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THE GENERALIZED WEINSTEIN–MOSER THEOREM

VIKTOR L. GINZBURG AND BAŞAK Z. GÜREL

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ABSTRACT. In this notice, we outline a proof of the existence of periodic orbits on all low energy levels for a twisted geodesic flow whenever the magnetic field form is symplectic and spherically rational. This result is established as a consequence of a more general theorem concerning periodic orbits of autonomous Hamiltonian flows near Morse–Bott non-degenerate, symplectic extrema. Namely, partially generalizing the Weinstein–Moser theorem, we show that all energy levels near such extrema carry periodic orbits, provided that the ambient manifold meets certain topological requirements. The proof of the generalized Weinstein–Moser theorem is a combination of a Sturm–theoretic argument and a Floer homology calculation.

1. Introduction and main results

In the early 1980s, V.I. Arnold proved, as a consequence of the Conley–Zehnder theorem, [6], the existence of periodic orbits of a twisted geodesic flow on $\mathbb{T}^2$ with symplectic magnetic field for all energy levels when the metric is flat and low energy levels for an arbitrary metric, [2]. This result initiated an extensive study of the existence problem for periodic orbits of general twisted geodesic flows via Hamiltonian dynamical systems methods and in the context of symplectic topology, mainly focusing on low energy levels. In particular, for a symplectic magnetic field the problem was cast in [29] in the framework of the generalized Weinstein–Moser theorem (conjectural, in general) concerning periodic orbits of autonomous Hamiltonians with Morse–Bott non-degenerate, symplectic extrema.

In the present announcement, we outline a proof of a version of this theorem, given in detail in [21]. Namely, we show that all energy levels of a Hamiltonian near a Morse–Bott non-degenerate, symplectic extremum carry periodic orbits, provided that the ambient manifold meets certain topological requirements. This result is a (partial) generalization of the Weinstein–Moser theorem, [34, 44], asserting that a certain number of distinct periodic orbits exist on every energy level near a non-degenerate extremum. As a consequence of the generalized Weinstein–Moser theorem, we establish the existence of periodic orbits of a twisted geodesic flow on all low energy levels and in all dimensions whenever the magnetic field form is symplectic and spherically rational. Here, a new point is that, in contrast with

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other results of this type, we do not require any compatibility conditions on the Hamiltonian and the magnetic field. The proof of the generalized Weinstein–Moser theorem is a combination of a Sturm–theoretic argument utilizing convexity of the Hamiltonian in the direction normal to the critical submanifold and of a Floer–homological calculation that guarantees "dense existence" of periodic orbits with certain index.

1.1. **The generalized Weinstein–Moser theorem.** Throughout the paper, M will stand for a closed symplectic submanifold of a symplectic manifold (P, ω). We denote by $[\omega]$ the cohomology class of ω and by $c_1(TP)$ the first Chern class of P equipped with an almost complex structure compatible with ω. The integrals of these classes over a 2-cycle u will be denoted by $\langle \omega, u\rangle$ and, respectively, $\langle c_1(TP), u\rangle$. Recall also that P is said to be spherically rational if the integrals $\langle \omega, u\rangle$ over all $u \in \pi_2(P)$ are commensurate, i.e., $\lambda_0 = \inf\{|\langle \omega, u\rangle| \mid u \in \pi_2(P)\} > 0$ or, equivalently, $\langle \omega, \pi_2(P)\rangle$ is a discrete subgroup of $\mathbb{R}$.

The key result of [21] is

Theorem 1.1 (Generalized Weinstein–Moser theorem, [21])**.** *Let $K\colon P \to \mathbb{R}$ be a smooth function on a symplectic manifold (P, ω), which attains its minimum $K = 0$ along a closed symplectic submanifold $M \subset P$. Assume in addition that the critical set M is Morse–Bott non-degenerate and one of the following cohomological conditions is satisfied:*

(i) *M is spherically rational and $c_1(TP) = 0$, or*

(ii) *$c_1(TP) = \lambda[\omega]$ for some $\lambda \neq 0$.*

Then for every sufficiently small $r^2 > 0$ the level $K = r^2$ carries a contractible in P periodic orbit of the Hamiltonian flow of K with period bounded from above by a constant independent of r.

