_1 = \{\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3\}$ and $\mathscr{B}_2 = \{\boldsymbol{x}_1', \boldsymbol{x}_2', \boldsymbol{x}_3'\}$ is the Hall basis determined by $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3$.

Now let $A = A_1$ be a 3×3 matrix in $SL_3(\mathbb{Z})$ that has eigenvalues $\alpha_1, \alpha_2, \alpha_3$, none of which have modulus one. The matrix A and the basis $\mathscr{B}_1$ define the linear map

$f_1 \colon V_1 \to V_1$. The linear map f_1 induces an automorphism f^A of $\mathfrak{f}_{3,2}(\mathbb{R})$. Let A_2 denote the matrix representing the restriction of f^A to $V_2(\mathbb{R})$ with respect to the basis $\mathscr{B}_2$.

The matrix A_1 has characteristic polynomial

$$p_1(x) = (x-\alpha_1)(x-\alpha_2)(x-\alpha_3).$$

A short calculation shows that A_2 is similar to A_1^{-1} and has characteristic polynomial

$$p_2(x) = (x-\alpha_2\alpha_3)(x-\alpha_1\alpha_3)(x-\alpha_1\alpha_2) = (x-\alpha_1^{-1})(x-\alpha_2^{-1})(x-\alpha_3^{-1}).$$

Neither A_1 nor A_2 have any eigenvalues of modulus one, so f^A is an Anosov automorphism of $\mathfrak{f}_{3,2}(\mathbb{R})$.

2.3. **Polynomials and algebraic numbers.** We will call a monic polynomial *Anosov* if it has integer coefficients, it has constant term ±1, and it has no roots with modulus one. The roots of an Anosov polynomial are algebraic units.

We can identify each monic polynomial p in $\mathbb{Z}[x]$ of degree n with an automorphism of the free nilpotent Lie algebra $\mathfrak{f}_{n,r}(\mathbb{R}) = \bigoplus_{i=1}^r V_i(\mathbb{R})$ with generating set $\mathscr{B}_1 = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\}$. Suppose $p = q_1 q_2 \cdots q_s$ is a factorization of p into irreducibles. Let A_i be the companion matrix for q_i, for $i = 1, \ldots, s$, and define the matrix A_p to be block diagonal with the matrices $A_1, \ldots, A_s$ down the diagonal. As already described, the matrix A_p and the basis $\mathscr{B}_1$ together define an automorphism of the free r-step nilpotent Lie algebra on n generators.

If E is a nontrivial rational invariant subspace for an Anosov automorphism f of an Anosov Lie algebra, we will let p_E denote the characteristic polynomial for the restriction of f to E. If p and q are polynomials in $\mathbb{Z}[x]$, we define the polynomial $p \wedge q$ to be the characteristic polynomial of the matrix $A_p \wedge A_q$.

Next, we illustrate how an Anosov polynomial determines a class of Anosov automorphisms.

EXAMPLE 2.6. Let p be an Anosov polynomial of degree n that is a product of two irreducible factors r_1 and r_2 of degrees d_1 and d_2, respectively. The companion matrices B_1 and B_2 to the polynomials r_1 and r_2 are in $GL_{d_1}(\mathbb{Z})$ and $GL_{d_2}(\mathbb{Z})$, respectively. Putting these matrices together in a block diagonal matrix gives a matrix

$$A_p = A_1 = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}$$

in $GL_n(\mathbb{Z})$ with characteristic polynomial p.

Let $\mathfrak{f}_{n,2}(\mathbb{R}) = V_1(\mathbb{R}) \oplus V_2(\mathbb{R})$ be the real free two-step nilpotent Lie algebra on n generators with generating set $\mathscr{B}_1$. The matrix A_1 and the basis $\mathscr{B}_1$ of $V_1(\mathbb{R})$ define a linear map $f_1 \colon V_1(\mathbb{R}) \to V_1(\mathbb{R})$ which induces an automorphism f^A of $\mathfrak{f}_{n,2}(\mathbb{R})$. Let $f_2 = f^A|_{V_2(\mathbb{R})}$. The map f_2 may be represented by a matrix A_2 that is block diagonal with matrices $B_1 \wedge B_1, B_1 \wedge B_2$ and $B_2 \wedge B_2$ along the diagonal. Let $\alpha_1, \ldots, \alpha_{d_1}$ denote the roots of r_1 and let $\beta_1, \ldots, \beta_{d_2}$ denote the roots of r_2. It can be shown that the matrix A_2 has characteristic polynomial

$$p_2 = (r_1 \wedge r_1)(r_1 \wedge r_2)(r_2 \wedge r_2),$$

where

$$\begin{aligned}(r_1 \wedge r_1)(x) &:= \prod_{1 \le i < j \le d_1} (x - \alpha_i \alpha_j),\\ (r_1 \wedge r_2)(x) &:= \prod_{\substack{1 \le i \le d_1\\ 1 \le j \le d_2}} (x - \alpha_i \beta_j), \quad \text{and}\\ (r_2 \wedge r_2)(x) &:= \prod_{1 \le i < j \le d_2} (x - \beta_i \beta_j).\end{aligned}$$

As long as none of the roots of p_2 have modulus one, the map f^A is Anosov.

Later we will need to know when polynomials in $\mathbb{Z}[x]$ have roots of modulus one and will use the following observation.

REMARK 2.7. Suppose that an irreducible polynomial p in $\mathbb{Z}[x]$ of degree $n \geq 2$ has a root α_1 with modulus one. Since $n \neq 1$, α_1 is not real. The complex conjugate $\bar{\alpha}_1 = \alpha_1^{-1}$ is also a root of p. Then, because complex conjugation is a nontrivial automorphism of the splitting field for p, the associated Galois group G has even order. By the transitivity of the Galois group, since α_1 and its inverse α_1^{-1} are roots of p, the rest of the roots of p come in inverse pairs of the form $\sigma(\alpha_1)$ and $\sigma(\alpha_1^{-1})$, where $\sigma \in G$. A polynomial is a called *self-reciprocal* when the roots come in inverse pairs like this.

3. POLYNOMIALS ASSOCIATED TO AUTOMORPHISMS

3.1. Properties of the r-tuple of characteristic polynomials.

Now we present some properties of the tuple of polynomials associated to an automorphism of a free nilpotent Lie algebra.

PROPOSITION 3.1. *Let f^A be a semisimple automorphism of the free nilpotent Lie algebra $\mathfrak{f}_{n,r}(\mathbb{R}) = \bigoplus_{i=1}^r V_i(\mathbb{R})$ defined by a matrix A in $GL_n(\mathbb{Z})$ and the Hall basis $\mathscr{B}$ defined by generating set $\mathscr{B}_1 = \{\boldsymbol{x}_i\}_{i=1}^n$. Let $(p_1, p_2, \ldots, p_r)$ be the r-tuple of polynomials associated to f, let $\alpha_1, \ldots, \alpha_n$ denote the roots of p_1, and let K denote the splitting field for p_1. Let $\mathscr{C}_1 = \{\boldsymbol{z}_i\}_{i=1}^n$ be a f_K^A-eigenvector basis of $V_1(K) < \mathfrak{f}_{n,r}(K)$ compatible with the rational structure defined by $\mathscr{B}$ and let $\mathscr{C} = \bigcup_{i=1}^r \mathscr{C}_i$ be the Hall basis of $\mathfrak{f}_{n,r}(K)$ associated to $\mathscr{C}_1$. For $i = 1, \ldots, n$, let α_i denote the eigenvalue for $\boldsymbol{z}_i$.*

1. *Each standard Hall monomial of degree i on $\boldsymbol{z}_1, \ldots, \boldsymbol{z}_n$ in the set $\mathscr{C}_i$ is an eigenvector for $f_K^A|_{V_i(K)}$ whose eigenvalue is the corresponding Hall word in $\alpha_1, \ldots, \alpha_n$.*
2. *For $i = 1, \ldots, r$, let $p_i = r_{i,1} \cdots r_{i,d_i}$ be a factorization of p_i into d_i irreducible monic polynomials in $\mathbb{Z}[x]$, and let $V_i(\mathbb{R}) = \bigoplus_{j=1}^{e_i} E_{i,j}$ be a decomposition of $V_i(\mathbb{R})$ into e_i minimal nontrivial rational f^A-invariant subspaces. For all $i = 1, \ldots, r$, $d_i = e_i$ and the map that sends E_{ij} to the characteristic polynomial of $f|_{E_{ij}}$ is a one-to-one correspondence between the set of rational subspaces $\{E_{ij}\}_{j=1}^{d_i}$ and the set of factors $\{r_{i,j} : j = 1, \ldots, e_i\}$ of p_i.*

It follows from Part (1) of the proposition that if the matrix A is diagonalizable over $\mathbb{C}$, then the automorphism f^A is semisimple. In particular, if the polynomial p_1 is separable over $\mathbb{Q}$, then f^A is semisimple. (Here, we say that a polynomial is *separable* over $\mathbb{Q}$ if all of its roots are distinct.)

REMARK 3.2. As a consequence of the second part of the proposition, if f is unimodular, the characteristic polynomial for the restriction of f to a rational invariant subspace E has a unit constant term, hence the restriction of f to any rational invariant subspace E is unimodular. Therefore, the second of the four Auslander–Scheuneman conditions is automatic.

It is well known that there are no Anosov Lie algebras of type $(n_1, \dots, n_r)$, where $n_1 = 2$ and $r > 1$. This follows from Part 1 of the proposition. Henceforth we shall only consider nilpotent Lie algebras where $n_1 \geq 3$ and $r \geq 2$.

Proof of Proposition 3.1. The first part is elementary. Hall words of degree i in $\boldsymbol{z}_i$ span $V_i(K)$, for $i = 1, \dots, r$. Because f_K^A is an automorphism of $\mathfrak{f}_{n,r}(K)$, a Hall word in $\boldsymbol{z}_1, \dots, \boldsymbol{z}_n$ is an eigenvector for f_K^A whose eigenvalue is the same Hall word in $\alpha_1, \dots, \alpha_n$.

The last part of the proposition follows from the existence of the elementary divisors rational canonical forms for matrices. The fact that the matrix is semisimple implies that the elementary divisors are irreducible. ☐

3.2. **The polynomials p_2 and p_3.** Let A be a semisimple matrix in $GL_n(\mathbb{Z})$, and let K be the splitting field for the characteristic polynomial p_1 of A. Let f^A be the semisimple automorphism of $\mathfrak{f}_{n,r}(\mathbb{R}) = \bigoplus_{i=1}^r V_i(\mathbb{R})$, where $n \geq 3$, induced by A and a basis $\mathscr{B}_1$ for $V_1(\mathbb{R})$. Let $\mathscr{C}_1 = \{\boldsymbol{z}_i\}_{i=1}^n$ be an eigenvector basis for $V_1(K) < \mathfrak{f}_{n,r}(K)$ compatible with the rational structure determined by $\mathscr{B}_1$, where $f^A(\boldsymbol{z}_j) = \alpha_j \boldsymbol{z}_j$ for $j = 1, \dots, n$, and let $\mathscr{C} = \bigcup_{i=1}^r \mathscr{C}_i$ be the Hall basis of $\mathfrak{f}_{n,r}(K)$ determined by $\mathscr{C}_1$.

In Example 2.1 we described Hall words of length two and three. By Proposition 3.1, the eigenvalue for an element $[\boldsymbol{z}_j, \boldsymbol{z}_i]$ of $\mathscr{C}_2$ is $\alpha_i \alpha_j$, so

$$p_2(x) = \prod_{1 \leq j < i \leq n} (x - \alpha_i \alpha_j). \tag{2}$$

Let $\mathscr{C}_3'$ and $\mathscr{C}_3''$ be as defined in Example 2.1. By Proposition 3.1, an element $[[\boldsymbol{z}_j, \boldsymbol{z}_i], \boldsymbol{z}_j]$ of $\mathscr{C}_3'$ is an eigenvector for f_K^A with eigenvalue $\alpha_i \alpha_j^2$ and an element $[[\boldsymbol{z}_i, \boldsymbol{z}_j], \boldsymbol{z}_k]$ of $\mathscr{C}_3''$ is an eigenvector for f_K^A with eigenvalue $\alpha_i \alpha_j \alpha_k$. Define the polynomials q_1 and q_2 by

$$q_1(x) = \prod_{1 \leq i, j \leq n, i \neq j} (x - \alpha_i \alpha_j^2), \qquad \text{and} \qquad q_2(x) = \prod_{1 \leq i < j < k \leq n} (x - \alpha_i \alpha_j \alpha_k). \tag{3}$$

Recall that G is the Galois group of the splitting field of p_1. Because they are invariant under the action of G, q_1 and q_2 have integral coefficients. The polynomial $p_3 = q_1 q_2^2$ is the characteristic polynomial for the restriction of the automorphism f_L^A to $V_3(L)$ for any extension L of $\mathbb{Q}$.

3.3. **Anosov polynomials and their roots.** In this section, we discuss Anosov polynomials and their properties.

PROPOSITION 3.3. *Let p_1 be an Anosov polynomial in $\mathbb{Z}[x]$ of degree $n \geq 3$. Let $(p_1,\ldots,p_r)$ be the associated r-tuple of polynomials.*

1. *If p_1 has constant term one, then its reciprocal polynomial $(p_1)_R$ is a factor of p_{n-1} and $(p_2)_R$ is a factor of p_{n-2}. If the constant term of p_1 is -1, then $(p_1)_R(-x)$ is a factor of p_{n-1} and $(p_2)_R(-x)$ is a factor of p_{n-2}.*
2. *If the roots $\alpha_1,\ldots,\alpha_n$ of p_1 are viewed as indeterminates, then the constant term of p_i is $(\alpha_1\cdots\alpha_n)^{D(i)}$, with the exponent $D(i)$ given by*

$$D(i) = \frac{1}{ni}\sum_{d|i}\mu(d)n^{i/d},$$

where μ is the Möbius function.

Proof. Let $\alpha_1,\ldots,\alpha_n$ be the roots of p_1. Suppose the constant term of p_1 is $(-1)^n$, so $\alpha_1\cdots\alpha_n = 1$. The reciprocal of α_j, for $j = 1,\ldots,n$, is $\alpha_1\cdots\hat{\alpha}_j\cdots\cdots\alpha_n$, the Hall word $\alpha_n\cdots\hat{\alpha}_j\cdots\cdots\alpha_1$ of length $n-1$. By Proposition 3.1, Part (1), this number is a root of p_{n-1}. Thus $(p_1)_R$ is a factor of p_{n-1}. Similarly, the reciprocal of the root $\alpha_{j_1}\alpha_{j_2}$ of $(p_2)_R$, where $1 \leq j_1 < j_2 \leq n$, is $\alpha_1\cdots\hat{\alpha}_{j_1}\cdots\hat{\alpha}_{j_2}\cdots\alpha_n$, which is a permutation of a Hall word on $n-2$ letters and therefore is a root of p_{n-2}. Thus, $(p_2)_R$ is a factor of p_{n-2}. If the constant term of p_1 is $(-1)^{n+1}$, then $\alpha_1\cdots\alpha_n = -1$ and the same argument shows that whenever α is a root of p_1, $-\alpha^{-1}$ is a root of p_{n-1}, and when α is a root of p_2, then $-\alpha^{-1}$ is a root of p_{n-2}.

The dimension of $V_i(\mathbb{R})$ is a Dedekind number $\frac{1}{i}\sum_{d|i}\mu(d)n^{i/d}$, where μ is the Möbius function [22, Corollary 4.14]. Each of $\alpha_1,\ldots,\alpha_n$ must occur the same number of times in the constant term of p_i by [1, Lemma 1]. Therefore, the constant term of p_i, for $1 \leq i \leq r$, is $\alpha_1\cdots\alpha_n$ to the power $1/n\cdot\dim(V_i(\mathbb{R}))$, as claimed. ☐

The next lemma helps identify roots of modulus one for automorphisms whose first polynomial p_1 has Galois group of odd order.

