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# NONLINEAR BOUNDARY VALUE PROBLEMS WITH MULTIPLE POSITIVE SOLUTIONS

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**Abstract.** We give conditions on $f(y)$ which guarantee that the boundary value problem $T_n y(x) = f(y(x))$, $0 < x < A$, $y^{(2j)}(0) = 0 = y^{(2j+1)}(A)$, $j = 0, 1, \cdots, n-1$, where $T_n$ is the $n$th iterate of the operator $Ty(x) = -\frac{1}{w(x)}(p(x)\, y'(x))'$, have a prescribed number of multiple positive solutions. Our main tool is a fixed point theorem of Krasnosel'skiĭ.

1. **Introduction.** Let $T_n$ be the $n$th iterate of the Sturm-Liouville operator

$$Ty(x) = -\frac{1}{w(x)}(p(x)\, y'(x))'.$$

Thus $T_1 = T$, $T_n = T(T_{n-1})$, for $n \geq 2$. We study boundary value problems of the form

$$T_n y(x) = f(y(x)), \qquad x \in (0, A), \tag{1}$$

$$y^{(2j)}(0) = 0 = y^{(2j+1)}(A), \qquad j = 0, 1, \cdots, n-1. \tag{2}$$

We shall assume throughout that $w, p \in C(0, A]$ and strictly positive on $(0, A]$. It is also assumed that

$$\int_0^A w(x)\, dx < \infty, \qquad \int_0^A \frac{1}{p(x)}\, dx < \infty. \tag{3}$$

Thus, we allow singularities, albeit restricted by the conditions (3), at the left endpoint. We shall always assume that $f$ is nonnegative. For $N$ given, we seek conditions on $f$ which guarantee the existence of $N$ distinct positive solutions of the boundary value problem (1), (2).

Our problem (1), (2) has a close connection with a different boundary value problem. If our $w, p$ are continuous on $0 < x < 2A$ and are symmetric about $x = A$ (for example, if $w, p$ are constant), then any solution of (1), (2) may be extended symmetrically about $x = A$ to get a symmetric solution of the boundary value problem

$$T_n y(x) = f(y(x)), \qquad x \in (0, 2A), \tag{4}$$

$$y^{(2j)}(0) = 0 = y^{(2j)}(2A), \qquad j = 0, 1, \cdots, n-1. \tag{5}$$

This last problem, with $A = 1/2$, has been a recent focus of attention. Henderson and Thompson [5] used the Leggett-Williams triple fixed point theorem for the case $w \equiv p \equiv 1$ to get $N = 3$ positive solutions, with the possibility that one of these three solutions is trivial. Then Baxley and Haywood [1] used shooting methods, still in the case $w \equiv p \equiv 1$, to get any number $N$ of positive solutions, but only for

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the second order case ($n = 1$). In [2], they extended the results of [1] to the general case of $w, p$, still with $n = 1$. Finally, Graef, Qian, and Yang [4] have applied a form of Krasnosel'skiĭ's fixed point theorem to the case $w \equiv p \equiv 1$ and general $n$ to get any number $N$ of positive solutions. Both [5] and [4] considered only the problem (4), (5) and made critical use of Green's functions for the simple operators involved. In [1], [2], the authors solved (4), (5) by considering instead (1), (2) and extending the solutions symetrically across $A = 1/2$.

Our methods here utilize a version of Krasnosel'skiĭ's fixed point theorem on cones in Banach spaces, which is a modification of that used by Graef, Qian, and Yang in [4]. Our main result below includes as corollaries essentially all of the work discussed above.

Recall that a cone in a Banach space $\mathbf{B}$ is a closed set $\mathbf{K}$ containing the zero element of $\mathbf{B}$ satisfying

1. For any $u$, $v \in \mathbf{K}$, $au + bv \in \mathbf{K}$ for any nonnegative real numbers $a, b$;
2. For each $0 \neq u \in \mathbf{K}$, at least one of $u, -u$ does not belong to $\mathbf{K}$.

Any cone $\mathbf{K}$ in a Banach space can be used to impose a partial ordering as follows: $u \leqslant v$ if $v - u \in \mathbf{K}$. The following theorem appears in [6, pp. 137-139, 148-149] in the case that $\Omega_1$, $\Omega_2$ are balls; part (1) is [3, Exercise 1, p. 253].