When M is a point, Theorem 1.1 turns into the Weinstein–Moser theorem (see [44] and [34]) on the existence of periodic orbits near a non-degenerate extremum, albeit without the lower bound $1 + \dim P/2$ on the number of periodic orbits.

Remark 1.2. The assertion of the theorem is local and concerns only a neighborhood of M in P. Hence, in (i) and (ii), we can replace $c_1(TP)$ by $c_1(TP|_M) = c_1(TM) + c_1(TM^\perp)$ and $[\omega]$ by $[\omega|_M]$. Also note that in (ii) we do not require λ to be positive, i.e., M need not be monotone. (However, this condition does imply that M is spherically rational.) We also emphasize that we do need conditions (i) and (ii) in their entirety – the weaker requirements $c_1(TP)|_{\pi_2(P)} = 0$ or $c_1(TP)|_{\pi_2(P)} = \lambda[\omega]|_{\pi_2(P)}$, common in symplectic topology, are not sufficient for the proof.

Although conditions (i) and (ii) enter our argument in an essential way, their role is probably technical, and one may expect the theorem to hold without any cohomological restrictions on P. For instance, this is the case whenever $\operatorname{codim} M = 2$; see [15]. Furthermore, when $\operatorname{codim} M \geq 2$, the theorem holds without (i) and (ii), provided that ω and the normal direction Hessian $d^2_M K$ meet certain geometrical compatibility requirements; [22, 23, 29]. On the other hand, the condition that the extremum M is Morse–Bott non-degenerate is essential; see [20].

1.2. **Periodic orbits of twisted geodesic flows.** Let M be a closed Riemannian manifold and let σ be a closed 2-form on M (a magnetic field). Equip T^*M with the twisted symplectic structure $\omega = \omega_0 + \pi^*\sigma$, where ω_0 is the standard symplectic form on T^*M and $\pi\colon T^*M \to M$ is the natural projection. Denote by K the standard kinetic energy Hamiltonian on T^*M corresponding to the Riemannian metric on M. The Hamiltonian flow of K on T^*M describes the motion of a charge on M in the magnetic field σ and is sometimes referred to as a twisted geodesic flow; see, e.g., [16] and references therein for more details. Clearly, $c_1(T(T^*M)) = 0$ and M is a Morse–Bott non-degenerate minimum of K. Furthermore, M is a symplectic submanifold of T^*M when the form σ symplectic. Hence, as an immediate application of case (i) of Theorem 1.1, we obtain

Theorem 1.3 ([21])**.** *Assume that σ is symplectic and spherically rational. Then for every sufficiently small $r^2 > 0$ the level $K = r^2$ carries a contractible in T^*M periodic orbit of the twisted geodesic flow with period bounded from above by a constant independent of r.*

Note that, as the example of the horocycle flow shows, a twisted geodesic flow with symplectic magnetic field need not have periodic orbits on all energy levels; see, e.g., [10, 16] for a detailed discussion of this example. Similar examples also exist for twisted geodesic flows when $\dim M > 2$, [17, Section 4].

The question of the existence of periodic orbits of twisted geodesic flows on (low) energy levels for magnetic fields on surfaces is studied in, e.g., [35, 36, 42, 43] in the context of Morse–Novikov theory; see also [8, 9, 10, 18]. For twisted geodesic flows on surfaces with exact magnetic fields, existence of periodic orbits on all energy levels is proved in [10]. In the framework of the generalized Weinstein–Moser theorem or of twisted geodesic flows, the problems of almost existence and dense existence of periodic orbits are investigated in, e.g., [5, 8, 9, 13, 20, 24, 30, 33, 31, 32, 40], following the original work of Hofer and Zehnder and of Struwe, [26, 27, 28, 41]. We refer the reader to [16, 18, 21] for a detailed review of relevant results and further references.