LEMMA 3.4. *Suppose that the characteristic polynomial p_1 of a semisimple hyperbolic matrix A in $GL_n(\mathbb{Z})$, where $n \geq 3$, has Galois group of odd order. Let $f^A\colon \mathfrak{f}_{n,r}(\mathbb{R}) \to \mathfrak{f}_{n,r}(\mathbb{R})$ be the automorphism of the free r-step nilpotent Lie algebra on n generators induced by A. If λ is an eigenvalue of f^A with modulus one, then $\lambda = 1$ or $\lambda = -1$.*

Proof. Let p_1 and f^A be as in the statement of the lemma. Let $(p_1,\ldots,p_r)$ be the r-tuple of polynomials associated to f^A.

If a monic irreducible nonlinear polynomial in $\mathbb{Z}[x]$ has a root of modulus one, by Remark 2.7, its Galois group G has even order.

By Proposition 3.1, if α is an eigenvalue of f^A, it is a root of an irreducible factor q of p_i for some $i = 1,\ldots,r$. Since the splitting field for q is a subfield of the splitting field for p_1, the Galois group H for q is the quotient of G by a normal subgroup; hence if q is nonlinear, H has odd order. Then, either q is linear and $\alpha = \pm 1$, or q is nonlinear and $|\alpha| \neq 1$. ☐

3.4. **The full rank condition.** Suppose p_1 is an Anosov polynomial in $\mathbb{Z}[x]$ with roots $\alpha_1, \dots, \alpha_n$. We will want to know when the equation

$$\alpha_1^{d_1} \alpha_2^{d_2} \cdots \alpha_n^{d_n} = 1 \tag{4}$$

has integer solutions $d_1, \dots, d_n$. Note that if p_1 has constant term $(-1)^n$, then $\alpha_1 \cdots \alpha_n = 1$, and $d_1 = \cdots = d_n = d$ is a solution for any integer d.

DEFINITION 3.5. Let $\Lambda = \{\alpha_1, \dots, \alpha_n\}$ be the set of roots of a polynomial p in $\mathbb{Z}[x]$ with constant term $(-1)^n$ and degree $n \geq 2$. The set Λ is said to be of *full rank* if the only integral solutions to Equation (4) are of the form $d_1 = d_2 = \cdots = d_n$.

The next proposition describes how multiplicative relationships among the roots of some polynomials in $\mathbb{Z}[x]$ depend on their Galois groups.

PROPOSITION 3.6. *Suppose $\alpha_1, \dots, \alpha_n$ are roots of a degree n irreducible monic polynomial p in $\mathbb{Z}[x]$ with constant term $(-1)^n$, and suppose none of $\alpha_1, \dots, \alpha_n$ are roots of unity. Let G denote the Galois group for p.*

The set $\{\alpha_1, \dots, \alpha_n\}$ is of full rank in the following situations.

1. *When the permutation representation of G on $\mathbb{Q}^n$ is the sum of the principal representation and a representation that is irreducible over $\mathbb{Q}$.*
2. *When the action of G on the set of roots of p is doubly transitive.*
3. *When p is Anosov, and precisely one of its roots α_1 has modulus greater than one.*

An algebraic number α_1 as in Part (3) of the proposition is a *PV-number*. Properties of PV-numbers were first investigated by Pisot and Vijayaraghavan. (See [18] and [2] for background on PV-numbers.) The proof of Part (3) of the proposition is due to Bell and Hare [3]; we repeat it here for the sake of completeness.

The action of the the Galois group G of $p_1 \in \mathbb{Z}[x]$ on the set $\{\alpha_1, \dots, \alpha_n\}$ of enumerated roots of p_1 gives an identification of G with a subgroup of S_n, and we can define a permutation representation ρ of G on $\mathbb{Q}^n$, with

$$\rho(g)(\beta_1, \dots, \beta_n) = (\beta_{g(1)}, \dots, \beta_{g(n)}) \tag{5}$$

for $g \in G$ and $(\beta_1, \dots, \beta_n) \in \mathbb{Q}^n$.

Proof. Fix $\alpha_1, \dots, \alpha_n$ as in the statement of the theorem. If $(d_1, \dots, d_n) \in \mathbb{Q}^n$ is a solution to Equation (4), and σ is in G, then

$$\sigma(\alpha_1)^{d_1} \sigma(\alpha_2)^{d_2} \cdots \sigma(\alpha_n)^{d_n} = 1,$$

which may be alternately expressed as

$$\alpha_1^{d_{\sigma^{-1}(1)}} \alpha_2^{d_{\sigma^{-1}(2)}} \cdots \alpha_n^{d_{\sigma^{-1}(n)}} = 1. \tag{6}$$

Thus, the set of integral solutions to Equation (4) is invariant for the permutation representation ρ: for all σ in G, $(d_1, \dots, d_n) \in \mathbb{Q}^n$ is a solution to Equation (4) if and only if $(d_{\sigma^{-1}(1)}, \dots, d_{\sigma^{-1}(n)})$ is a solution to Equation (4). It is easy to see that the set S of solutions to Equation (4) in $\mathbb{Z}^n$ is closed under addition and subtraction. Therefore, if $(d_1,\dots,d_n)$ is an integral solution to Equation (4), then any vector in

$$\operatorname{span}_{\mathbb{Z}}\{\rho(\sigma)(d_1,\dots,d_n) : \sigma \in G\}$$

is also a solution to Equation (4).

Suppose ρ decomposes as the sum of the trivial representation on $\mathbb{Q}(1,1,\dots,1)$ and an irreducible representation on $W = (1,1,\dots,1)^{\perp}$. We will show by contradiction that $\{\alpha_1,\dots,\alpha_n\}$ has full rank. Suppose that $(d_1,\dots,d_n)$ is an n-tuple of integers such that Equation (4) holds and that $(d_1,\dots,d_n)$ is not a scalar multiple of $(1,1,\dots,1)$. After subtracting the appropriate multiple of $(1,1,\dots,1)$, we may assume that the solution $(d_1,\dots,d_n)$ is a nontrivial vector in W. The representation of G on W is irreducible over $\mathbb{Q}$, and $(d_1,\dots,d_n)$ is a nontrivial element of W, so the invariant subspace $\operatorname{span}_{\mathbb{Q}}\{\rho(\sigma)(d_1,\dots,d_n) : \sigma \in G\}$ is all of W. Then $\operatorname{span}_{\mathbb{Q}} S = \mathbb{Q}^n$, implying that $(1,0,\dots,0)$ is a $\mathbb{Q}$-linear combination of solutions $(d_1,\dots,d_n)$ to Equation (4). But then there exists an integer N such that $\alpha_1^N = 1$, a contradiction. Hence, every solution to Equation (4) is a scalar multiple of $(1,1,\dots,1)$.

If the action of G is two-transitive, then the permutation representation of G on $\mathbb{C}^n$ is the sum of the trivial representation on $\mathbb{C}(1,1,\dots,1)$ and a representation on the orthogonal complement that is irreducible over $\mathbb{C}$ [23, Exercise 2.6]. Then the permutation representation of G on $\mathbb{Q}^n$ is the sum of the trivial representation on $\mathbb{Q}(1,1,\dots,1)$ and a representation on the orthogonal complement that is irreducible over $\mathbb{Q}$, and the set of roots is of full rank, by Part (1). Therefore, if G is two-transitive, then the set of roots is of full rank.

Now assume that p is irreducible, and without loss of generality that $\alpha_1 > 0$. Suppose that the roots of p satisfy

$$\alpha_1 > 1 > |\alpha_2| \geq \cdots \geq |\alpha_n|,$$

and that Equation (4) holds for $d_1, d_2, \dots, d_n$. Let m be the index such that d_m achieves the minimum of the set $\{d_i : i = 1,\dots,n\}$. Since $(d_m,\dots,d_m)$ is a solution to Equation (4), the n-tuple

$$(e_1,\dots,e_n) = (d_1 - d_m, d_2 - d_m, \dots, d_n - d_m)$$

is a solution to Equation (4) with $e_m = 0$ and $e_i \geq 0$ for all $i = 1,\dots,m$.

Because p is irreducible, there exists a permutation σ in G such that $\sigma(\alpha_1) = \alpha_m$, or if we identify G with a subgroup of S_n in the natural way, $\sigma(1) = m$. We then have

$$\rho(\sigma)(e_1,\dots,e_n) = (e_{\sigma(1)}, e_{\sigma(2)}, \dots, e_{\sigma(n)}) = (0, e_{\sigma(2)}, \dots, e_{\sigma(n)})$$

is also a solution to the equation, so

$$\alpha_2^{e_{\sigma(2)}} \alpha_3^{e_{\sigma(3)}} \cdots \alpha_n^{e_{\sigma(n)}} = 1.$$

But $|\alpha_i| < 1$ for $i = 2,\dots,n$, and all the exponents are nonnegative, hence all the exponents e_i must be zero. Then $d_1 = d_2 = \cdots = d_n$ as desired, and Part (3) holds. □

The next lemma shows that when the set of roots of an Anosov polynomial p_1 is of full rank, there are strong restrictions on the Galois groups for irreducible factors of polynomials $p_2, p_3, \dots$ associated to p_1.

LEMMA 3.7. *Let p_1 be an Anosov polynomial of degree n with constant term $(-1)^n$. Suppose that the set of roots $\{\alpha_1, \dots, \alpha_n\}$ of p_1 has full rank. Let G denote the Galois group of p_1, and let $(p_1, \dots, p_r)$ be the r-tuple of polynomials associated to p_1 for some $r > 1$.*

Fix p_i for some $i = 2, \dots, r$, and suppose that q is an irreducible nonlinear factor of p_i over $\mathbb{Z}$ with root $\beta = \alpha_1^{d_1} \cdots \alpha_s^{d_s}$ for $s \leq n-1$. The Galois group G acts on the splitting field $\mathbb{Q}(p_1)$ for p_1, and the splitting field $\mathbb{Q}(q)$ for q is a subfield of $\mathbb{Q}(p_1)$. Let $H < G$ be the stabilizer of $\mathbb{Q}(q)$. Any element σ in H has the properties

1. *σ permutes the set $\{\alpha_{s+1}, \dots, \alpha_n\}$.*
2. *For $j, k = 1, \dots, s$, $\sigma(\alpha_j) = \alpha_k$ only if $d_j = d_k$. That is, σ permutes the sets of roots having the same exponent in the expression for β in terms of $\alpha_1, \dots, \alpha_n$.*

Thus, H is isomorphic to a subgroup of the direct product

$$S_{k_1} \times S_{k_2} \cdots \times S_{k_{m-1}} \times S_{n-s}, \qquad (k_1 + \cdots + k_{m-1} = s)$$

of $m \geq 2$ symmetric groups.

Proof. Let $\beta = \alpha_1^{d_1} \cdots \alpha_n^{d_n}$, and let σ be in the stabilizer H of $\mathbb{Q}(q)$. Then $\sigma(\beta) = \beta$ implies that

$$\alpha_1^{d_{\sigma^{-1}(1)}} \alpha_2^{d_{\sigma^{-1}(2)}} \cdots \alpha_n^{d_{\sigma^{-1}(n)}} = \alpha_1^{d_1} \cdots \alpha_n^{d_n},$$

so then

$$\alpha_1^{d_{\sigma^{-1}(1)} - d_1} \alpha_2^{d_{\sigma^{-1}(2)} - d_2} \cdots \alpha_s^{d_{\sigma^{-1}(n)} - d_n} = 1.$$

By definition of full rank,

$$d_{\sigma^{-1}(1)} - d_1 = d_{\sigma^{-1}(2)} - d_2 = \cdots = d_{\sigma^{-1}(n)} - d_n.$$

We also know that because σ permutes the set $\{1, 2, \dots, n\}$,

$$d_1 + d_2 + \cdots + d_n = d_{\sigma^{-1}(1)} + d_{\sigma^{-1}(2)} + \cdots + d_{\sigma^{-1}(n)}.$$

Therefore, $d_{\sigma^{-1}(i)} = d_i$ for all $i = 1, \dots, n$; in other words, if $\sigma(i) = j$, then $d_i = d_j$. Hence σ permutes each set of α_i's for which the exponents d_i agree, including the nonempty set $\{\alpha_{s+1}, \dots, \alpha_n\}$ where $d_i = 0$. ☐

The following lemma describes in terms of the properties of p_1 how the polynomials in the r-tuple $(p_1, p_2, \dots, p_r)$ can factor.

LEMMA 3.8. *Suppose p_1 is a monic polynomial in $\mathbb{Z}[x]$ of degree $n \geq 3$ with constant term $(-1)^n$. Let G denote the Galois group for p_1. Let $(p_1, p_2, \dots, p_r)$ be an r-tuple of polynomials associated to p_1. Let q_1 and q_2 be as defined in Equation (3).*

1. *Assume that p_1 is separable. If p_2 or q_1 factors over $\mathbb{Z}$ as a power of an irreducible polynomial r, then the degree of r is $n-1$ or more. If q_2 factors over $\mathbb{Z}$ as a power of an irreducible polynomial r, then the degree of r is $n-2$ or more.*

2. *Suppose that the set of roots of p_1 is of full rank.*
   (a) *For $i = 2, \ldots, r$, the degree k of any factor of p_i satisfies $k = \frac{d \cdot n}{i}$ for some positive integer $d \leq D(i)$, where $D(i)$ is as defined in Proposition 3.3. Therefore, if $\gcd(n,i) = r$, then n/r divides k.*
   (b) *If q is a nonlinear irreducible factor of p_i over $\mathbb{Z}$ for some $i = 2, \ldots, r$, then the normal subgroup N of G of automorphisms fixing $\mathbb{Q}(q)$ does not act transitively on the roots of p_1.*
   (c) *If $p_2 = r^s$ or $q_1 = r^s$ for an irreducible monic polynomial $r \in \mathbb{Z}[x]$, then $s = 1$ when $n \geq 3$; and when $n \geq 4$, if $q_2 = r^s$ for an irreducible r, then $s = 1$.*

Proof. Let $\alpha_1, \ldots, \alpha_n$ denote the roots of p_1. If a polynomial r is irreducible with m distinct roots, then r^s has m distinct roots each of multiplicity s. Thus, to prove the the first part, we simply count the number of roots of each polynomial p_2, q_1, and q_2 that are guaranteed to be distinct, and the degree of r is necessarily greater or equal to that number if p_2, q_1, or q_2 is of the form r^s. For p_2, roots of the form $\alpha_1\alpha_j, j = 2, \ldots, n$ are distinct; for q_1, roots of the form $\alpha_1^2\alpha_j, j = 2, \ldots, n$ are distinct; and for p_2, roots of the form $\alpha_1\alpha_2\alpha_j, j = 3, \ldots, n$ are distinct. This proves Part (1).

Now suppose that the set of roots of p_1 has full rank. Let q be a degree-k factor of p_i over $\mathbb{Z}[x]$ for some $i = 2, \ldots, n$. The constant term of q is ± 1, and by the full rank property, of the form $(\alpha_1\alpha_2 \cdots \alpha_n)^d$ for some positive integer d. The constant term of p_i is $(\alpha_1 \cdots \alpha_n)^{D(i)}$, by Proposition 3.3, so $d \leq D(i)$. On the other hand, roots of p_i are Hall words of degree i in $\alpha_1, \alpha_2, \ldots, \alpha_n$, so the constant term of q is the product of k i-letter words in $\alpha_1, \alpha_2, \ldots, \alpha_n$. Thus, $ki = nd$, and the degree k is an integral multiple of n/i as desired. This proves Part (2a).

Now suppose that q is nonlinear and irreducible. Let β be a root of q. Using the identity $\alpha_1 \cdots \alpha_n = 1$, we may write β in the form $\alpha_1^{d_1} \cdots \alpha_{n-1}^{d_n}$ where no α_n appears, and Lemma 3.7 applies. Then by the lemma, the action of N on $\{\alpha_1 \cdots \alpha_n\}$ is not transitive.

