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ON THE VARIATIONAL THEORY OF TRAFFIC FLOW: WELL-POSEDNESS, DUALITY AND APPLICATIONS

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Abstract. This paper describes some simplifications allowed by the variational theory of traffic flow(VT). It presents general conditions guaranteeing that the solution of a VT problem with bottlenecks exists, is unique and makes physical sense; i.e., that the problem is well-posed. The requirements for well-posedness are mild and met by practical applications. They are consistent with narrower results available for kinematic wave or Hamilton-Jacobi theories. The paper also describes some duality ideas relevant to these theories. Duality and VT are used to establish the equivalence of eight traffic models. Finally, the paper discusses how its ideas can be used to model networks of multi-lane traffic streams.

1. **Introduction.** Consider an infinite one-directional road on which vehicles cannot pass and move in the direction of increasing distance, x. If at some location $x = 0$ we assign consecutive integers to the vehicles we observe as time increases from $-\infty$ to $+\infty$ then the space-time trajectories of all the vehicles are completely defined by the integer contours of a surface. The idea of such a surface was proposed in [29] and further elaborated in [26]. The surface is characterized by a continuous function with contour levels n, $N(t, x) = n$. The floor $\lfloor n \rfloor$ is the number of the last vehicle to have advanced beyond x by time t. Since passing is not allowed the ordering of the vehicles is preserved everywhere. Therefore we can assume without loss of generality that $N(t, x)$ is non-decreasing in t for every x. Moreover, since vehicles move in the direction of increasing x, we can also assume that N is non-increasing in x for every t.

The simplest model of traffic flow further assumes that N is differentiable almost everywhere (except possibly along some curves that would form ridges in the surface defined by N) and that the first partial derivatives of N are related by a function; i.e.:

$$\partial N/\partial t = Q(-\partial N/\partial x, t, x). \tag{1}$$

This is a Hamilton-Jacobi (HJ) equation with Q as the Hamiltonian. Note that $\partial N/\partial t$ (abbreviated q) is the flow and $-\partial N/\partial x$ (abbreviated k) is the density, and that meaningful solutions require flow and density to be non-negative.

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