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VARYING DOMAINS: STABILITY OF THE DIRICHLET AND THE POISSON PROBLEM

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Dedicated to Professor Edward Norman Dancer
on the occasion of his 60th birthday

Abstract. For Ω a bounded open set in $\mathbb{R}^N$ we consider the space $H_0^1(\bar\Omega) = \{u_{|\Omega}\colon u \in H^1(\mathbb{R}^N)\colon u(x) = 0 \text{ a.e. outside } \bar\Omega\}$. The set Ω is called *stable* if $H_0^1(\Omega) = H_0^1(\bar\Omega)$. Stability of Ω can be characterised by the convergence of the solutions of the Poisson equation

$$-\Delta u_n = f \quad \text{in } \mathcal{D}(\Omega_n)', \qquad u_n \in H_0^1(\Omega_n)$$

and also the Dirichlet Problem with respect to Ω_n if Ω_n converges to Ω in a sense to be made precise. We give diverse results in this direction, all with purely analytical tools not referring to abstract potential theory as in Hedberg's survey article [Expo. Math. **11** (1993), 193–259]. The most complete picture is obtained when Ω is supposed to be Dirichlet regular. However, stability does not imply Dirichlet regularity as Lebesgue's cusp shows.

1. **Introduction.** There are two natural ways to define Dirichlet boundary conditions on a bounded open set $\Omega \subset \mathbb{R}^N$ for functions in the Sobolev space $H^1(\Omega) := \{u \in L^2(\Omega)\colon D_j u \in L^2(\Omega), j = 1, \dots N\}$. The most common one consists in considering the space $H_0^1(\Omega)$ defined as the closure of the test functions $\mathcal{D}(\Omega)$ in $H^1(\Omega)$. The other natural space is

$$H_0^1(\bar\Omega) := \{u_{|\Omega}\colon u \in H^1(\mathbb{R}^N),\, u(x) = 0 \text{ a.e. on } \mathbb{R}^N \setminus \Omega\}.$$

The last one always contains $H_0^1(\Omega)$ but the inclusion is strict in general.

The purpose of this article is to characterise when both spaces coincide. We then say that Ω is *stable*. We may associate two realizations Δ_Ω and $\Delta_{\bar\Omega}$ of the Laplacian on $L^2(\Omega)$ associated with the form domains $H_0^1(\Omega)$ and $H_0^1(\bar\Omega)$ corresponding to a "minimal" and a "maximal" realisation of the Laplacian with Dirichlet boundary conditions. Both can be obtained by approximation, the first from the inside, the second from the outside. If $\Omega_n \uparrow \Omega$, that is, if $\Omega_n \subset \Omega_{n+1}$ are open and $\bigcup_{n\in\mathbb{N}} \Omega_n = \Omega$, then

$$e^{t\Delta_{\Omega_n}} \to e^{t\Delta_\Omega}$$

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