As was pointed out above, requirements (i) and (ii) of Theorem 1.1 can certainly be relaxed. For instance, every low energy level of K carries a periodic orbit whenever $\operatorname{codim} M = 2$ or provided that the normal direction Hessian $d^2_M K$ and ω meet certain geometrical compatibility conditions, which are automatically satisfied when $\operatorname{codim} M = 2$ or M is a point; see [14, 15, 22, 23, 29, 34, 44] and references therein. Moreover, under these conditions, non-trivial lower bounds on the number of distinct periodic orbits have also been obtained.

1.3. **Infinitely many periodic orbits.** The multiplicity results from [2, 14, 15, 22, 23, 29] rely (implicitly in some instances) on the count of "short" periodic orbits of the Hamiltonian flow on the level $K = r^2$. The resulting lower bounds on the number of periodic orbits can be viewed as a "crossing-over" between the Weinstein–Moser type lower bounds in the normal direction to M and the Arnold conjecture type lower bounds along M. This approach encounters serious technical difficulties unless ω and $d^2_M K$ meet certain geometrical compatibility requirements, for otherwise even identifying the class of short orbits is problematic. However, looking at the question from the perspective of the Conley conjecture (see [12, 19, 25, 39]) rather than of the Arnold conjecture, one can expect every low level of K to carry infinitely many periodic orbits (not necessarily short), provided that $\dim M \geq 2$ and M is symplectically aspherical. An indication that this may indeed be the case is given by

Proposition 1.4 ([21])**.** *Assume that M is symplectically aspherical and not a point,* $\operatorname{codim} M = 2$ *and the normal bundle to M in P is trivial. Then every* *level $K = r^2$, where $r > 0$ is sufficiently small, carries infinitely many distinct, contractible in P periodic orbits of K.*

This proposition does not rely on Theorem 1.1 and is an immediate consequence of the results of [2, 14] and the Conley conjecture; see [19] and also [12, 25, 39]. In a similar vein, in the setting of Theorem 1.3 with $M = \mathbb{T}^2$ and K arising from a flat metric, the level $K = r^2$ carries infinitely many periodic orbits for every (not necessarily small) $r > 0$.

2. Outline of the proof

The proof of Theorem 1.1 hinges on an interplay of two counterparts: a version of the Sturm comparison theorem and a Floer homological calculation. Namely, on the one hand, a Floer homological calculation along the lines of [20] guarantees that almost all low energy levels of K carry periodic orbits with Conley–Zehnder index depending only on $\dim P$ and $\dim M$. On the other hand, since K is fiber-wise convex in a tubular neighborhood of M, a Sturm theoretic argument ensures that periodic orbits with large period must have large index. (Strictly speaking, the orbits in question are degenerate and the Conley–Zehnder index is not defined. Hence, we work with the mean index Δ, [39].) Thus, the orbits detected by Floer homology have period *a priori* bounded from above and the existence of periodic orbits on all low levels follows from the Arzela–Ascoli theorem.

In this section we outline the proof of Theorem 1.1 in the particular case where P is geometrically bounded and symplectically aspherical (i.e., $\omega|_{\pi_2(P)} = 0 = c_1(TP)|_{\pi_2(P)}$). Referring the reader to, e.g., [3, 5, 20] for the definition and a detailed discussion of geometrically bounded manifolds, here we only mention that twisted cotangent bundles are geometrically bounded.

2.1. Sturm theory for the Salamon–Zehnder invariant.

Denote by $\Delta(\Phi)$ the mean index of a path $\Phi\colon [a, b] \to \mathrm{Sp}(V)$, introduced in [39] and referred to in what follows as the Salamon–Zehnder invariant. (Here $\mathrm{Sp}(V)$ stands for the group of symplectic linear transformations of a symplectic vector space V.) Recall that $\Delta(\Phi)$ is invariant under homotopy of Φ with fixed end-points. (In particular, Δ gives rise to a continuous map $\widetilde{\mathrm{Sp}}(V) \to \mathbb{R}$, where $\widetilde{\mathrm{Sp}}(V)$ is the universal covering of $\mathrm{Sp}(V)$.) Furthermore, the map Δ is additive with respect to concatenation of paths and $\Delta(\Psi^{-1}\Phi\Psi) = \Delta(\Phi)$ for any two continuous paths Φ and Ψ in $\mathrm{Sp}(V)$.