Finally, to show irreducibility of p_2, q_1 and q_2, use the full rank condition to show that the roots of p_2 and q_1 are distinct when $n \geq 3$, and the roots of q_2 are distinct when $n \geq 4$. □

We obtain a corollary that describes the dimensions of the steps of certain sorts of Anosov Lie algebras.

COROLLARY 3.9. *Let $\mathfrak{n}$ be an r-step nilpotent Lie algebra of type $(n_1, \ldots, n_r)$, where n_1 is prime and $1 < r < n_1$, that admits an Anosov automorphism f. Let $(p_1, p_2, \ldots, p_n)$ be the r-tuple of polynomials associated to an automorphism f of $\mathfrak{f}_{n,r}(\mathbb{R})$ that has $\bar{f}$ as a quotient. Suppose that the polynomial p_1 is irreducible. Then n_1 divides n_i for all $i = 2, \ldots, r$.*

Proof. Let G denote the Galois group of the polynomial p_1 associated to the Anosov automorphism f. Because p_1 is irreducible, its roots are distinct, and f is semisimple. The degree n_1 of p_1 is prime, hence it divides the order of the Galois group G for the splitting field of p_1. By Cauchy's Theorem, there is a subgroup H of G isomorphic to C_{n_1}.

Let $\rho\colon G \to GL_{n_1}(\mathbb{Q})$ be the permutation representation of G on $\mathbb{Q}^n$. Let g be a generator for the subgroup H. The element $\rho(g)$ in $GL_{n_1}(\mathbb{Q})$ has characteristic polynomial $r(x) = x^{n_1} - 1$. Because n_1 is prime, the polynomial $r(x)$ factors as $(x-1)\Phi_{n_1}(x)$, where $\Phi_{n_1}(x)$ is the irreducible cyclotomic polynomial of degree $n_1 - 1$. Now we refer to the theory of rational canonical forms, from which it follows that the $\mathbb{Q}^{n_1}$ decomposes as the direct sum of the $\rho(g)$-invariant subspace $\mathbb{Q}(1,1,\ldots,1)$ and a complementary rational $\rho(g)$-invariant subspace E. The characteristic polynomial for the restriction of $\rho(g)$ to E is the irreducible polynomial $\Phi_{n_1}(x)$, and E has no nontrivial proper rational $\rho(g)$-invariant subspaces. Hence, the permutation representation ρ of G on $\mathbb{Q}^n$ is the sum of the principal representation and a representation that is irreducible over $\mathbb{Q}$. By Proposition 3.6, the set of roots of p_1 has full rank.

Now we use the correspondence between irreducible polynomials and rational invariant subspaces for $f\colon \mathfrak{f}_{n_1,r}(\mathbb{R}) \to \mathfrak{f}_{n_1,r}(\mathbb{R})$ established in Part (2) of Proposition 3.1. Let F denote a minimal rational invariant subspace of $V_i(\mathbb{R})$ of dimension $k \geq 1$, where $1 \leq i \leq r$. Let q denote the irreducible characteristic polynomial for the restriction of f to F. Because n_1 is prime and $i < n_1$, we have $\gcd(n_1, i) = 1$. Therefore, by Part (2a) of Lemma 3.8, n_1 divides $k = \dim(F)$. Thus, for all $i = 2,\ldots,r$, the subspace $V_i(\mathbb{R})$ is the direct sum $V_i(\mathbb{R}) = \bigoplus_{j=1}^{e_i} E_{i,j}$ of minimal nontrivial rational f-invariant subspaces $E_{i,j}$, all of whose dimensions are multiples of n_1.

Now suppose that $\bar{f}$ is an Anosov automorphism defined by f and some ideal $\mathfrak{i}$ satisfying the Auslander–Scheuneman conditions. The ideal $\mathfrak{i}$ is the direct sum of minimal nontrivial rational invariant subspaces of the form $E_{i,j}$ as above. Therefore, the dimension n_i of the subspace $V_i(\mathbb{R})/(V_i(\mathbb{R}) \cap \mathfrak{i})$ of $\mathfrak{n}$ is a multiple of n_1 for all $i = 2,\ldots,r$. □

3.5. **The existence of Anosov polynomials with given Galois group.**

In the next proposition, we summarize some results on the existence of Anosov polynomials with certain properties.

PROPOSITION 3.10. *There exist irreducible Anosov polynomials in $\mathbb{Z}[x]$ satisfying the following conditions.*

1. *For all $n \geq 2$, for all $r = 1,\ldots n-1$, there exists an irreducible Anosov polynomial p of degree n such that precisely r of the roots have modulus larger than one.*
2. *For all $n \geq 2$, there exists an irreducible Anosov polynomial of degree n with Galois group S_n.*
3. *For all prime $n \geq 2$, there exists an irreducible Anosov polynomial of degree n with Galois group C_n.*
4. *Suppose that the group $G \subset S_n$ is the Galois group of an irreducible polynomial in $\mathbb{Z}[x]$ of degree n, where $2 \leq n \leq 5$, and G is not isomorphic to the*

alternating group A_5. *Then there exists an Anosov polynomial of degree* n *having Galois group* G.

D. Fried showed how to use the geometric version of the Dirichlet Unit Theorem and the results from [1] to construct Anosov automorphisms with given spectral properties [9]; these methods can also be used to prove the first part of the proposition. We provide an alternate proof that shows the actual polynomials defining the automorphisms.

Proof. The polynomial $p(x) = x^n + a_1x^{n-1} + \cdots + a_{n-1}x \pm 1$ has r roots greater than one in modulus and $n-r$ roots less than one in modulus when

$$|a_r| > 2 + |a_1| + \cdots + |a_{r-1}| + |a_{r+1}| + \cdots + |a_{n-1}|$$

([17], see [25]). By letting $a_r = 3$ and $a_i = 0$ for $i \neq r$ in $\{1,\ldots,n-1\}$, we get a polynomial of degree n with precisely r roots greater than one in modulus that is irreducible by Eisenstein's Criterion. By Remark 2.7, if $n \neq 2r$, the polynomial can not have any roots of modulus one. If $n = 2r$, then the polynomial is $(x^r)^2 + 3x^r \pm 1$; one can check by hand that neither of these polynomials have roots of modulus one. This proves Part (1).

The irreducible polynomial $x^n - x - 1$ has Galois group S_n [24]. As it is not self-reciprocal, by Remark 2.7, it has no roots of modulus one.

Now suppose that K is a Galois extension of $\mathbb{Q}$ with Galois group C_n, with $n \geq 3$ prime. It is well known that such a field exists (see the Kronecker–Weber Theorem in [11]). Let η be a Dirichlet fundamental unit for K, and let p denote its minimal polynomial. Since η is a unit, the constant term of p is ± 1. Because $\eta \notin \mathbb{Q}$, the degree m of η is greater than one. Because m divides n, and n is prime, m must equal n. Thus, the minimal polynomial p for η has degree n and splitting field K. The degree of p is odd, so by Lemma 3.4, p has no roots of modulus one. Thus we have shown that p is an Anosov polynomial.

Now we prove Part (4). Recall that if G is the Galois group of an irreducible polynomial of degree n, it can be represented as a subgroup of S_n that acts transitively on $\{1,2,\ldots,n\}$. Such subgroups can be enumerated in low dimensions. It is simplest to list examples of Anosov polynomials of each kind; see Table 1. By Remark 2.7, the only polynomial in the table that could possibly have roots of modulus one is the self-reciprocal polynomial with Galois group V_4. An easy calculation shows that it does not have roots of modulus one. □

Many of the examples in Table 1 were taken from the appendix of [19]; the reader may find there a great many more examples of Anosov polynomials of degree $n \geq 6$ with a variety of Galois groups. To our knowledge, it is not known whether the existence of a polynomial in $\mathbb{Z}[x]$ with Galois group G guarantees the existence of a polynomial in $\mathbb{Z}[x]$ with constant term ± 1 and Galois group G. In addition, we do not know of an example of an Anosov polynomial with Galois group A_5.

Degree	Galois group	Anosov polynomial
2	C_2	$p(x)=x^2-x-1$
3	C_3	$p(x)=x^3-3x-1$
3	S_3	$p(x)=x^3-x-1$
4	C_4	$p(x)=x^4+x^3-4x^2-4x+1$
4	V_4	$p(x)=x^4+3x^2+1$
4	D_8	$p(x)=x^4-x^3-x^2+x+1$
4	A_4	$p(x)=x^4+2x^3+3x^2-3x+1$
4	S_4	$p(x)=x^4-x-1$
5	C_5	$p(x)=x^5+x^4-4x^3-3x^2+3x+1$
5	D_{10}	$p(x)=x^5-x^3-2x^2-2x-1$
5	F_{20}	$p(x)=x^5+x^4+2x^3+4x^2+x+1$
5	A_5	Q: What is an example with small coefficients?
5	S_5	$p(x)=x^5-x-1$

TABLE 1. The inverse Galois problem for Anosov polynomials of low degree.

4. ACTIONS OF THE GALOIS GROUP

4.1. **Definitions of actions.** In this section, we associate the $\mathbb{Q}$-linear action of a finite group to an automorphism of a free nilpotent Lie algebra that preserves the rational structure defined by a Hall basis.

DEFINITION 4.1. Let A be a matrix in $GL_n(\mathbb{Z})$, let p_1 be the characteristic polynomial of A, and let K be the splitting field for p_1. Suppose that f is the automorphism of $\mathfrak{f}_{n,r}(K)$ determined by A and a set $\mathscr{B}_1=\{\boldsymbol{x}_i\}_{i=1}^n$ of generators for $\mathfrak{f}_{n,r}(K)$. Let $\mathscr{B}=\bigcup_{i=1}^r \mathscr{B}_i$ be the Hall basis determined by $\mathscr{B}_1$. Write $\mathfrak{f}_{n,r}(K)=\bigoplus_{i=1}^r V_i(K)$, and for $i=1,\ldots,r$, let $n_i=\dim_K V_i(K)$.

Let G denote the Galois group for the field K. Let $\boldsymbol{x}^i_1, \dots, \boldsymbol{x}^i_{n_i}$ denote the n_i elements of the basis $\mathscr{B}_i$ of $V_i(K)$; they determine an identification $V_i(K) \cong K^{n_i}$. For $i = 1, \dots, r$, the G-action on K extends to a diagonal G-action on $V_i(K) \cong K^{n_i}$. In particular, if $\boldsymbol{w} = \sum_{j=1}^{n_i} \beta_j \boldsymbol{x}^i_j$ is an element in $V_i(K)$, where $\beta_1, \dots, \beta_{n_i} \in K$ and $g \in G$, then $g \cdot \boldsymbol{w}$ is defined by

$$g \cdot \boldsymbol{w} = \sum_{j=1}^{n_i} (g \cdot \beta_j)\, \boldsymbol{x}^i_j \in V_i(K).$$

A G-action on the free nilpotent Lie algebra $\mathfrak{f}_{n,r}(K) = \bigoplus_{i=1}^r V_i(K)$ is defined by extending each of the G-actions on $V_i(K)$, for $i = 1, \dots, r$.

Next, we describe properties of the action.

PROPOSITION 4.2. *Let A be a semisimple matrix in $GL_n(\mathbb{Z})$, let p_1 be the characteristic polynomial of A, and let K be the splitting field for p_1 over $\mathbb{Q}$. Let $\mathscr{B}_1$ be a generating set for the free nilpotent Lie algebra $\mathfrak{f}_{n,r}$ and let $\mathscr{B}$ be the Hall basis that it determines. Let f be the automorphism of $\mathfrak{f}_{n,r}(K)$ induced by the Hall basis $\mathscr{B}$ and the matrix A. Let G be the Galois group for K, and let $G \times \mathfrak{f}_{n,r}(K) \to \mathfrak{f}_{n,r}(K)$ be the action defined in Definition 4.1. Then*

1. *The G-action is $\mathbb{Q}$-linear, preserving $\mathfrak{f}_{n,r}(\mathbb{Q}) < \mathfrak{f}_{n,r}(K)$, and it preserves the decomposition $\mathfrak{f}_{n,r}(K) = \bigoplus_{i=1}^r V_i(K)$ of $\mathfrak{f}_{n,r}(K)$ into steps.*
2. *The G-action on $\mathfrak{f}_{n,r}(K)$ commutes with the Lie bracket.*
3. *The function $f : \mathfrak{f}_{n,r}(K) \to \mathfrak{f}_{n,r}(K)$ is G-equivariant.*
4. *The G-action permutes the eigenspaces of f,* i.e., *an element $g \in G$ sends the α eigenspace for f to the $g \cdot \alpha$ eigenspace for f.*

The proof is elementary, so we leave it to the reader. Now we use the action defined in Definition 4.1 to describe certain important kinds of subspaces and ideals of Anosov Lie algebras.

DEFINITION 4.3. Let G be a finite group that acts on the free nilpotent Lie algebra $\mathfrak{f}_{n,r}(K)$ over the field K, and let $\boldsymbol{z} \in \mathfrak{f}_{n,r}(K)$. Define the subspace $E^K_G(\boldsymbol{z})$ of $\mathfrak{f}_{n,r}(K)$ to be the K-span of the G-orbit of $\boldsymbol{z}$:

$$E^K_G(\boldsymbol{z}) = \operatorname{span}_K\{g \cdot \boldsymbol{z} : g \in G\}. \tag{7}$$

EXAMPLE 4.4. Let $\mathfrak{f}_{n,2}(K) = V_1(K) \oplus V_2(K)$ be the free two-step Lie algebra on n generators over the field K. Let $\mathscr{C}_1 = \{\boldsymbol{z}_i\}_{i=1}^n$ be a set of n generators for $\mathfrak{f}_{n,2}(K)$. Let G be the cyclic group of order n that acts on $\mathfrak{f}_{n,2}(K)$ through the natural action of $G \cong C_n$ on $\mathscr{C}_1$. For each $j = 2, \dots, \lfloor n/2 \rfloor + 1$, define the ideal $\mathfrak{i}_j < V_2(K)$ by

$$\mathfrak{i}_j = E^K_G([\boldsymbol{z}_j, \boldsymbol{z}_1]) = \operatorname{span}_K\{[\boldsymbol{z}_s, \boldsymbol{z}_t] : s - t = j - 1 \mod n\}.$$

For example, when $n = 4$,

$$\begin{aligned} \mathfrak{i}_2 &= \operatorname{span}_K\{[\boldsymbol{z}_2, \boldsymbol{z}_1], [\boldsymbol{z}_3, \boldsymbol{z}_2], [\boldsymbol{z}_4, \boldsymbol{z}_3], [\boldsymbol{z}_1, \boldsymbol{z}_4]\}, \quad \text{and} \\ \mathfrak{i}_3 &= \operatorname{span}_K\{[\boldsymbol{z}_3, \boldsymbol{z}_1], [\boldsymbol{z}_4, \boldsymbol{z}_2]\}. \end{aligned}$$

For distinct j_1 and j_2 in $\{2,\ldots,\lfloor n/2\rfloor+1\}$, the subspaces $\mathfrak{i}_{j_1}$ and $\mathfrak{i}_{j_2}$ are independent, hence $V_2(K)=\bigoplus_{j=2}^{\lfloor n/2\rfloor+1}\mathfrak{i}_j$. When $n=n_1$ is odd, there are $\frac{1}{2}(n-1)$ such subspaces, each of dimension n. When $n=n_1$ is even, the subspaces $\mathfrak{i}_j$, $j=2,\ldots,n/2$ are of dimension n, and the subspace $\mathfrak{i}_{n/2+1}$ is of dimension $n/2$.

For any proper subset S of $\{2,\ldots,\lfloor n/2\rfloor+1\}$, there is an ideal $\mathfrak{i}(S)$ of $\mathfrak{f}_{n,2}(K)$ defined by $\bigoplus_{j\in S}\mathfrak{i}_j$, and this ideal defines a two-step Lie algebra $\mathfrak{n}_S=\mathfrak{f}_{n,2}(K)/\mathfrak{i}_S$.

We define ideals of free nilpotent Lie algebras arising from group actions, as the Lie algebra $\mathfrak{n}_S$ in the previous example arises from the action of a cyclic group.