**Theorem 1.** (Krasnosel'skiĭ) Suppose $F$ is a completely continuous operator on a Banach space $\mathbf{K}$, and let $\mathbf{K}$ be a cone in $\mathbf{B}$. If there exist bounded open sets $\Omega_1$, $\Omega_2$ containing the zero element of $\mathbf{B}$ such that $\overline{\Omega}_1 \subset \Omega_2$, $F\colon \mathbf{K} \cap (\overline{\Omega}_2 - \Omega_1) \to \mathbf{K}$, and either:

1. $u \nleqslant Fu$ for $u \in \mathbf{K} \cap \partial\Omega_1$ and $Fu \nleqslant u$ for $u \in \mathbf{K} \cap \partial\Omega_2$; or
2. $Fu \nleqslant u$ for $u \in \mathbf{K} \cap \partial\Omega_1$ and $u \nleqslant Fu$ for $u \in \mathbf{K} \cap \partial\Omega_2$,

then there is an element $u_0 \in \Omega_2 - \overline{\Omega}_1$ satisfying $Fu_0 = u_0$.

2. **Preparatory Work.** Our work will take place in the Banach space $\mathbf{B} = C[0, A]$ with norm

$$\|u\| = \sup_{x \in [0,A]} |u(x)|$$

We shall use the cone in $\mathbf{B}$ defined by

$$\mathbf{K} = \{u \in \mathbf{B} : u(0) = 0,\ u \text{ is nondecreasing on } [0, A],\ u(A) \leq \sigma u(A/q)\},$$

for some fixed real number $q > 1$. Here, $\sigma$ is given by

$$\sigma = 1 + \int_{A/q}^{A} \frac{1}{p(t)}\, dt \left( \int_0^{A/q} \frac{1}{p(t)}\, dt \right)^{-1}.$$

Now, define an operator $L\colon \mathbf{B} \to \mathbf{B}$ by

$$Lu(x) = \int_0^x \frac{1}{p(s)} \int_s^A w(t)u(t)\, dt\, ds.$$

Our integrability assumptions guarantee that the operator $L$ is well-defined. For any continuous, bounded, nonnegative real function $f$, we define the operators $L_n : \mathbf{B} \to \mathbf{B}$ recursively by

$$L_1 u = \int_0^x \frac{1}{p(s)} \int_s^A w(t) f(u(t))\, dt\, ds,$$

$L_j u = L(L_{j-1} u)$. It follows from the definition of the operators that if $y = L_n u$ for some $u \in \mathbf{B}$, then $y \in C^{(2n)}[0, A]$, $T_n y = f(u)$, and $y$ satisfies the boundary

conditions (5). We prepare for the application of Theorem 1 with the following lemmas.

**Lemma 1.** *Each operator in the sequence $\{L_n\}$ is completely continuous on the Banach space* $\mathbf{B}$.

Proof: Let $u \in \mathbf{B}$. Since $f$ is uniformly continuous on any compact set, given $\epsilon > 0$, there exists $\delta > 0$ so that $|f(u(t)) - f(v(t))| < \epsilon$ whenever $v \in \mathbf{B}$ and $\|u - v\| < \delta$. Using induction, it is easy to verify that

$$\|L_n u - L_n v\| < \epsilon Q^n, \text{ for } n = 1, 2, 3, \cdots, \tag{6}$$

where

$$Q = \int_0^A \frac{1}{p(s)} \int_s^A w(t)\, dt\, ds.$$

The continuity of each $L_n$ follows from (6). Further, one may verify by induction that

$$\|L_n u\| \le M Q^n, \text{ for } n = 1, 2, 3, \cdots, \tag{7}$$

from which we also get

$$|L_n u(x_2) - L_n u(x_1)| \le M Q^{n-1} \int_{x_1}^{x_2} \frac{1}{p(s)} \int_s^A w(t)\, ds\, dt, \text{ for } n = 1, 2, 3, \cdots. \tag{8}$$

Now if $\{u_n\}$ is a bounded sequence in $\mathbf{B}$, the inequalities (7), (8) allow us to conclude that this sequence is mapped by $L_n$ to a sequence which is uniformly bounded and equicontinuous. Then Ascoli's theorem gives the conclusion 2 that $L_n$ is completely continuous.