One additional property of Δ important for the proof of Theorem 1.1 is that $\Delta\colon \widetilde{\mathrm{Sp}}(V) \to \mathbb{R}$ is a quasi-morphism, i.e., for any two elements Φ and Ψ in $\widetilde{\mathrm{Sp}}(V)$ the difference $|\Delta(\Psi\Phi) - \Delta(\Psi) - \Delta(\Phi)|$ is bounded from above by a constant depending only on $\dim V$. Equivalently, for any continuous path Φ in $\mathrm{Sp}(V)$, not necessarily originating at the identity, and for any $A \in \mathrm{Sp}(V)$

$$|\Delta(A\Phi) - \Delta(\Phi)| \leq C, \tag{2.1}$$

where the constant $C \geq 0$ is independent of Φ and A. (Throughout the rest of the section C will always denote such a constant. However, the value of C can vary from one formula to another.)

Finally, assume that $\Phi(0) = I$ and $\Phi(T) - I$ is non-degenerate. (Here $\Phi\colon [0, T] \to \mathrm{Sp}(V)$.) Then the Conley–Zehnder index $\mu_{\mathrm{CZ}}(\Phi)$ is defined (see [7] and also, e.g., [38, 39]) and, as is shown in [39],

$$|\,\mu_{\mathrm{CZ}}(\Phi) - \Delta(\Phi)| \leq \dim V/2. \tag{2.2}$$

Let now $H(t)$ be a time-dependent, quadratic Hamiltonian on V. Denote by $\Phi_H(t) \in \mathrm{Sp}(V)$ the time-dependent flow generated by H via the Hamilton equation. Once V is identified with $\mathbb{R}^{2n} = \mathbb{C}^n$, this equation takes the form

$$\dot{\Phi}_H = -JH(t)\Phi_H(t),$$

where J is the standard complex structure.

Proposition 2.1 (Sturm Comparison Theorem, [21]). *Assume that $H_1(t) \geq H_0(t)$ for all t. Then*

$$\Delta(\Phi_{H_1}) \leq \Delta(\Phi_{H_0}) + C$$

as functions of t, where C depends only on $\dim V$.

This result is yet another version of the comparison theorem in (symplectic) Sturm theory, similar to those established in, e.g., [1, 4, 11]. The proposition can be easily verified by combining the construction of the generalized Maslov index, [37], with the Arnold comparison theorem, [1], and utilizing (2.1) and (2.2).

Example 2.2. Let $H(t)$ be a quadratic Hamiltonian on $\mathbb{R}^{2n}$ such that $H(t)(X) \geq \alpha\|X\|^2$ for all t, where $\|X\|$ stands for the standard Euclidean norm of $X \in \mathbb{R}^{2n}$ and α is a constant. Then, for all t,

$$\Delta(\Phi_H) \leq -2n\alpha \cdot t + C.$$

More generally, let $H(t)$ be a quadratic Hamiltonian on $\mathbb{R}^{2n_1} \times \mathbb{R}^{2n_2}$ such that $H(t)((X,Y)) \geq \alpha\|X\|^2 - \beta\|Y\|^2$ for all t, where $X \in \mathbb{R}^{2n_1}$ and $Y \in \mathbb{R}^{2n_2}$ and α and β are constants. Then

$$\Delta(\Phi_H) \leq -2(n_1\alpha - n_2\beta)t + C.$$

These inequalities readily follow from Proposition 2.1 by a direct calculation.