DEFINITION 4.5. Let G be a finite group that acts on the free r-step nilpotent Lie algebra $\mathfrak{f}_{n,r}(K)=\bigoplus_{i=1}^{r}V_i(K)$ over K, where the field K is an extension of $\mathbb{Q}$. Let L be another extension of $\mathbb{Q}$, typically $\mathbb{R}$ or $\mathbb{C}$. Suppose that Hall bases $\mathscr{B}\subset\mathfrak{f}_{n,r}(K)$ and $\mathscr{B}'\subset\mathfrak{f}_{n,r}(L)$ define rational structures on $\mathfrak{f}_{n,r}(K)$ and $\mathfrak{f}_{n,r}(L)$ respectively, and let $E\to E^L$ be the correspondence of rational invariant subspaces defined in Definition 2.3 resulting from an identification of $\mathscr{B}'$ and $\mathscr{B}$.

A rational ideal of $\mathfrak{f}_{n,r}(L)$ generated by sets of the form $(E_G(\boldsymbol{w}))^L$, where $E_G^K(\boldsymbol{w})$ is as defined in Definition 4.3 for $\boldsymbol{w}\in\mathfrak{f}_{n,r}(K)$, is called *an ideal of type G defined over L*. We use $\mathfrak{i}(G,\boldsymbol{w})$ to denote the ideal of $\mathfrak{f}_{n,r}(L)$ generated by the subspace $(E_G^K(\boldsymbol{w}))^L$.

A nilpotent Lie algebra of the form $\mathfrak{f}_{n,r}(L)/\mathfrak{i}$, where $\mathfrak{i}$ is of type G, will be called a *nilpotent Lie algebra of type G*.

Now we describe ideals of symmetric type for two- and three-step free nilpotent Lie algebras.

EXAMPLE 4.6. Suppose that $\mathfrak{f}_{n,r}(K)$ has generating set $\mathscr{C}_1=\{\boldsymbol{z}_j\}_{j=1}^n$ and that the action of $G\cong S_n$ on $\mathfrak{f}_{n,r}(K)$ is defined by permuting elements $\boldsymbol{z}_1,\ldots,\boldsymbol{z}_n$ of the generating set $\mathscr{C}_1$. Let $\mathscr{C}=\bigcup_{i=1}^{r}\mathscr{C}_i$ be the Hall basis determined by $\mathscr{C}_1$.

For any $\boldsymbol{w}=[\boldsymbol{z}_i,\boldsymbol{z}_j]$ in $\mathscr{C}_2$, the subspace $E_G^K(\boldsymbol{w})$ is all of $V_2(K)$:

$$E_G^K(\boldsymbol{w})=\operatorname{span}_K\{[\boldsymbol{z}_j,\boldsymbol{z}_i]\}_{1\le i<j\le n}=V_2(K).$$

Hence, an ideal $\mathfrak{i}<\mathfrak{f}_{n,r}(K)$ of type S_n that intersects $V_2(K)$ nontrivially must contain all of $V_2(K)$.

When $r\ge 3$, there are two sets of the form $E_G^K(\boldsymbol{w})$ with $\boldsymbol{w}\in\mathscr{C}_3$, the subspaces F_1 and F_2 defined in Equation (1):

$$F_1(K)=E_G^K([[\boldsymbol{z}_2,\boldsymbol{z}_1],\boldsymbol{z}_1])\quad\text{and}\quad F_2(K)=E_G^K([[\boldsymbol{z}_2,\boldsymbol{z}_1],\boldsymbol{z}_3]).$$

Therefore, for any ideal $\mathfrak{i}$ of type S_n, the subspace $\mathfrak{i}\cap V_3(K)$ is one of the following: $\{0\}$, $F_1(K)$, $F_2(K)$, or $V_3(K)$.

Next, an example showing nilpotent Lie algebras arising from dihedral groups.

EXAMPLE 4.7. Let A be a semisimple matrix in $GL_n(\mathbb{Z})$ whose characteristic polynomial has splitting field K and Galois group G isomorphic to the dihedral group D_{2n} of order $2n$. Let f be the automorphism of $\mathfrak{f}_{n,2}(K)$ induced by A and a Hall basis $\mathscr{B}$. Let $\boldsymbol{z}_1,\dots,\boldsymbol{z}_n$ denote a set of eigenvectors of $f|_{V_1(K)}$ spanning $V_1(K)$ compatible with the rational structure, and let $\alpha_1,\dots,\alpha_n$ denote the corresponding eigenvalues.

The group D_{2n} is isomorphic to the group of symmetries of a regular n-gon. Enumerate the vertices of such an n-gon in counterclockwise order, so $D_{2n} \cong \langle r,s\rangle$, where r is counterclockwise rotation by $2\pi/n$ and s is reflection through a line through center of the n-gon and the first vertex. Let X_n be the complete graph on n vertices obtained by adding edges connecting all distinct vertex pairs of the n-gon. Identify the roots of p_1 with the n vertices and the roots of p_2 with the $\binom{n}{2}$ edges in such a way that the eigenvalue $\alpha_i\alpha_j$ corresponds the to edge connecting vertices corresponding to eigenvalues α_i and α_j. The G-action on $\mathfrak{f}_{n,2}(K)$ can then be visualized through the D_{2n} action on the graph X_n. For example, the G-orbit of the eigenvector $[\boldsymbol{z}_2,\boldsymbol{z}_1]$ is

$$G\cdot[\boldsymbol{z}_2,\boldsymbol{z}_1] = \{[\boldsymbol{z}_2,\boldsymbol{z}_1],[\boldsymbol{z}_3,\boldsymbol{z}_2],\dots,[\boldsymbol{z}_1,\boldsymbol{z}_n]\},$$

corresponding to the n “external” edges of the graph X_2. The subspace $E_G^K([\boldsymbol{z}_2,\boldsymbol{z}_1])$ defined by Definition 4.3 is the K-span of this set, an n-dimensional subspace of $V_2(K)$. Other orbits depend on the value of n: if $n=3$ there are no other orbits, and if $n=4$ or 5, there is one more orbit coming from “interior” edges on the graph, yielding a subspace $E_G^K([\boldsymbol{z}_3,\boldsymbol{z}_1])$ of $V_2(K)$ that is complementary to $E_G^K([\boldsymbol{z}_2,\boldsymbol{z}_1])$. When $n\ge 6$, there are two more orbits coming from interior edges.

5. RATIONAL INVARIANT SUBSPACES

Given an automorphism of a free nilpotent Lie algebra, the next theorem describes how orbits of the G-action on $\mathfrak{f}_{n,r}(\mathbb{R})$ relate to the factorization of the polynomials $p_1,\dots,p_r$. These restrictions on factorizations yield restrictions on the existence of Anosov quotients. Roughly speaking, when the Galois group of p_1 is highly transitive, the rational invariant subspaces for associated Anosov automorphisms tend to be big also, and when the group is small, the rational invariant subspaces are small. The larger rational invariant subspaces are, the fewer Anosov quotients there may be. The field L in the theorem is typically $\mathbb{R}$ or $\mathbb{C}$.

THEOREM 5.1. *Let A be a semisimple matrix in $GL_n(\mathbb{Z})$. Let $(p_1,\dots,p_r)$ be the r-tuple of polynomials associated to f, let K be the splitting field of p_1, and let G be the Galois group for K. Let f be the semisimple automorphism of $\mathfrak{f}_{n,r}(K)$ defined by a Hall basis $\mathscr{B}=\bigcup_{i=1}^r \mathscr{B}_i$ of $\mathfrak{f}_{n,r}(K)$ and the matrix A. Let $\mathscr{C}=\bigcup_{i=1}^r \mathscr{C}_i$ be the Hall basis for $\mathfrak{f}_{n,r}(K)$ determined by a set of $\mathscr{C}_1=\{\boldsymbol{z}_j\}_{j=1}^n$ of eigenvectors for $f|_{V_1(K)}$ that is compatible with the rational structure defined by $\mathscr{B}$.*

For all $i=1,\dots,r$, the vector subspace $V_i(K)$ of $\mathfrak{f}_{n,r}(K)$ decomposes as the direct sum of rational invariant subspaces of the form $E_G^K(\boldsymbol{z})$, where $\boldsymbol{z}\in\mathscr{C}_i$, and $E_G^K(\boldsymbol{z})$ is as defined in Definition 4.3. The characteristic polynomial p_E for the restriction of f to $E_G^K(\boldsymbol{z})$ is of the form $p_E=r^s$, where r is a polynomial that is irreducible over $\mathbb{Z}$.

Suppose that the field L is an extension of $\mathbb{Q}$*, and that* f *is a semisimple automorphism of* $\mathfrak{f}_{n,r}(L)$ *defined by a Hall basis* $\mathscr{B}' = \bigcup_{i=1}^r \mathscr{B}_i'$ *and the matrix A. Since the subspaces* $E_G^K(\boldsymbol{z})$ *of* $\mathfrak{f}_{n,r}(K)$ *are rational, for all* $i = 1, \dots, r$*, there is a decomposition* $V_i(L) = \bigoplus (E_G^K(\boldsymbol{z}))^L$ *of* $V_i(L) < \mathfrak{f}_{n,r}(L)$ *into rational* f*-invariant subspaces, through the correspondence defined in Definition 2.3 induced by the identification of the rational Hall bases* $\mathscr{B}$ *and* $\mathscr{B}'$ *of* $\mathfrak{f}_{n,r}(K)$ *and* $\mathfrak{f}_{n,r}(L)$*.*

Before giving the proof, we illustrate the theorem by considering a special case of Example 4.7.

EXAMPLE 5.2. Let A be a semisimple hyperbolic matrix in $GL_5(\mathbb{Z})$ such that the splitting field K for the characteristic polynomial p_1 for A has Galois group G isomorphic to the dihedral group D_{10} of order 10. Let f_K be the automorphism of $\mathfrak{f}_{5,2}(K) = V_1(K) \oplus V_2(K)$ induced by A and a Hall basis $\mathscr{B}$. Let $\mathscr{C}_1 = \{\boldsymbol{z}_1, \dots, \boldsymbol{z}_5\}$ be the set of eigenvectors of $f_K|_{V_1(K)}$ and let $\alpha_1, \dots, \alpha_5$ denote the corresponding eigenvalues. Let $\mathscr{C} = \mathscr{C}_1 \cup \mathscr{C}_2$ be the Hall basis of $\mathfrak{f}_{5,2}(K)$ determined by $\mathscr{C}_1$.

We saw in Example 4.7 that the D_{10} action on $\mathfrak{f}_{5,2}(K)$ has two orbits each of K-dimension five. By Theorem 5.1, these two orbits yield rational f_K-invariant subspaces

$$\mathfrak{i}_1(K) = E_G([\boldsymbol{z}_2, \boldsymbol{z}_1]) \quad \text{and} \quad \mathfrak{i}_2(K) = E_G([\boldsymbol{z}_3, \boldsymbol{z}_1])$$

and a decomposition $V_2(K) = \mathfrak{i}_1(K) \oplus \mathfrak{i}_2(K)$ of $V_2(K) < \mathfrak{f}_{5,2}(K)$ into rational invariant subspaces.

Let $f_{\mathbb{R}}$ be the automorphism of $\mathfrak{f}_{5,2}(\mathbb{R})$ induced by A and a Hall basis $\mathscr{B}' = \bigcup_{i=1}^r \mathscr{B}_i'$ of $\mathfrak{f}_{5,2}(\mathbb{R})$. Letting $L = \mathbb{R}$ in Theorem 5.1, and using the correspondence between $\mathfrak{f}_{5,2}(K)$ and $\mathfrak{f}_{5,2}(\mathbb{R})$ as in Definition 2.3, we get a rational $f_{\mathbb{R}}$-invariant decomposition of $V_2(\mathbb{R}) < \mathfrak{f}_{5,2}(\mathbb{R})$ into rational $f_{\mathbb{R}}$-invariant subspaces $\mathfrak{i}_1(\mathbb{R}) = (\mathfrak{i}_1(K))^{\mathbb{R}}$ and $\mathfrak{i}_2(\mathbb{R}) = (\mathfrak{i}_2(K))^{\mathbb{R}}$.

The polynomial $p_2(x)$ factors as $p_2 = r_1 r_2$, where the two quintic factors r_1 and r_2 are characteristic polynomials of the restriction of f to the rational invariant ideals $\mathfrak{i}_1(K)$ and $\mathfrak{i}_2(K)$. respectively. Neither r_1 nor r_2 have roots of modulus one by Remark 2.7. By Lemma 3.8, Part 2a, both r_1 and r_2 are irreducible.

As a result, the only ideals $\mathfrak{i}$ of $\mathfrak{f}_{5,2}(\mathbb{R})$ satisfying the Auslander–Scheuneman conditions for $f : \mathfrak{f}_{n,r}(\mathbb{R}) \to \mathfrak{f}_{n,r}(\mathbb{R})$, and $\mathscr{B}$ and defining a two-step Anosov quotient $\mathfrak{f}_{5,2}/\mathfrak{i}$ of type $(5, n_2)$ are the trivial ideal, the ideal $\mathfrak{i}_1(\mathbb{R})$ and the ideal $\mathfrak{i}_2(\mathbb{R})$.

Now, we would like to write out the ideals $\mathfrak{i}_1^{\mathbb{R}}$ and $\mathfrak{i}_2^{\mathbb{R}}$ in terms of generators and relations. We need to consider two cases. In the first case, all of the eigenvalues of $f_{\mathbb{R}}$ are real, so the quotient algebras $\mathfrak{n}_1 = \mathfrak{f}_{5,2}/\mathfrak{i}_1(\mathbb{R})$ and $\mathfrak{n}_1' = \mathfrak{f}_{5,2}/\mathfrak{i}_2(\mathbb{R})$ may be written with generators $\boldsymbol{z}_1, \boldsymbol{z}_2, \boldsymbol{z}_3, \boldsymbol{z}_4, \boldsymbol{z}_5$ in $\mathfrak{f}_{5,2}(\mathbb{R})$ and relations

$$[\boldsymbol{z}_1, \boldsymbol{z}_2] = [\boldsymbol{z}_2, \boldsymbol{z}_3] = [\boldsymbol{z}_3, \boldsymbol{z}_4] = [\boldsymbol{z}_4, \boldsymbol{z}_5] = [\boldsymbol{z}_5, \boldsymbol{z}_1] = 0$$

for the first Lie algebra $\mathfrak{n}_1$, and relations

$$[\boldsymbol{z}_1, \boldsymbol{z}_3] = [\boldsymbol{z}_2, \boldsymbol{z}_4] = [\boldsymbol{z}_3, \boldsymbol{z}_5] = [\boldsymbol{z}_4, \boldsymbol{z}_1] = [\boldsymbol{z}_5, \boldsymbol{z}_2] = 0$$

for the second Lie algebra. These Lie algebras are clearly isomorphic.

In the second case, there is an eigenvector $\boldsymbol{z}_1$ with a real eigenvalue and there are two complex eigenvalue pairs yielding eigenvector pairs $\boldsymbol{z}_2, \boldsymbol{z}_3 = \boldsymbol{x}_2 \pm i\,\boldsymbol{y}_2$ and

$\boldsymbol{z}_4, \boldsymbol{z}_5 = \boldsymbol{x}_3 \pm i\,\boldsymbol{y}_3$, where the vectors $\boldsymbol{x}_i, \boldsymbol{y}_i < \mathfrak{f}_{5,2}(\mathbb{R})$ $(i = 1,2)$. (If there were only one complex eigenvalue pair, the Galois group would be S_5.) The reader may check that the ideal $\mathfrak{i}_1^{\mathbb{R}}$ is generated by the elements

$$[\boldsymbol{z}_1, \boldsymbol{x}_2], [\boldsymbol{x}_2, \boldsymbol{x}_3] - [\boldsymbol{y}_2, \boldsymbol{y}_3], [\boldsymbol{x}_2, \boldsymbol{y}_3] + [\boldsymbol{y}_2, \boldsymbol{x}_3], [\boldsymbol{x}_3, \boldsymbol{y}_3], [\boldsymbol{z}_1, \boldsymbol{y}_2]$$

of $V_2(\mathbb{R}) < \mathfrak{f}_{5,2}(\mathbb{R})$. The ideal $\mathfrak{i}_2^{\mathbb{R}}$ yields another ideal isomorphic to the first. These ideals yield isomorphic Lie algebras $\mathfrak{n}_2$ and $\mathfrak{n}_2'$.