**Lemma 2.** *Suppose $g(t)$ is a nonnegative continuous function and let*

$$y(x) = \int_0^x \frac{1}{p(s)} \int_s^A w(t) g(t)\, dt\, ds.$$

*Then $y' \ge 0$, $(p\, y')' \le 0$ on $[0, A]$ and $y \in \mathbf{K}$.*

Proof: Clearly $y \in \mathbf{B}$ and $y(0) = 0$. Differentiating gives

$$y'(x) = \frac{1}{p(x)} \int_x^A w(t) g(t)\, dt \ge 0.$$

Thus $y(x)$ is nondecreasing on $[0, A]$. Differentiating again gives

$$(p(x) y'(x))' = -w(x) g(x) \le 0,$$

so $py'$ is nonincreasing. Thus

$$y(A) - y(A/q) = \int_{A/q}^A \frac{p(t) y'(t)}{p(t)}\, dt \le p(A/q) y'(A/q) \int_{A/q}^A \frac{1}{p(t)}\, dt.$$

Similarly,

$$y(A/q) = \int_0^{A/q} \frac{p(t) y'(t)}{p(t)}\, dt \ge p(A/q) y'(A/q) \int_0^{A/q} \frac{1}{p(t)}\, dt.$$

Combining these last two inequalities, we obtain

$$y(A) - y(A/q) \le y(A/q) \int_{A/q}^A \frac{1}{p(t)}\, dt \left( \int_0^{A/q} \frac{1}{p(t)}\, dt \right)^{-1}$$

and it follows that $y \in \mathbf{K}$.

**Lemma 3.** *Each of the operators $L_n$ maps the cone $\mathbf{K}$ into itself.*

Proof: Let $u \in \mathbf{K}$ and apply Lemma 2 to $g(t) = f(u(t)) \geq 0$ to conclude that $L_1 u \in \mathbf{K}$. Using induction, assume that $L_n u \in \mathbf{K}$. Then $g(t) = L_n u(t) \geq 0$ and Lemma 2 implies that $L_n u \in \mathbf{K}$, and we are finished.

We must now introduce some auxiliary functions which are needed in the proof of our main theorem. Let sequences of functions $\{\alpha_n\}$, $\{\beta_n\}$ be defined as follows:

$$\begin{aligned} \alpha_1(x) &= \int_0^x \frac{1}{p(s)} \int_s^A w(t)dt\, ds; \\ \alpha_n(x) &= L\alpha_{n-1}(x), \qquad n = 2, 3, 4, \cdots \end{aligned}$$

$$\beta_1(x) = \begin{cases} \int_{A/q}^A w(t)\, dt \int_0^x \frac{1}{p(t)}\, dt, & 0 \leq x < A/q \\ \int_0^{A/q} \frac{1}{p(t)}\, dt \int_{A/q}^A w(t)\, dt + \int_{A/q}^x \frac{1}{p(s)} \int_s^A w(t)\, dt\, ds, & A/q \leq x \leq A \end{cases}$$

$$\beta_n(x) = L\beta_{n-1}(x), \quad n = 2, 3, 4, \cdots$$

These functions will serve as bounds for the functions $y_n$. We will also assume that there exist positive real numbers $a_0, a_1, \cdots, a_m, b_1, \cdots, b_m, K, k$ such that

$$a_0 < b_1 < \frac{a_1}{\sigma} < \frac{b_2}{\sigma} < \frac{a_2}{\sigma^2} < \cdots < \frac{b_m}{\sigma^{m-1}} < \frac{a_m}{\sigma^m}$$

and that the function $f$ satisfies the following conditions:

$$f(u) \leq Ka_0,\ f(u) \not\equiv Ka_0, \qquad 0 \leq u \leq a_0; \tag{9}$$

$$f(u) \geq kb_j, \qquad b_j \leq u \leq \sigma b_j, \quad j = 1, 2, \cdots, m; \tag{10}$$

$$f(u) \leq Ka_j, \qquad 0 \leq u \leq a_j, \quad j = 1, 2, \cdots, m. \tag{11}$$

The following sequences of open sets in $\mathbf{B}$ will then be needed:

$$\Omega_{0,1} = \{u \in \mathbf{B} : \|u\| < a_0, \},$$

$$\Omega_{j,1} = \{u \in \mathbf{B} : \|u\| < a_j, \}, \qquad j = 1, 2, 3, \cdots, m,$$

$$\Omega_{j,2} = \{u \in \mathbf{B} : \|u\| < \sigma b_j\} \cap \{u \in \mathbf{B} : |u(A/q)| < b_j\}, \qquad j = 1, 2, 3, \cdots, m.$$

These sets are related to the growth conditions (9)-(11) in a clear way. In particular, note that $\overline{\Omega}_{j-1,1} \subset \Omega_{j,2}$ and that $\overline{\Omega}_{j,2} \subset \Omega_{j,1}$ for each $j = 1, 2, \cdots, m$.