Turning to the definition of the Salamon–Zehnder invariant for integral curves of Hamiltonian flows, let us first set notation and conventions. Let H be a T-periodic in time (e.g., autonomous) Hamiltonian on a symplectic manifold (P, ω). Recall that a capping of a contractible loop $\gamma\colon S^1_T \to P$, where $S^1_T = \mathbb{R}/T\mathbb{Z}$, is an extension of γ to a map $v\colon D^2 \to P$. The action of H on a capped loop (γ, v) is defined by

$$A_H(\gamma, v) = -\int_v \omega + \int_{S^1_T} H_t(\gamma(t))\, dt,$$

where $H_t = H(t, \cdot)$. When $\omega|_{\pi_2(P)} = 0$, the action $A_H(\gamma, v)$ is independent of the choice of v and we will use the notation $A_H(\gamma)$.

The least action principle asserts that the critical points of A_H on the space of all (capped) contractible loops $\gamma\colon S^1_T \to P$ are exactly (capped) contractible T-periodic orbits of the time-dependent Hamiltonian flow φ^t_H of H. The Hamiltonian vector field X_H of H, generating this flow, is given by $i_{X_H}\omega = -dH$.

Let $\gamma\colon [0, T] \to P$ be an integral curve φ^t_H and let ξ be a symplectic trivialization of TP along γ. The trivialization ξ gives rise to a symplectic identification of $T_{\gamma(t)}P$ with $T_{\gamma(0)}P$, and hence the linearization of φ^t_H along γ can be viewed as a family $\Phi(t) \in \mathrm{Sp}(T_{\gamma(0)}P)$. We set $\Delta_\xi(\gamma) := \Delta(\Phi)$. This is the Salamon–Zehnder invariant of γ with respect to ξ. Clearly, $\Delta_\xi(\gamma)$ depends on ξ. Assume now that γ is a contractible T-periodic orbit of H. A capping v of γ gives rise to a symplectic trivialization of TP along v and hence along γ, unique up to homotopy, and we denote by $\Delta_v(\gamma)$ the Salamon–Zehnder invariant of γ evaluated with respect to this trivialization. Note that $\Delta_v(\gamma)$ is determined entirely by the homotopy class of v and, as is well known, adding a sphere $w \in \pi_2(P)$ to v results in the Salamon–Zehnder invariant changing by $2\int_w c_1(TP)$. In particular, $\Delta(\gamma) := \Delta_v(\gamma)$ is independent of v whenever $c_1(TP)|_{\pi_2(P)} = 0$.

When γ is non-degenerate, i.e., $d\varphi_H^T : T_{\gamma(0)}P \to T_{\gamma(0)}P$ does not have one as an eigenvalue, the Conley–Zehnder index $\mu_{\mathrm{CZ}}(\gamma, v)$ is defined similarly to $\Delta_v(\gamma)$ by using a trivialization along γ; see [7, 38, 39]. Then inequality (2.2) turns into

$$|\,\mu_{\mathrm{CZ}}(\gamma, v) - \Delta_v(\gamma)| \leq \dim P/2. \tag{2.3}$$

When $c_1(TP)|_{\pi_2(P)} = 0$, the choice of capping is immaterial (see, e.g., [38]) and we will use the notation $\Delta(\gamma)$ and $\mu_{\mathrm{CZ}}(\gamma)$.

Example 2.3. Let $K : \mathbb{R}^{2n} \to \mathbb{R}$ be a convex autonomous Hamiltonian such that $d^2K \geq \alpha \cdot I$ at all points, where α is a constant. Then, as is easy to see from Example 2.2, $\Delta(\gamma) \leq -2n\alpha \cdot T + C$ for any integral curve $\gamma : [0, T] \to \mathbb{R}^{2n}$. Note that here $\Delta(\gamma)$ is evaluated with respect to the standard Euclidean trivialization and we are not assuming that the curve γ is closed.