It can be shown that $\mathfrak{n}_1$ and $\mathfrak{n}_2$ are not isomorphic, because the the D_{10} symmetry of $\mathfrak{f}_{5,2}(K)/\mathfrak{i}_1$ is preserved when moving to $\mathfrak{n}_1$, but it is lost when moving to $\mathfrak{n}_2$. (In particular, the $\boldsymbol{z}_1$ coset in $\mathfrak{f}_{5,2}(\mathbb{R})/\mathfrak{i}_2^{\mathbb{R}}$ is the unique element having a three-dimensional centralizer.)

In summary, in addition to $\mathfrak{f}_{5,2}(\mathbb{R})$, there are exactly two isomorphic two-step nilpotent quotients, both of type $(5,5)$ to which f descends as an Anosov map, and there are exactly two nonisomorphic Anosov Lie algebras of type $(5,5)$ with Anosov automorphisms yielding Galois group D_{10}.

Proof of Theorem 5.1. Fix an element $\boldsymbol{w}$ in the basis $\mathscr{C}_i$ for $V_i(K) < \mathfrak{f}_{n,r}(K)$. By Proposition 3.1, it is an eigenvector for f; let α denote its eigenvalue. When represented with respect to the rational basis $\mathscr{B}_i$ for $V_i(K)$, the vector $\boldsymbol{w}$ has coordinates in K^{n_i}, where $n_i = \dim V_i$. Let $E_G^K(\boldsymbol{w})$ be the subspace of $V_i(K)$ generated by $\boldsymbol{w}$ and G as in Definition 4.3.

First, we show that $E_G^K(\boldsymbol{w})$ is invariant under f. By Proposition 4.2, part (4), for all g in G, the vector $g\cdot\boldsymbol{w}$ is an eigenvector with eigenvalue $g\cdot\alpha$. An element $\boldsymbol{u}$ of $V_i(K)$ is in $E_G^K(\boldsymbol{w})$ if and only if it is of the form $\boldsymbol{u} = \sum_{g\in G} c_g\,(g\cdot\boldsymbol{w})$, where $c_g \in K$ for $g \in G$. Then

$$f(\boldsymbol{u}) = \sum_{g\in G} c_g\, f(g\cdot\boldsymbol{w}) = \sum_{g\in G} c_g\,(g\cdot\alpha)\, g\cdot\boldsymbol{w},$$

so $f(\boldsymbol{u})$ is also in $E_G^K(\boldsymbol{w})$.

Now we will show that any nontrivial subspace of $\mathfrak{f}_{n,r}(K)$ that is G-invariant and f-invariant must also be rational. Let E be such a subspace. Let $\boldsymbol{v}_1$ be an eigenvector for f in E. Then the vector $\boldsymbol{w}_1 = \sum_{g\in G} g\cdot\boldsymbol{v}_1$ has rational coefficients relative to a rational basis $\mathscr{B}$ for $\mathfrak{f}_{n,r}(K)$. The set $S_1 = \{f^i(\boldsymbol{w}_1) \mid i = 0,1,2,\ldots\}$ is a finite set of rational vectors. Clearly, the subspace $E_1 = \operatorname{span}_K(S_1)$ is a rational and f-invariant subspace of E. If $E = E_1$, we are done. If not, we may proceed inductively, and since f is semisimple, we will eventually be able to write E as the direct sum of rational f-invariant subspaces. Hence, E itself is rational. Thus, any subspace of $\mathfrak{f}_{n,r}$ that is G-invariant and f-invariant is rational. In particular, any subspace of the form $E_G(\boldsymbol{w})$ is rational.

Now we show that the characteristic polynomial for the restriction of f to $E_G^K(\boldsymbol{w})$ is a power of an irreducible. Let $q(x) = \prod_{i=1}^k (x-\alpha_i)$ denote the minimal polynomial for the eigenvalue $\alpha = \alpha_1$ for the eigenvector $\boldsymbol{w}$. By Proposition 3.1, the splitting field $L = \mathbb{Q}(q)$ is intermediate to $\mathbb{Q}$ and the splitting field $\mathbb{Q}(p_1)$ of p_1.

The Galois group for L over $\mathbb{Q}$ is G/N, where N is the normal subgroup of automorphisms in G that fix L. Because q is irreducible, G/N acts transitively on $\{\alpha_i\}_{i=1}^k$, hence the G-orbit $\{g \cdot \alpha_1\}_{g\in G}$ coincides with the set $\{\alpha_i\}_{i=1}^k$ of roots of q. But the G-orbit $\{g \cdot \alpha_1\}_{g\in G}$ is also equal to the set of eigenvalues for the eigenvectors spanning $E_G^K(\boldsymbol{w})$. Thus, the characteristic polynomial p_E for the restriction of f to $E_G^K(\boldsymbol{w})$ is a power of the irreducible polynomial q.

The last part of the theorem is obvious. □

We will describe a some rational invariant subspaces that exist for any automorphism of a free nilpotent Lie algebra preserving a rational structure. First, we need to make a definition.

DEFINITION 5.3. Let K be a field. Let $\mathscr{C}_1 = \{\boldsymbol{z}_j\}_{j=1}^n$ be a generating set for the free r-step nilpotent Lie algebra $\mathfrak{f}_{n,n}(K)$, and let $\mathscr{C}$ be the associated Hall basis. Define the ideal $\mathfrak{j}_{n,r}$ of $\mathfrak{f}_{n,r}(K)$ to be the ideal generated by all elements $\boldsymbol{w}$ of the Hall basis $\mathscr{C}$ having the property that there is a single number k such that for all $j = 1,\ldots,n$, the letter $\boldsymbol{z}_j$ occurs exactly k times in the Hall word $\boldsymbol{w}$.

For example, when $n = 3$, the ideal $\mathfrak{j}_{3,2} < \mathfrak{f}_{3,2}(K)$ is $\{0\}$, and the ideal $\mathfrak{j}_{3,3} < \mathfrak{f}_{3,3}$ is given by

$$\mathfrak{j}_{3,3} = \operatorname{span}_K\{[[\boldsymbol{z}_2,\boldsymbol{z}_1],\boldsymbol{z}_3],[[\boldsymbol{z}_3,\boldsymbol{z}_1],\boldsymbol{z}_2]\} = F_2(K) < V_3(K),$$

where $F_2(K)$ is as defined in Equation (1), and the ideal $\mathfrak{j}_{4,3}$ in the four-step free nilpotent Lie algebra $\mathfrak{f}_{4,3}(K)$ on three generators is given by

$$\mathfrak{j}_{4,3} = \mathfrak{j}_{3,3} \oplus [\mathfrak{j}_{3,3},\mathfrak{f}_{4,3}],$$

where we map $\mathfrak{j}_{3,3}$ into $\mathfrak{f}_{4,3}$ in the natural way.

REMARK 5.4. Since the product of the roots of an Anosov polynomial is always ±1, any ideal $\mathfrak{i} < \mathfrak{f}_{n,r}$ satisfying the Auslander–Scheuneman conditions for some f must contain the ideal $\mathfrak{j}_{n,r}$ (defined relative to an eigenvector basis) when $r \geq n$.

PROPOSITION 5.5. *Let A be a semisimple matrix in $GL_n(\mathbb{Z})$ whose characteristic polynomial has splitting field K. Let $\mathfrak{f}_{n,r}(K) = \bigoplus_{i=1}^r V_i(K)$ be the free r-step nilpotent Lie algebra on $n \geq 3$ generators over K, endowed with the rational structure defined by a Hall basis $\mathscr{B}$. Let f be the semisimple automorphism of $\mathfrak{f}_{n,r}(K)$ defined by the matrix A and the Hall basis $\mathscr{B}$. Let $\mathscr{C}$ be the Hall basis of $\mathfrak{f}_{n,r}(K)$ determined by a set of eigenvectors $\mathscr{C}_1$ for $f|_{V_1(K)}$ that is compatible with the rational structure.*

The ideal $\mathfrak{j}_{n,r}$ defined in Definition 5.3 is a rational invariant subspace, and when $r \geq 3$, the subspace $V_3(K)$ is the direct sum $F_1(K) \oplus F_2(K)$ of rational invariant subspaces where $F_1(K)$ and $F_2(K)$ are as in Equation (1), while $F_2(K)$ decomposes as the direct sum $F_2(K) = F_{2a}(K) \oplus F_{2b}(K)$ where $F_{2a}(K)$ and $F_{2b}(K)$ are rational invariant subspaces each of dimension $\binom{n}{3}$. The characteristic polynomial for the restrictions of f to $F_1(K)$ is q_1 and the characteristic polynomials for the restriction of f to $F_{2a}(K)$ and $F_{2b}(K)$ are both q_2.

Proof. Let G denote the Galois group of the polynomial p_1 associated to f. Suppose that $\boldsymbol{w}$ is a k-fold bracket of elements in $\mathscr{C}_1 = \{\boldsymbol{z}_j\}_{j=1}^{n_1}$.

Recall from Example 2.1 that $\mathscr{C}_3$ is the union of the set $\mathscr{C}_3'$ of standard Hall monomials of the first type and the set $\mathscr{C}_3''$ of standard Hall monomials of the second type. Let $g \in S_n$. It is easy to see that if $\boldsymbol{w} \in \mathscr{C}_3'$ then $g \cdot \boldsymbol{w} \in \mathscr{C}_3'$ or $-(g \cdot \boldsymbol{w}) \in \mathscr{C}_3'$, and if $\boldsymbol{w} \in \mathscr{C}_3''$ then $g \cdot \boldsymbol{w} \in \mathscr{C}_3''$, $-(g \cdot \boldsymbol{w}) \in \mathscr{C}_3''$, or $g \cdot \boldsymbol{w}$ is a linear combination of elements of $\mathscr{C}_3''$ through the Jacobi Identity. The action of the group G on $\mathfrak{f}_{n,r}(K)$ therefore preserves the subspaces F_1 and F_2, so if $\boldsymbol{w}$ in $\mathscr{C}_2$ is in $F_1(K)$ or $F_2(K)$, then $E_G^K(\boldsymbol{w}) < F_1$ or $E_G^K(\boldsymbol{w}) < F_2(K)$, respectively. The space $F_1(K)$ is the sum of the rational invariant spaces $E_G^K(\boldsymbol{w})$ as $\boldsymbol{w}$ varies over elements of $\mathscr{C}_3'$, so is rational and invariant. By the same reasoning $F_2(K)$ is rational and f-invariant also.

The characteristic polynomial for the restriction of f to $F_2(K)$ is q_2^2, where q_2 is as defined in Equation (3). The pair of elements of the form $[[\boldsymbol{z}_i, \boldsymbol{z}_j], \boldsymbol{z}_k]$ and $[[\boldsymbol{z}_k, \boldsymbol{z}_j], \boldsymbol{z}_i]$, where $1 \le j < i < k \le n$, in $\mathscr{C}_3$ have the same eigenvalue, $\alpha_i \alpha_j \alpha_k$, where $\alpha_i, \alpha_j, \alpha_k$ are the eigenvalues of $\boldsymbol{z}_i, \boldsymbol{z}_j$, and $\boldsymbol{z}_k$, respectively. These yield one basis vector for F_{2a} and one basis vector for F_{2b}. Each factor q_2 in the characteristic polynomial q_2^2 for $F_2(K)$ yields one rational invariant subspace of $F_2(K)$ of dimension $\deg q_2 = \binom{n}{3}$, each spanned by elements of $\mathscr{C}_3''$. Call these subspaces $F_{2a}(K)$ and $F_{2b}(K)$, so $F_2(K) = F_{2a}(K) \oplus F_{2b}(K)$.

Similarly, the set of k-fold brackets $\boldsymbol{w}$ of $\mathscr{C}$ having the property that each element $\boldsymbol{z}_i$ occurs the same number of times in $\boldsymbol{w}$ is clearly invariant under the action of G, so the ideal that it generates, $\mathfrak{j}_{n,r}$, is G-invariant, hence rational. ☐

The following elementary proposition yields restrictions on possible dimensions of rational invariant subspaces for semisimple automorphisms of nilpotent Lie algebras.

PROPOSITION 5.6. *Let $\mathfrak{f}_{n,r}(\mathbb{R})$ be a free nilpotent Lie algebra, and let $f\colon \mathfrak{f}_{n,r}(\mathbb{R}) \to \mathfrak{f}_{n,r}(\mathbb{R})$ be a semisimple automorphism defined by a matrix A in $GL_n(\mathbb{Z})$ and a Hall basis $\mathscr{B}$ of $\mathfrak{f}_{n,r}(\mathbb{R})$. Suppose that the characteristic polynomial p_1 of A is irreducible with Galois group G. Let m be the dimension of a minimal nontrivial rational invariant subspace $E < \mathfrak{f}_{n,r}(\mathbb{R})$ for f. Then G has a normal subgroup N such that G/N acts faithfully and transitively on a set of m elements.*

Note that the subspace E is one-dimensional if and only if $N = G$.

Proof. Suppose E is a minimal nontrivial invariant subspace of dimension m. The characteristic polynomial p_E for the restriction of f to E is irreducible. Since the roots of p_E are contained in the splitting field $\mathbb{Q}(p_1)$, the Galois group for p_E is the quotient G/N of G by the normal subgroup of elements of G fixing $\mathbb{Q}(p_E)$. The group G/N acts faithfully and transitively on the m roots of p_E since it is the Galois group of an irreducible polynomial. ☐

The previous proposition can be used to find all possible dimensions of minimal nontrivial rational invariant subspaces for any Anosov automorphism whose associated Galois group is some fixed group G. One simply needs to find all numbers m such that there exists a normal subgroup N of G and there exists a faithful transitive action of $H = G/N$ on a set of m elements. Every faithful transitive action of a group H on a set X is conjugate to the action of H on the cosets

$X' = \{hK\}_{h\in H}$ of a subgroup K of H such that K contains no nontrivial normal subgroups of H. To find all faithful transitive actions of a group $H = G/N$, one must list all subgroups of H and eliminate any that contain nontrivial normal subgroups. The cardinalities $|H|/|K|$ of the set $\{hK\}_{h\in H}$ are admissible values for the cardinality m of a set X on which H acts faithfully and transitively. In our situation, where G is the Galois group of p_1, the number m could be the degree of a polynomial having Galois group G/N, and m could be the dimension of a rational invariant subspace of the corresponding automorphism of $\mathfrak{f}_{n,r}(\mathbb{R})$.

$\deg p_1$	G	full rank?	$N \neq G$	G/N	dimension m of E
3	S_3	yes	$C_3^*, \{1\}$	C_2^*, S_3	$2^*; 3, 6$
3	C_3	yes	$\{1\}$	C_3	3
4	S_4	yes	$A_4^*, V_4^*, \{1\}$	C_2^*, S_3^*, S_4	$2^*; 3^*, 6^*; 4, 6, 8, 12, 24$
4	A_4	yes	$V_4^*, \{1\}$	C_3^*, A_4	$3^*; 4, 12$
4	D_8	no	$C_4, C_2, \{1\}$	C_2, V_4, D_8	$2; 4; 4, 8$
4	C_4	no	$C_2, \{1\}$	C_2, C_4	$2; 4$
4	V_4	no	$C_2, \{1\}$	C_2, V_4	$2; 4$

TABLE 2. Possible dimensions $m > 1$ for rational invariant subspaces E for an Anosov automorphism of an r-step nilpotent Lie algebra $\mathfrak{n} = \mathfrak{f}_{n,r}(\mathbb{R})/\mathfrak{i}$ when $n = 3$ or 4 and p_1 is irreducible. An asterisk indicates that the marked values of $N, G/N$ and $\dim E$ cannot occur by Lemma 3.8, Part 2b.