Consider now two autonomous Hamiltonians H and K on a symplectic manifold P such that H is an increasing function of K, i.e., $H = f \circ K$, where $f : \mathbb{R} \to \mathbb{R}$ is an increasing function. Let γ be a periodic orbit of K lying on an energy level, which is regular for both K and H. Then γ can also be viewed, up to a change of time, as a periodic orbit of H. Fixing a trivialization of TP along γ, we have the Salamon–Zehnder invariants, $\Delta(\gamma, K)$ and $\Delta(\gamma, H)$ of γ defined for the flows of K and H. The following result is nearly obvious:

Lemma 2.4 ([21])**.** *Under the above assumptions,* $\Delta(\gamma, K) = \Delta(\gamma, H)$.

Let, as in Theorem 1.1, $K : P \to \mathbb{R}$ be an autonomous Hamiltonian attaining its Morse-Bott non-degenerate minimum $K = 0$ along a closed symplectic submanifold $M \subset P$. The key to the proof of Theorem 1.1 is a version of the Sturm comparison theorem for K:

Proposition 2.5 ([21])**.** *Assume that* $c_1(TP) = 0$*. Then there exist constants* $a > 0$ *and* c *and* $r_0 > 0$ *such that, whenever* $0 < r < r_0$,

$$-\Delta(\gamma) \geq a \cdot T - c \tag{2.4}$$

for every contractible T*-periodic orbit* γ *of* K *on the level* $K = r^2$.

A similar lower bound on $-\Delta(\gamma)$ holds without the assumption that $c_1(TP) = 0$ and is essential for the proof of the general case of Theorem 1.1. Fix a closed 2-form σ with $[\sigma] = c_1(TP)$. For instance, we can take as σ the Chern–Weil form representing c_1 with respect to a Hermitian connection on TP.

Proposition 2.6 ([21])**.** *There exist constants* $a > 0$ *and* c *and* $r_0 > 0$ *such that, whenever* $0 < r < r_0$,

$$-\Delta_v(\gamma) \geq a \cdot T - c - 2\int_v \sigma \tag{2.5}$$

for every contractible T*-periodic orbit* γ *of* K *on the level* $K = r^2$ *with capping* v.

The idea of the proof of these propositions is that the fiber contribution to $\Delta(\gamma)$ is of order T and positive, while the base contribution is of order $r \cdot T$. To be more specific, first note that the length $l(\gamma)$ is bounded from above by

$const \cdot r \cdot T$, where the constant is independent of γ and r, when $r > 0$ is sufficiently small. Furthermore, it is easy to prove (2.4) for the Euclidean trivialization and any integral curve (not necessarily closed) contained entirely in a Darboux chart, cf. Examples 2.2 and 2.3. Then, we cover γ by a finite collection of such charts. The required number of charts is of order $l(\gamma) \sim r \cdot T$. Within every chart, we have the lower bound (2.4) on $-\Delta$ with respect to the Euclidean trivialization. Combined, these trivializations can be viewed as an approximation to a Hermitian-parallel trivialization ξ along $\gamma\colon [0, T] \to P$. (We do not assume that $\xi(0) = \xi(T)$.) Moreover, within every chart the discrepancy between the values of Δ for the Euclidean and Hermitian-parallel trivializations is bounded by a constant independent of γ and r. As a consequence, the difference between $\Delta_\xi(\gamma)$ and the total Salamon–Zehnder invariant for Euclidean chart-wise trivializations is of order $N \sim r \cdot T$, and we conclude that (2.4) holds for $\Delta_\xi(\gamma)$. By the Gauss–Bonnet theorem, the effect of replacing ξ by a trivialization associated with a capping is captured by the integral term in (2.5). Finally, (2.4) follows from (2.5) and the upper bound $l(\gamma) \leq const \cdot r \cdot T$.

2.2. **Proof of Theorem 1.1 in a particular case.** Now we are in a position to prove Theorem 1.1 in the particular case where P is geometrically bounded and symplectically aspherical. The proof of the theorem uses two major ingredients. One is the Sturm comparison theorem for K (Proposition 2.5). The other is a calculation of the filtered Floer homology for a suitably reparametrized flow of K.