In Table 2 we analyze the possible dimensions of rational invariant subspaces of Anosov Lie algebras $\mathfrak{n}$ for which p_1 is irreducible and degree 3 or 4. The first two columns in the table give all possible Galois groups G for irreducible polynomials of degrees three and four, grouped by degree. The fourth and fifth columns list the isomorphism class of proper normal subgroups N of each group G, and the quotients G/N. (Since $\mathfrak{n}$ is Anosov, there are no one-dimensional rational invariant subspaces, and we omit the case $N = G$.) The quotient groups G/N are potential Galois groups for characteristic polynomials of rational invariant subspaces of an Anosov Lie algebra with polynomial p_1 having Galois group G. The last column gives the cardinalities of sets on which each G/N can act faithfully and transitively, found by the procedure described above. The numbers in the last column are listed in the same order as the subgroups in the second column, with the numbers for different subgroups separated by semicolons. In the third column, we show when the roots of a polynomial with given Galois group must have full rank by Proposition 3.6. When the set of roots has full rank, some possibilities for the normal subgroup N may be prohibited by Lemma 3.8: these subgroups and the corresponding dimensions are indicated in the table with asterisks.

From the table we obtain the following corollary to Theorem 5.1.

COROLLARY 5.7. *Let f be a semisimple automorphism of $\mathfrak{f}_{n,r}(\mathbb{R})$ induced by a hyperbolic matrix in $GL_n(\mathbb{Z})$ and a Hall basis $\mathscr{B}$. Let $(p_1,\dots,p_r)$ be the r-tuple of polynomials associated to f. If p_1 is irreducible, then the dimension of any minimal nontrivial invariant subspace of $\mathfrak{f}_{n,r}(\mathbb{R})$ is* 3 *or* 6 *if $n=3$ and is one of* $2,4,6,8,12,24$ *if $n=4$.*

6. AUTOMORPHISMS WITH CYCLIC AND SYMMETRIC GALOIS GROUPS

In this section we use Theorem 5.1 to analyze the structure of Anosov Lie algebras whose associated polynomial p_1 has either a small Galois group, such as a cyclic group, or a large Galois group, such as a symmetric group. The following theorem describes Anosov automorphisms associated to Galois groups whose actions on the roots of p_1 is highly transitive.

THEOREM 6.1. *Suppose that $\mathfrak{n}$ is a real r-step Anosov Lie algebra admitting an Anosov automorphism defined by a semisimple matrix A in $GL_n(\mathbb{Z})$, a Hall basis $\mathscr{B}$, and an ideal $\mathfrak{i} < \mathfrak{f}_{n,r}(\mathbb{R})$ satisfying the Auslander–Scheuneman conditions. Suppose that the polynomial p_1 associated to f is irreducible with Galois group G. Let $(p_1,\dots,p_r)$ be the r-tuple of polynomials associated to p_1.*

1. *If the action of G on the roots of p_1 is two-transitive, then the polynomial p_2 is irreducible and Anosov, and if $r=2$, then $\mathfrak{n}$ is isomorphic to the free nilpotent algebra $\mathfrak{f}_{n,2}(\mathbb{R})$.*
2. *If the action of G on the roots of p_1 is three-transitive and $r=3$, then $\mathfrak{n}$ is isomorphic to $\mathfrak{f}_{n,3}(\mathbb{R})/\mathfrak{i}$, where $\mathfrak{i}$ is trivial or a sum of $F_1(\mathbb{R})$, $F_{2a}(\mathbb{R})$, and $F_{2b}(\mathbb{R})$, where $F_1(\mathbb{R})$ is as defined in Equation* (1)*, and $F_{2a}(\mathbb{R})$ and $F_{2b}(\mathbb{R})$ are as in Proposition 5.5. If $n=3$, then $\mathfrak{i}$ contains $F_2(\mathbb{R})$.*
3. *A Lie algebra $\mathfrak{f}_{n,r}(\mathbb{R})/\mathfrak{i}$ of type S_n is Anosov so long as the ideal $\mathfrak{i}$ contains the ideal $\mathfrak{j}_{n,r}$ as defined in Definition 5.3*

Note that when G is S_n, the Anosov Lie algebra need not be of type S_n as in Definition 4.5: the Lie algebra $\mathfrak{f}_{n,3}/F_{2a}(\mathbb{R})$ is not of type S_n although it admits an Anosov automorphism with symmetric Galois group.

Proof. Let $\alpha_1,\dots,\alpha_n$ denote the roots of p_1 and $\mathscr{C}_1=\{\boldsymbol{z}_1,\dots,\boldsymbol{z}_n\}$ a set of corresponding eigenvectors of $f|_{V_1(K)}$ that is compatible with the rational structure defined by $\mathscr{B}$. Let $\mathscr{C}=\bigcup_{i=1}^r \mathscr{C}_i$ be the Hall basis of $\mathfrak{f}_{n,r}(K)$ determined by $\mathscr{C}_1$.

Suppose that the action of the Galois group G on $\{\alpha_1,\dots,\alpha_n\}$ is two-transitive. If the action of G on the roots of p_1 is two-transitive, then the set of roots of p_1

has full rank by Lemma 3.8. Since the action of the group G sends an eigenvector $\boldsymbol{z}_i$ to a multiple of another eigenvector $\boldsymbol{z}_j$, the G-action sends an element of $\mathscr{C}_2 = \{[\boldsymbol{z}_k, \boldsymbol{z}_j]\}_{1 \le j < k \le n}$ to a scalar multiple of an element of $\mathscr{C}_2$. Because the action is doubly transitive, for any $\boldsymbol{w}$ in $\mathscr{C}_2$, the rational invariant subspace $E_G^K(\boldsymbol{w})$ is all of $V_2(\mathbb{R})$. By Theorem 5.1, the characteristic polynomial p_2 for the restriction of f to $V_2(\mathbb{R})$ is a power of an irreducible polynomial. But actually, p_2 itself is irreducible by Part (2c) of Lemma 3.8. Thus, the only proper rational invariant subspace of $V_2(\mathbb{R})$ is trivial, and when $r = 2$, the only two-step Anosov quotient of $\mathfrak{f}_{n,2}(\mathbb{R})$ is itself. This proves the first part of the theorem.

Now suppose $r = 3$ and that the action of the Galois group G on $\{\alpha_1, \dots, \alpha_n\}$ is three-transitive. Then G is two-transitive, and by the argument above, the only proper rational invariant subspace of $V_2(\mathbb{R})$ is $\{0\}$. Recall from Proposition 5.5 that $V_3(\mathbb{R})$ is the direct sum $V_3(\mathbb{R}) = F_1 \oplus F_{2a} \oplus F_{2b}$ of the rational invariant subspaces F_1, F_{2a}, and F_{2b} where $F_2 = F_{2a} \oplus F_{2b}$, and the characteristic polynomial for the restrictions of f to F_1 is q_1 and the characteristic polynomials for the restriction of f to F_{2a} and F_{2b} are both q_2. By the three-fold transitivity of G, the subspaces F_1 and F_2 are all single G-orbits; hence q_1 and q_2^2 are powers of irreducibles. But by Lemma 3.8, when $n \ge 3$, q_1 is irreducible, so F_1 has no nontrivial rational invariant subspaces, and when $n > 3$, the polynomial q_2 is irreducible; hence the subspaces F_{2a} and F_{2b} are minimal nontrivial invariant subspaces. If $n = 3$, then $\alpha_1\alpha_2\alpha_3 = \pm 1$, so $q_2(x) = x \pm 1$, and $\mathfrak{i}$ must contain F_2. Therefore, in order for an ideal $\mathfrak{i}$ of $\mathfrak{f}_{n,3}(\mathbb{R})$ to satisfy the Auslander–Scheuneman conditions relative to f, it is necessary for $\mathfrak{i}$ to be a sum of $\{0\}, F_1, F_{2a}$ F_{2b}. Thus, the second part of the theorem holds.

Now we consider the case that G is symmetric. Assume that $\mathfrak{i}_0$ is an ideal of $\mathfrak{f}_{n,r}(\mathbb{R})$ of type S_n relative to some Hall basis $\mathscr{D}$, and $\mathfrak{f}_{n,r}(\mathbb{R})/\mathfrak{i}_0$ is a Lie algebra of type S_n. Assume also that $\mathfrak{i}_0$ contains $\mathfrak{j}_{n,r}$. We need to show that $\mathfrak{i}_0$ is satisfies the Auslander–Scheuneman conditions relative to some automorphism f of $\mathfrak{f}_{n,r}(\mathbb{R})$.

Let A be the companion matrix to an Anosov polynomial with Galois group S_n, such as $p_1(x) = x^n - x - 1$ as in Proposition 3.10. Together, A and a set of generators $\mathscr{B}_1$ determine an automorphism f of $\mathfrak{f}_{n,r}(\mathbb{R})$ that is rational relative to the Hall basis $\mathscr{B}$ determined by $\mathscr{B}_1$. Let $\mathscr{C}_1$ denote the set of eigenvectors for f in $V_1(\mathbb{R})$, and let $\mathscr{C}$ be the corresponding Hall basis. There is an ideal $\mathfrak{i}$ isomorphic to $\mathfrak{i}_0$ that is the image of $\mathfrak{i}_0$ under the isomorphism $g\colon \mathfrak{f}_{n,r}(\mathbb{R}) \to \mathfrak{f}_{n,r}(\mathbb{R})$ defined by a bijection from $\mathscr{D}$ to $\mathscr{C}$. The ideal $\mathfrak{i}$ is invariant under f by Theorem 5.1. By Remark 3.2, the restriction of f to $\mathfrak{i}$ is unimodular. The third of the Auslander–Scheuneman condition holds by the theory of rational canonical forms. The last condition is that all roots of modulus one or minus one are in $\mathfrak{i}$: this holds because the set of roots of p_1 has full rank by Proposition 3.6, and $\mathfrak{j}_{n,r} < \mathfrak{i}$. Thus, f descends to an Anosov automorphism of $\mathfrak{f}_{n,r}(\mathbb{R})/\mathfrak{i} \cong \mathfrak{f}_{n,r}(\mathbb{R})/\mathfrak{i}_0$. ☐

We can completely describe Anosov Lie algebras whose associated polynomials p_1 are irreducible of prime degree $n \ge 3$ with cyclic Galois group.

THEOREM 6.2. *Suppose that* $\mathfrak{n} = \mathfrak{f}_{n,r}(\mathbb{R})/\mathfrak{i}$ *is an* r*-step Anosov Lie algebra of type* $(n_1, \dots, n_r)$ *admitting an Anosov automorphism defined by a semisimple hyperbolic matrix in* $GL_n(\mathbb{Z})$*, a rational Hall basis* $\mathscr{B}$*, the resulting automorphism* f *in* $\operatorname{Aut}(\mathfrak{f}_{n,r})$*, and an ideal* $\mathfrak{i}$ *satisfying the Auslander–Scheuneman conditions. Let* $(p_1, \dots, p_r)$ *be the associated* r*-tuple of polynomials. Suppose that* p_1 *is irreducible of prime degree* $n = n_1 \geq 3$*, and that* p_1 *has cyclic Galois group* G*. Then the ideal* $\mathfrak{i}$ *is of cyclic type as in Definition 4.5, and* $\mathfrak{i}$ *contains* $\mathfrak{j}_{n,r}$*, where* $\mathfrak{j}_{n,r}$ *is as in Definition 5.3. Furthermore,* $n = n_1$ *divides* n_i *for all* $i = 2, \cdots, r$*.*

Conversely, for any prime $n \geq 3$*, a Lie algebra* $\mathfrak{f}_{n,r}(\mathbb{R})/\mathfrak{i}$ *of cyclic type is Anosov, as long as the ideal* $\mathfrak{i}$ *contains* $\mathfrak{j}_{n,r}$*.*

Proof. Let $\mathfrak{i}, \mathfrak{f}_{n,r}$ and $(p_1, \dots, p_r)$ be as in the statement of the theorem. By Remark 5.4, the ideal $\mathfrak{j}_{n,r}$ is contained in $\mathfrak{i}$. Recall that any irreducible polynomial in $\mathbb{Z}[x]$ of prime degree $n \geq 3$ and cyclic Galois group has totally real roots; hence f has real spectrum.

Let $\mathscr{C}_1$ be a basis of eigenvectors for $f|_{V_1(\mathbb{R})}$ that is compatible with the rational structure, and let $\mathscr{C} = \bigcup_{i=1}^r \mathscr{C}_i$ be the Hall basis defined by $\mathscr{C}_1$. By Theorem 5.1, for any $\boldsymbol{w}$ in $\mathscr{C}_i$, the orbit

$$E_G^K(\boldsymbol{w}) = \operatorname{span}_K\{\sigma \cdot \boldsymbol{w} : \sigma \in G\}$$

is a rational invariant subspace whose characteristic polynomial is a power r^s of an irreducible polynomial r. The dimension d of $E_G^K(\boldsymbol{w})$ is n or less because $|G| = n$. The dimension d also satisfies $d = s \cdot \deg r$. Because the splitting field for r is a subfield of $\mathbb{Q}(p_1)$ and the Galois group G for p_1 is isomorphic to the simple group C_n, either r is linear or r is irreducible of degree n and has Galois group G. Hence, either $\mathfrak{i}(G, \boldsymbol{w})$ is contained in $\mathfrak{i}$, or it is n-dimensional and its intersection with $\mathfrak{i}$ is trivial. The ideal $\mathfrak{i}$ is then a direct sum of subspaces of the form $E_G^K(\boldsymbol{w})$, hence is of cyclic type. Since each step V_i decomposes as the direct sum of $\mathfrak{i} \cap V_i$ and n-dimensional subspaces of the form $E_G^K(\boldsymbol{w})$, for $\boldsymbol{w} \in \mathscr{C}_i$, the dimension of the ith step of the quotient $\mathfrak{n}$ is divisible by n, for all $i = 2, \dots, n$.

Now let $\mathfrak{n}$ be the quotient of $\mathfrak{f}_{n,r}(\mathbb{R}) = \bigoplus_{i=1}^r V_i$ by an ideal $\mathfrak{i}_0$ of cyclic type relative to some Hall basis $\mathscr{D} = \bigcup_{i=1}^r \mathscr{D}_i$, where $n \geq 3$ is prime, and suppose that $\mathfrak{i}_0 > \mathfrak{j}_{n,r}$ where $\mathfrak{j}_{n,r}$ is defined relative to $\mathscr{D}$. We will show that $\mathfrak{n}$ is Anosov. By Proposition 3.10, there exists an Anosov polynomial p_1 whose Galois group is cyclic of order n. By using the companion matrix to p_1 and a Hall basis $\mathscr{B}$, we can define an automorphism f of $\mathfrak{f}_{n,r}(\mathbb{R})$. Let $\mathscr{C}_1$ be an eigenvector basis for V_1 that is compatible with the rational structure defined by $\mathscr{B}$, and let $\mathscr{C}$ be the Hall basis defined by $\mathscr{C}_1$. Then there is an ideal $\mathfrak{i}$ of $\mathfrak{f}_{n,r}(\mathbb{R})$ that is cyclic relative to the G-action on the Hall basis $\mathscr{C}$ and is isomorphic to $\mathfrak{i}_0$.

The ideal $\mathfrak{i}$ is rational and invariant by Theorem 5.1. All we need to show is that the quotient map $\bar{f}$ on $\mathfrak{f}_{n,r}(\mathbb{R})/\mathfrak{i}$ has no roots of modulus one. But we have already shown in the first part of the proof that irreducible factors for the characteristic polynomial for f corresponding to minimal rational invariant subspaces are either linear or of odd degree. By reasoning as in the proof of Corollary 3.9, it can be seen that the set of roots of p_1 is of full rank, so the rational invariant subspaces corresponding to linear factors are in $\mathfrak{j}_{n,r} < \mathfrak{i}$. If an irreducible factor has odd degree, by Remark 2.7, the polynomial has no roots of modulus one. Hence $\bar{f}$ is Anosov. □

7. ANOSOV AUTOMORPHISMS IN LOW DIMENSIONS

In this section we describe Anosov automorphisms of some nilpotent Lie algebras that arise from Anosov polynomials of low degree.