Let $K\colon P \to \mathbb{R}$ be as in Theorem 1.1. Pick sufficiently small $r > 0$ and $\epsilon > 0$. Let $H\colon [r^2 - \epsilon, r^2 + \epsilon] \to [0, \infty)$ be a smooth decreasing function such that

$$H \equiv \max H \text{ near } r^2 - \epsilon \text{ and } H \equiv 0 \text{ near } r^2 + \epsilon.$$

Consider now the Hamiltonian equal to $H \circ K$ within the shell bounded by the levels $K = r^2 - \epsilon$ and $K = r^2 + \epsilon$ and extended to the entire manifold P as a locally constant function. Abusing notation, we denote the resulting Hamiltonian by H again. Clearly, $\min H = 0$ on P and the maximum, $\max H$, is attained on the entire domain $K \leq r^2 - \epsilon$. Denote by $\mathrm{HF}_*^{(a,b)}(H)$ the filtered Floer homology of H for the interval (a, b). (A detailed list of references to the original papers and expository accounts of Floer theory can be found in, e.g., [20, 21, 38, 39].)

Proposition 2.7 ([20])**.** *Assume that P is geometrically bounded and symplectically aspherical and that $r > 0$ is sufficiently small. Then, once $\max H \geq C(r)$ where $C(r) \to 0$ as $r \to 0$, we have*

$$\mathrm{HF}_{n_0}^{(a,\, b)}(H) \neq 0$$

for $n_0 = 1 + (\operatorname{codim} M - \dim M)/2$ and some interval $(a,\, b)$ with $a > \max H$ and $b < \max H + C(r)$.

From Proposition 2.7, it is easy to see that H has a non-trivial contractible one-periodic orbit γ with

$$1 - \dim M = n_0 - \dim P/2 \leq \Delta(\gamma, H) \leq n_0 + \dim P/2 = 1 + \operatorname{codim} M. \tag{2.6}$$

Furthermore, since H is a function of K, we may also view γ, keeping the same notation for the orbit, as a T-periodic orbit of K. Note that H is a decreasing function of K, but otherwise the requirements of Lemma 2.4 are met. Hence, $\Delta(\gamma, K) = -\Delta(\gamma, H)$ and (2.6) turns into

$$1 - \dim M \leq -\Delta(\gamma, K) \leq 1 + \operatorname{codim} M.$$

On the other hand, by Proposition 2.5,

$$-\Delta(\gamma, K) \geq a \cdot T - c,$$

where the constants are independent of H and r and $\epsilon > 0$. Hence, we have an *a priori* bound on T:

$$T \leq T_0 = (1 + c + \operatorname{codim} M)/a.$$

Passing to the limit as $\epsilon \to 0$, we see that the T-periodic orbits γ of K converge, by the Arzela-Ascoli theorem, to a periodic orbit of K on the level $K = r^2$ with period bounded from above by T_0. This completes the proof of Theorem 1.1 in the particular case.

The proof of Theorem 1.1 in full generality relies on an extension of Proposition 2.7, using Floer–Novikov homology, to a broader class of manifolds including those satisfying conditions (i) or (ii). (The key to such an extension is the fact that the filtered Floer–Novikov homology of a compactly supported Hamiltonian on a spherically rational manifold P is defined for any short action interval (a, b) that does not intersect $\langle \omega, \pi_2(P) \rangle$, even if P is not geometrically bounded and monotone.) Then, when (i) holds, the proof of Theorem 1.1 is identical word-for-word to the proof for geometrically bounded, symplectically aspherical manifolds. To establish the theorem when (ii) holds, we use, in addition to the Sturm comparison theorem for K, the action bounds from the extension of Proposition 2.7 to control the effect of capping on the Salamon–Zehnder invariant. We refer the reader to [21] for details.

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VG: Department of Mathematics, UC Santa Cruz, Santa Cruz, CA 95064, USA
E-mail address: ginzburg@math.ucsc.edu

BG: Centre de recherches mathématiques, Université de Montréal, P.O. Box 6128, Centre–ville Station, Montréal, QC, H3C 3J7, CANADA
E-mail address: gurel@crm.umontreal.ca