7.1. When p_1 is a product of quadratics.

We analyze Anosov automorphisms for which the associated polynomial p_1 is a product of quadratic polynomials. To do this we need to define a family of two-step nilpotent Lie algebras.

DEFINITION 7.1. Let $\mathfrak{f}_{2n,2}(\mathbb{R}) = V_1(\mathbb{R}) \oplus V_2(\mathbb{R})$ be the free two-step Lie algebra on $2n$ generators, where $n \geq 2$. Let $\mathscr{C}_1 = \{\boldsymbol{z}_1, \ldots \boldsymbol{z}_{2n}\}$ be a set of generating vectors spanning $V_1(\mathbb{R})$, and let $\mathscr{C}$ be the Hall basis determined by $\mathscr{C}_1$. Let S_1 and S_2 be subsets of the set $\{\{i,j\} : 1 \leq i < j \leq n\}$ of subsets of $\{1, 2, \ldots, n\}$ of cardinality two. To the subsets S_1 and S_2, associate the ideal $\mathfrak{i}(S_1, S_2)$ of $\mathfrak{f}_{n,2}(\mathbb{R})$ defined by

$$
\begin{aligned}
(8) \quad \mathfrak{i}(S_1,S_2) = \bigoplus_{i=1}^{n} \operatorname{span}\{[\boldsymbol{z}_{2i-1}, \boldsymbol{z}_{2i}]\} \oplus \bigoplus_{\{i,j\}\in S_1} & \operatorname{span}\{[\boldsymbol{z}_{2i-1}, \boldsymbol{z}_{2j-1}], [\boldsymbol{z}_{2i}, \boldsymbol{z}_{2j}]\} \oplus \\
& \bigoplus_{\{i,j\}\in S_2} \operatorname{span}\{[\boldsymbol{z}_{2i-1}, \boldsymbol{z}_{2j}], [\boldsymbol{z}_{2i}, \boldsymbol{z}_{2j-1}]\}
\end{aligned}
$$

Define the two-step nilpotent Lie algebra $\mathfrak{n}(S_1, S_2)$ to be $\mathfrak{f}_{2n,2}(\mathbb{R})/\mathfrak{i}(S_1, S_2)$. A two-step Lie algebra of this form will be said to be of *quadratic type.*

In the next theorem we classify two-step Anosov Lie algebras such that the polynomial p_1 is a product of quadratics.

THEOREM 7.2. *Suppose that the polynomial p_1 associated to a semisimple Anosov automorphism f of a two-step Anosov Lie algebra $\mathfrak{n}$ is the product of quadratic polynomials. Then $\mathfrak{n}$ is of quadratic type, as defined in Definition 7.1, and all the eigenvalues of f are real. Furthermore, every two-step nilpotent Lie algebra of quadratic type is Anosov.*

Proof. Let $\mathfrak{f}_{2n,2}(\mathbb{R}) = V_1(\mathbb{R}) \oplus V_2(\mathbb{R})$ be the free two-step nilpotent Lie algebra on $2n$ generators. Let $\bar{f}$ be an an Anosov automorphism of $\mathfrak{n} = \mathfrak{f}_{2n,2}(\mathbb{R})/\mathfrak{i}$ defined by automorphism f of $\mathfrak{f}_{2n,2}(\mathbb{R})$, a Hall basis $\mathscr{B}$ and an ideal $\mathfrak{i}$ satisfying the Auslander–Scheuneman conditions. Without loss of generality, assume that $\mathfrak{i} < V_2(\mathbb{R})$. Let (p_1, p_2) denote the pair of polynomials associated to f, and assume that the polynomial p_1 of degree $2n$ is the product of n quadratic Anosov polynomials $r_1, \ldots, r_n$.

By the quadratic equation, any roots of a quadratic Anosov polynomial are real. All the eigenvalues of the Anosov automorphism $\bar{f}$ are Hall words in the roots of p_1, hence are real. The subspace $V_1(\mathbb{R})$ decomposes as the direct sum $\bigoplus_{i=1}^{n} E_i$ of rational invariant subspaces such that the characteristic polynomial for $f|_{E_i}$ is r_i. For $i = 1, \ldots, n$, let $\boldsymbol{z}_{2i-1}$ and $\boldsymbol{z}_{2i}$ denote eigenvectors in E_i with eigenvalues α_{2i-1} and α_{2i}, respectively. We may assume without loss of generality that $\alpha_{2i-1} > 1 > \alpha_{2i} = \alpha_{2i-1}^{-1}$. As in Example 2.6, the polynomial p_2 may be written as

$$p_2 = \prod_{i=1}^{n} (r_i \wedge r_i) \times \prod_{1 \le i < j \le n} (r_i \wedge r_j),$$

and this factorization corresponds to a decomposition $V_2(\mathbb{R}) = \bigoplus_{1 \le i \le j \le n} [E_i, E_j]$ of $V_2(\mathbb{R})$ into rational invariant subspaces.

For all $i = 1, \ldots, n$, the polynomial $r_i \wedge r_i$ is linear with root $\alpha_{2i-1}\alpha_{2i} = 1$, so $(r_i \wedge r_i)(x) = x - 1$. Therefore, the ideal $\mathfrak{i}$ must contain the n-dimensional subspace $\bigoplus_{i=1}^{n} [E_i, E_i]$ of $V_2(\mathbb{R})$. For $i \neq j$, the polynomial $r_i \wedge r_j$ is given by

$$(r_i \wedge r_j)(x) = (x - \alpha_{2i-1}\alpha_{2j-1})(x - \alpha_{2i-1}\alpha_{2j-1}^{-1})(x - \alpha_{2i-1}^{-1}\alpha_{2j-1})(x - \alpha_{2i-1}^{-1}\alpha_{2j-1}^{-1}).$$

Minimal nontrivial invariant subspaces of $[E_i, E_j]$ correspond to factorizations of $r_i \wedge r_j$ over $\mathbb{Z}$.

If the splitting fields $\mathbb{Q}(r_i)$ and $\mathbb{Q}(r_j)$ do not coincide, then they are linearly disjoint, and $\mathbb{Q}(r_i \wedge r_j) = \mathbb{Q}(r_i)\mathbb{Q}(r_j)$ is a biquadratic extension of $\mathbb{Q}$. Therefore, $r_i \wedge r_j$ is irreducible, and $[E_i, E_j]$ has no nontrivial rational invariant subspaces.

Now suppose that the splitting fields $\mathbb{Q}(r_i)$ and $\mathbb{Q}(r_j)$ are equal, for $i \neq j$. Since the roots of Anosov quadratics are real, the field $\mathbb{Q}(r_i)$ is a totally real quadratic extension of $\mathbb{Q}$. By Dirichlet's Fundamental Theorem, there are units $\zeta = \pm 1$, and η, a fundamental unit in $\mathbb{Q}(r_i)$, such that any unit β in $\mathbb{Q}(r_i)$ can be expressed as $\beta = \zeta^a \eta^b$, where $a \in \{0, 1\}$ and $b \in \mathbb{Z}$. We may choose $\eta > 1$. The Galois group for r_1 is generated by the automorphism of $\mathbb{Q}(\eta)$ mapping η to η^{-1}.

We can write

$$\alpha_{2i-1} = \eta^{b_i}, \alpha_{2i} = \eta^{-b_i}, \alpha_{2j-1} = \eta^{b_j} \text{ and } \quad \alpha_{2j} = \eta^{-b_j},$$

where b_i and b_j are in $\mathbb{Z}^+$. The four roots of $(r_i \wedge r_j)(x)$ are then the numbers $\eta^{\pm b_i \pm b_j}$. Therefore, $r_i \wedge r_j$ factors over $\mathbb{Z}[x]$ as the product of two quadratics in $\mathbb{Z}[x]$

$$(x - \eta^{b_i + b_j})(x - \eta^{-b_i - b_j}) = (x - \alpha_{2i-1}\alpha_{2j-1})(x - \alpha_{2i}\alpha_{2j}) \quad \text{and}$$
$$(x - \eta^{b_i - b_j})(x - \eta^{-b_i + b_j}) = (x - \alpha_{2i-1}\alpha_{2j-1}^{-1})(x - \alpha_{2i-1}^{-1}\alpha_{2j-1}^{-1}).$$

The first polynomial is irreducible since $b_i + b_j > 0$, and η is not a root of unity. If $\alpha_{2i-1} = \alpha_{2j-1}$, then $b_i = b_j$ and the second polynomial is equal to $(x-1)^2$.

This analysis shows that $\mathfrak{i} \cap [E_i, E_j]$ must be one of the subspaces $\{0\}$, $[E_i, E_j]$,

$$E_{i,j} = \operatorname{span}\{[\boldsymbol{z}_{2i-1}, \boldsymbol{z}_{2j-1}], [\boldsymbol{z}_{2i}, \boldsymbol{z}_{2j}]\} \quad \text{or} \quad E'_{i,j} = \operatorname{span}\{[\boldsymbol{z}_{2i-1}, \boldsymbol{z}_{2j}], [\boldsymbol{z}_{2i}, \boldsymbol{z}_{2j-1}]\}.$$

Then $\mathfrak{i}$ is the direct sum of $\bigoplus_{i=1}^{n} [E_i, E_i]$, subspaces of the form $E_{i,j}$, and subspaces of the form $E'_{i,j}$. Therefore, $\mathfrak{n}$ is of quadratic type.

Conversely, we show that for any ideal $\mathfrak{i}$ of quadratic type in $\mathfrak{f}_{2n,2}(\mathbb{R})$, the quotient $\mathfrak{n} = \mathfrak{f}_{2n,2}(\mathbb{R})/\mathfrak{i}$ is an Anosov Lie algebra. Suppose that $\mathfrak{i}(S_1, S_2)$ is an ideal as in the definition of quadratic type. Fix a fundamental unit η in a totally real quadratic extension of $\mathbb{Q}$. Let $b_1, \ldots, b_n$ be distinct positive integers. For each $i = 1, \ldots, n$, define the polynomial r_i in $\mathbb{Z}[x]$ by

$$r_i(x) = (x - \eta^{b_i})(x - \eta^{-b_i})$$

and let $p_1 = r_1 \cdots r_n$. Then for all $i \neq j$, the polynomial $p_i \wedge p_j$ factors over $\mathbb{Z}$ as the product of pairs of irreducible quadratic polynomials

$$(x - \eta^{b_i + b_j})(x - \eta^{-b_i - b_j}) \quad \text{and} \quad (x - \eta^{b_i - b_j})(x - \eta^{-b_i + b_j})$$

in $\mathbb{Z}[x]$, neither having roots of modulus one. The two factors give a two rational invariant subspace of the form E_{ij} and E'_{ij}. Therefore the ideal

$$\mathfrak{i} = \bigoplus_{i=1}^{n} [E_i, E_i] \oplus \bigoplus_{\{i,j\} \in S_1} E_{i,j} \oplus \bigoplus_{\{i,j\} \in S_2} E'_{i,j}$$

satisfies the four Auslander–Scheuneman conditions and is of the form $\mathfrak{i}(S_1, S_2)$ with respect to the appropriate Hall basis of $\mathfrak{f}_{2n,2}(\mathbb{R})$. This completes the proof of the theorem. □

7.2. **When p_1 is a cubic.** We can classify all Anosov Lie algebras of type $(3, \ldots, n_r)$ with $r = 2$ or $r = 3$.

THEOREM 7.3. *If $\mathfrak{n}$ is a two-step Anosov Lie algebra of type $(3, n_2)$, then $\mathfrak{n} \cong \mathfrak{f}_{3,2}$. If $\mathfrak{n}$ is three-step Anosov Lie algebra of type $(3, n_2, n_3)$, then $\mathfrak{n}$ is isomorphic to the Anosov Lie algebra $\mathfrak{f}_{3,3}/F_2$ of type $(3,3,6)$, where $\mathfrak{i} = F_2$ is as defined in Equation (1), or $\mathfrak{i}_1 = F_2 \oplus \mathfrak{i}(C_3, [[\boldsymbol{z}_2, \boldsymbol{z}_1], \boldsymbol{z}_2])$ of type $(3,3,3)$, where $\mathfrak{i}(C_3, [[\boldsymbol{z}_2, \boldsymbol{z}_1], \boldsymbol{z}_2])$ is as in Definition 4.5.*

Proof. Suppose that A is a semisimple hyperbolic matrix in $GL_n(\mathbb{Z})$ with associated triple of polynomials (p_1, p_2, p_3). Let $\mathscr{B}_1$ be a generating set for $\mathfrak{f}_{3,3}$ and let $\mathscr{B}$ be the Hall basis determined by $\mathscr{B}$. Let $\alpha_1, \alpha_2, \alpha_3$ denote the roots of p_1. Since p_1 can not have 1 or -1 as a root, it is irreducible over $\mathbb{Z}$, and the Galois group G of the splitting field of p_1 is either C_3 or S_3.

As demonstrated in Example 2.5, the cubic polynomial p_2 is irreducible and Anosov, so there are no Anosov Lie algebras of type $(3, n_2)$ other than $\mathfrak{f}_{3,2}$.

Let $\mathfrak{i}$ be an ideal such that f descends to an Anosov automorphism of a three-step nilpotent Lie algebra $\mathfrak{f}_{3,2}/\mathfrak{i}$. If G is symmetric, then $\mathfrak{i} = F_2$ by Theorem 6.1. If G is cyclic, by Theorem 6.2, $\mathfrak{i}$ is either F_2 or it is an ideal of the form $F_2 \oplus \mathfrak{i}(C_3, \boldsymbol{w})$, for $\boldsymbol{w} \in \mathscr{C}'_3$.

Each Lie algebra that is listed may be realized by choosing the appropriate Anosov polynomial from Table 1 and using its companion matrix A to define an automorphism of $\mathfrak{f}_{3,2}$ or $\mathfrak{f}_{3,3}$. When $r = 3$, by Lemma 3.4, all vectors with eigenvalue ±1 will be in the kernel $F_2 = \mathfrak{j}_{3,3}$. □

7.3. **When p_1 is a quartic.** Now we consider the case that p_1 is a quartic Anosov polynomial. The next lemma is useful for understanding Anosov Lie algebras of type $(4, n_2)$.

$\mathfrak{n} = \mathfrak{f}_{n,r}(\mathbb{R})/\mathfrak{i}$	type	ideal $\mathfrak{i}$	reference for definition of $\mathfrak{i}$
$\mathfrak{f}_{3,2}$	$(3,3)$	$\{0\}$	
$\mathfrak{f}_{4,2}$	$(4,6)$	$\{0\}$	
$\mathfrak{f}_{4,2}/\mathfrak{i}$	$(4,4)$	$\mathfrak{i}(V_4,[z_2,z_1])$	Definition 4.5
$\mathfrak{f}_{4,2}/\mathfrak{i} \cong \mathfrak{h}_3 \oplus \mathfrak{h}_3$	$(4,2)$	$\mathfrak{i}(C_4,[z_2,z_1])$	Definition 4.5
$\mathfrak{f}_{5,2}$	$(5,10)$	$\{0\}$	
$\mathfrak{f}_{5,2}/\mathfrak{i}$	$(5,9)$	$\mathfrak{i}_1$	Definition 7.6
$\mathfrak{f}_{5,2}/\mathfrak{i}$	$(5,6)$	$\mathfrak{i}_1 \oplus \mathfrak{i}_2$	Definition 7.6
$\mathfrak{f}_{5,2}/\mathfrak{i}$	$(5,5)$	$\mathfrak{i}(C_5,[z_2,z_1])$	Definition 4.5
$\mathfrak{f}_{5,2}/\mathfrak{i}$	$(5,5)$	$\mathfrak{i}_2$	Example 5.2
$\mathfrak{f}_{5,2}/\mathfrak{i} \cong \mathbb{R}^2 \oplus \mathfrak{f}_{3,2}$	$(5,3)$	$\mathfrak{i}_1 \oplus \mathfrak{i}_3$	Definition 7.6

TABLE 3. Two-step Anosov Lie algebras of type (n_1, n_2) with $n_1 \leq 5$.

LEMMA 7.4. *Let (p_1, p_2) be the pair of polynomials associated to an irreducible Anosov polynomial p_1 of degree four. Let G denote the Galois group of the splitting field for p_1. Then*

1. *$G \cong S_4$ or $G \cong A_4$ if and only if p_2 is irreducible.*
2. *$G \cong C_4$ or $G \cong D_8$ if and only if p_2 has an irreducible quartic factor.*
3. *$G \cong V_4$ if and only if p_2 has no irreducible factors of degree three or more.*

Furthermore, roots of p_2 come in reciprocal pairs β and $\pm\beta^{-1}$.

Proof. Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ denote the four distinct roots of p_1. Then the roots of p_2 are the six numbers $\alpha_i\alpha_j$, where $1 \leq i < j \leq 4$. Because $\alpha_1\alpha_2\alpha_3\alpha_4 = \pm 1$, roots of p_2 come in pairs such as $\alpha_1\alpha_2$ and $\alpha_3\alpha_4 = \pm(\alpha_1\alpha_2)^{-1}$.

The resolvent cubic r for p_1 has roots

$$\beta_1 = \alpha_1\alpha_2 + \alpha_3\alpha_4, \quad \beta_2 = \alpha_1\alpha_3 + \alpha_2\alpha_4, \quad \text{and} \quad \beta_3 = \alpha_1\alpha_4 + \alpha_2\alpha_3.$$

Recall that one of three things must occur:

(1) none of $\beta_1, \beta_2, \beta_3$ lies in $\mathbb{Q}$, r is irreducible, and $G \cong S_4$ or $G \cong A_4$,

(2) exactly one of $\beta_1, \beta_2, \beta_3$ lies in $\mathbb{Q}$, r is the product of an irreducible quadratic and a linear factor, and $G \cong C_4$ or $G \cong D_8$, and
(3) $\beta_1, \beta_2, \beta_3$ all lie in $\mathbb{Q}$, r splits over $\mathbb{Q}$, and $G \cong V_4$.

Since $\alpha_1\alpha_2\alpha_3\alpha_4 = \pm 1$, $\beta_1 = \alpha_1\alpha_2 + \alpha_3\alpha_4 \in \mathbb{Q}$ if and only if $(x-\alpha_1\alpha_2)(x-\alpha_3\alpha_4)$ is a quadratic factor of p_2 over $\mathbb{Q}$. Factors that are the counterparts from β_2 and β_3,

$$(x-\alpha_1\alpha_3)(x-\alpha_2\alpha_4) \quad \text{and} \quad (x-\alpha_1\alpha_4)(x-\alpha_2\alpha_3), \tag{9}$$

are in $\mathbb{Q}[x]$ depending on whether β_2 and β_3 are in $\mathbb{Q}$. Therefore, when all of $\beta_1, \beta_2, \beta_3$ lie in $\mathbb{Q}$, p_2 factors as the product of quadratics in $\mathbb{Q}[x]$, establishing the claim in Case (3).

In Case (1), p_2 is irreducible by Theorem 6.1, Part (1).

In Case (2), when G is C_4 or D_8, there is a G-orbit of cardinality four, something like $\{\alpha_1\alpha_3, \alpha_1\alpha_4, \alpha_2\alpha_3, \alpha_2\alpha_4\}$, that by Theorem 5.1 yields a factor

$$q(x) = (x-\alpha_1\alpha_3)(x-\alpha_2\alpha_4)(x-\alpha_1\alpha_4)(x-\alpha_2\alpha_3)$$

of p_2, and an orbit of cardinality two corresponding to a factor $(x-\alpha_1\alpha_2)(x-\alpha_3\alpha_4)$ of p_2. Then $\beta_1 \in \mathbb{Q}$ and $\beta_2, \beta_3 \notin \mathbb{Q}$. By Theorem 5.1, if q were to factor, it would be a power of an irreducible. Knowing that $\beta_2, \beta_3 \notin \mathbb{Q}$ rules out the polynomials in Equation (9) as factors of q. The only other options for factors yield contradictions to the distinctness of roots of p_1. Thus, q is an irreducible quartic factor of p_2. □

Now we are ready to classify two-step Anosov Lie algebras of type $(4, n_2)$ using the methods we have established.

THEOREM 7.5. *If $\mathfrak{n}$ is an Anosov Lie algebra of type $(4, n_2)$, then $\mathfrak{n}$ is one of the Anosov Lie algebras listed in Table 3.*

Proof. Suppose that f is a semisimple automorphism of $\mathfrak{f}_{4,2}(\mathbb{R}) = V_1(\mathbb{R}) \oplus V_2(\mathbb{R})$ that projects to an Anosov automorphism of a two-step quotient $\mathfrak{n} = \mathfrak{f}_{4,2}/\mathfrak{i}$. Let (p_1, p_2) be the polynomials associated to f, and let K be the splitting field for p_1. Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ be the roots of p_1 and let $\boldsymbol{z}_1, \boldsymbol{z}_2, \boldsymbol{z}_3, \boldsymbol{z}_4$ be corresponding eigenvectors for f_K in $V_1(K)$.

If p_1 is reducible, since it is Anosov, it is a product of quadratics, and Theorem 7.2 describes the Anosov Lie algebras in that situation: $\mathfrak{i}$ is one of $\mathfrak{i}(V_4, [\boldsymbol{z}_2, \boldsymbol{z}_1])$ and $\mathfrak{i}(C_4, [\boldsymbol{z}_2, \boldsymbol{z}_1])$.

Assume that p_1 is irreducible. The Galois group G of p_1 is then a transitive permutation group of degree four.

By Lemma 7.4, if G is S_4 or A_4, the polynomial p_2 is irreducible, so $V_2(\mathbb{R})$ has no nontrivial proper rational invariant subspaces; hence, $\mathfrak{i} = \{0\}$, and $\mathfrak{n}$ is free. If G is D_8 or C_4, then by considering the action of G on the set of roots it can be seen that $\mathfrak{f}_{4,2}(K)$ is the direct sum of a four-dimensional rational f_K-invariant subspace $\mathfrak{i}_1$ and a two-dimensional rational f_K-invariant subspace $\mathfrak{i}_2$. If the roots are real, these may be represented as follows, renumbering the roots if necessary,

$$\mathfrak{i}_1 = \mathfrak{i}(C_4, [\boldsymbol{z}_2, \boldsymbol{z}_1]) = \mathfrak{i}(D_8, [\boldsymbol{z}_2, \boldsymbol{z}_1]), \quad \text{and} \quad \mathfrak{i}_2 = \mathfrak{i}(C_4, [\boldsymbol{z}_3, \boldsymbol{z}_1]) = \mathfrak{i}(D_8, [\boldsymbol{z}_3, \boldsymbol{z}_1]).$$

By Lemma 7.4, the minimal polynomial for $\mathfrak{i}_1$ is irreducible, so $\mathfrak{i}$ is a minimal nontrivial invariant subspace. If there are complex roots, a short computation shows that the rational invariant subspaces $\mathfrak{i}_1'$ and $\mathfrak{i}_2'$ of $\mathfrak{f}_{4,2}(K)$ yield rational invariant subspaces $(\mathfrak{i}_1')^{\mathbb{R}}$ and $(\mathfrak{i}_2')^{\mathbb{R}}$ of $\mathfrak{f}_{4,2}(\mathbb{R})$ isomorphic to $\mathfrak{i}_1$ and $\mathfrak{i}_2$. The characteristic polynomial for $\mathfrak{i}_2$ is either irreducible or it has roots ± 1, in which case $\mathfrak{i}_2$ must be contained in the ideal $\mathfrak{i}$. Thus, when $G = C_4$ or D_8, the ideal $\mathfrak{i}$ is $\{0\}, \mathfrak{i}_1$ or $\mathfrak{i}_2$.

Finally, if G is the Klein four-group, by the same reasoning, $V_2(K)$ is the direct sum of three two-dimensional ideals that are either minimal or must be contained in $\mathfrak{i}$. Then $\mathfrak{n}$ is of quadratic type.

Anosov automorphisms of all types $(4, n_2)$ may be realized by choosing an appropriate polynomial p_1 from Table 1. By Remark 2.7, the polynomial p_2 defined by such a p_1 can not have any nonreal roots of modulus one unless it is self-reciprocal. The only polynomial listed in Table 1 that is self-reciprocal is for V_4, so p_2 will not have roots of modulus one unless $G = V_4$, in which case the eigenspaces of those roots lie in the ideal $\mathfrak{i}$. □

7.4. **When p_1 is a quintic.**

First we define some two-step nilpotent Lie algebras on five generators.

DEFINITION 7.6. Let $\mathfrak{f}_{5,2} = V_1(\mathbb{R}) \oplus V_2(\mathbb{R})$ be a free Lie algebra on five generators $\{z_i\}_{i=1}^5$. Define subspaces E_1 and E_2 of $V_1(\mathbb{R})$ by $E_1 = \operatorname{span}_{\mathbb{R}}\{z_1, z_2\}$ and let $E_2 = \operatorname{span}_{\mathbb{R}}\{z_3, z_4, z_5\}$, and define ideals of $\mathfrak{f}_{5,2}$ by

$$\mathfrak{i}_1 = [E_1, E_1], \quad \mathfrak{i}_2 = [E_1, E_2], \quad \text{and} \quad \mathfrak{i}_3 = [E_2, E_2].$$

Define two-step Lie algebras by

$$\mathfrak{n}_1 = \mathfrak{f}_{5,2}/\mathfrak{i}_1, \quad \mathfrak{n}_2 = \mathfrak{f}_{5,2}/(\mathfrak{i}_1 \oplus \mathfrak{i}_2), \quad \text{and} \quad \mathfrak{n}_3 = \mathfrak{f}_{5,2}/(\mathfrak{i}_1 \oplus \mathfrak{i}_3).$$

These are of types $(5,9)$, $(5,3)$ and $(5,6)$, respectively. Note that $\mathfrak{n}_2 \cong \mathbb{R}^2 \oplus \mathfrak{f}_{3,2}$.

THEOREM 7.7. *Suppose that $\mathfrak{n}$ is a two-step nilpotent Lie algebra of type $(5, n_2)$ admitting an Anosov automorphism f. Then $\mathfrak{n}$ is one of the Lie algebras listed in Table 3. Furthermore, all of the Lie algebras of type $(5, n_2)$ in Table 3 are Anosov.*

Proof. Let (p_1, p_2) be the pair of polynomials associated to an automorphism f of $\mathfrak{f}_{5,2} = V_1(\mathbb{R}) \oplus V_2(\mathbb{R})$ that projects to an Anosov automorphism of an Anosov Lie algebra $\mathfrak{n} = \mathfrak{f}_{5,2}/\mathfrak{i}$, for some ideal $\mathfrak{i}$ of $\mathfrak{f}_{5,2}$ satisfying the Auslander–Scheuneman conditions. Without loss of generality we assume that the roots of p_1 have product 1.

First, suppose that p_1 is irreducible, so its Galois group G is isomorphic to S_5, A_5, D_{10}, C_5 or the holomorph $\mathrm{Hol}(C_5)$ of C_5. If G is isomorphic to one of S_5, A_5, and $\mathrm{Hol}(C_5)$, then the action of G on the roots of p_1 is two-transitive, so by Theorem 6.1, $\mathfrak{n}$ is isomorphic to $\mathfrak{f}_{5,2}$. The case that the Galois group is D_{10} was considered in Example 5.2, where it was found that either $\mathfrak{n}$ is free, $\mathfrak{n}$ is isomorphic to $\mathfrak{n}_1$ or $\mathfrak{n}_2$ in Example 5.2, both of type $(5,5)$. The case of C_5 is covered by Theorem 6.2. Thus, in all possible cases, $\mathfrak{n}$ is one of the Lie algebras listed in Table 3. Each example may be realized by choosing a polynomial p_1 from Table 1, and using its companion matrix to define an automorphism of $\mathfrak{f}_{5,2}$.

To get $\mathfrak{n}_1$, one needs to choose p_1 with all real roots, and to get $\mathfrak{n}_2$, one needs p_2 to have four nonreal roots. Examples of both kinds are in the table. The associated polynomials p_2 have no roots of modulus one by Lemma 3.4.

Now suppose that the Anosov polynomial p_1 is the product of a quadratic Anosov polynomial r_1 and a cubic Anosov polynomial r_2. Let E_1 and E_2 denote the rational invariant subspaces of $V_1(\mathbb{R})$ corresponding to r_1 and r_2 respectively. Because r_1 is quadratic, $r_1 \wedge r_1 = x \pm 1$, so $\mathfrak{i}_1 = [E_1, E_1]$ must be contained in $\mathfrak{i}$. As seen in Example 2.5, since r_2 is a cubic, the polynomial $r_2 \wedge r_2$ is irreducible and Anosov. Therefore, the set $\mathfrak{i}_3 = [E_2, E_2]$ is a minimal nontrivial invariant subspace of $V_2(\mathbb{R})$.

Let α_1 and α_2 denote the roots of r_1, while $\alpha_3, \alpha_4, \alpha_5$ are the roots of r_2. Then $\alpha_1\alpha_3$ is a root of $r_1 \wedge r_2$. By standard arguments, $[\mathbb{Q}(\alpha_1\alpha_3) : \mathbb{Q}] = 6$, so $r_1 \wedge r_2$ is the minimal polynomial of $\alpha_1\alpha_3$. Therefore $r_1 \wedge r_2$ is irreducible and $\mathfrak{i}_2 = [E_1, E_2]$ is a minimal nontrivial rational invariant subspace. The subspace $V_2(\mathbb{R})$ decomposes as the sum $\mathfrak{i}_1 \oplus \mathfrak{i}_2 \oplus \mathfrak{i}_3$ of minimal nontrivial invariant subspaces, where $\mathfrak{i}_1, \mathfrak{i}_2$ and $\mathfrak{i}_3$ are as in Definition 7.6, and the only possibilities for an ideal $\mathfrak{i}$ defining a two-step Anosov quotient are $\mathfrak{i}_1, \mathfrak{i}_1 \oplus \mathfrak{i}_2$ and $\mathfrak{i}_1 \oplus \mathfrak{i}_3$, as claimed.

Choosing r_1 and r_2 to be arbitrary Anosov polynomials of degree two and three, respectively, yields an Anosov polynomial $p_1 = r_1 r_2$ such that the corresponding automorphism f of $\mathfrak{f}_{5,2}$ admits quotients of all types listed in the table. By choosing r_2 so it has real roots, the roots of p_2 are real, and $r_1 \wedge r_2$ has no roots of modulus one. □

8. PROOFS OF MAIN THEOREMS

Now we provide proofs for the theorems presented in Section 1.

Proof of Theorem 1.1. Suppose that $\mathfrak{n}$ is a two-step Anosov Lie algebra of type (n_1, n_2) with associated polynomials (p_1, p_2). If $n_1 = 3, 4$, or 5, then $\mathfrak{n}$ is one of the Lie algebras in Table 3, by Theorems 7.3, 7.5 and 7.7. Therefore, Part (1) of Theorem 1.1 holds.

The second part follows immediately from by Part (1) of Theorem 6.1. □

Proof of Theorem 1.2. The first part of the theorem follows from Theorem 6.2. The second part is a consequence of Theorem 6.1. □

Proof of Theorem 1.3. Corollaries 3.9 and 5.7 imply the theorem. □

Proof of Theorem 1.4. Suppose the spectrum of f is in $\mathbb{Q}(\sqrt{b})$. Then the polynomial p_1 associated to f is a product of quadratics, each of whose roots lie in $\mathbb{Q}(\sqrt{b})$. Theorem 7.2 implies that $\mathfrak{n}$ is one of the Lie algebras in Definition 7.1. □

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