**Lemma 4.** *If $u \in \mathbf{K} \cap \partial\Omega_{0,1}$, then $L_n u(x) \leq Ka_0\alpha_n(x)$ on $[0, A]$, and the inequality is strict at $x = A$.*

Proof: Consider first the case $n = 1$. By definition,

$$L_1 u(x) = \int_0^x \frac{1}{p(s)} \int_s^A w(t) f(u(t))\, dt\, ds.$$

Since $u \in \mathbf{K} \cap \partial\Omega_{0,1}$, $u(x) \leq a_0$ for any $x \in [0, A]$ and $u(A) = a_0$. Thus $f(u)$ is bounded above by $Ka_0$ and also $f(u) \not\equiv Ka_0$. Using this bound, we obtain

$$L_1 u(x) = \int_0^x \frac{1}{p(s)} \int_s^A w(t) f(u(t))\, dtds \leq \int_0^x \frac{1}{p(s)} \int_s^A w(t) Ka_0\, dtds = Ka_0\alpha_1(x). \tag{12}$$

In particular, note that $L_1 u(A) \leq Ka_0\alpha_1(A)$. However, in order for equality to occur, the inequality (12) must in fact be equality for $x = A$. Hence $f(u(t)) = Ka_0$ for all $t \in [0, A]$ and thus, since $u(0) = 0$ and $u(A) = a_0$, $f(u) = Ka_0$ for all $u \in [0, a_0]$, which contradicts the assumption that $f(u) \not\equiv Ka_0$ on $[0, a_0]$. Therefore

$L_1u(A) < Ka_0\alpha_n(A)$. Continuing by induction, we assume that $L_nu(t) \le Ka_0\alpha_n(t)$ and then

$$L_{n+1}u(x) \le \int_0^x \frac{1}{p(s)} \int_s^A w(t)Ka_0\alpha_n(t)\,dt\,ds = Ka_0\alpha_{n+1}(x). \tag{13}$$

The strictness of (13) at $x = A$ is shown by noting that since $L_nu(A) < Ka_0\alpha_n(A)$ and $L_nu$ is continuous, strict inequality must also hold for values of $L_nu(t)$ in some small neighborhood of $A$.

**Corollary 1.** *If $K = (\alpha_n(A))^{-1}$, then $L_nu(A) < a_0$ for $u \in \mathbf{K} \cap \partial\Omega_{0,1}$.*

**Lemma 5.** *For any $u \in \mathbf{K} \cap \partial\Omega_{j,1}$, then $L_nu(x) \le Ka_j\alpha_n(x)$ on $[0, A]$, and this inequality is strict for $x = A$.*

Proof: Since $f(y) \not\equiv Ka_j$ on $[0, a_j]$, the proof is identical to that of Lemma 4.

**Corollary 2.** *If $K = (\alpha_n(A))^{-1}$, then $L_nu(A) < a_j$ for all $u \in \mathbf{K} \cap \partial\Omega_{j,1}$.*

**Lemma 6.** *If $u \in \mathbf{K}\cap\partial\Omega_{j,2}$, then $L_nu(x) \ge kb_j\beta_n(x)$ on $[0, A]$, with strict inequality for the interval $[A/q, A]$.*

Proof: If $u \in \mathbf{K} \cap \partial\Omega_{j,2}$ and $\|u\| = u(A) = \sigma b_j$, then $\sigma b_j = u(A) \le \sigma u(A/q)$. Thus $u(A/q) \ge b_j$, so $u(A/q) = b_j$. Hence $b_j \le u(x) \le \sigma b_j$ for $A/q \le x \le A$. For any $u \in \mathbf{K} \cap \partial\Omega_{j,2}$,

$$L_1u(x) = \int_0^x \frac{1}{p(s)} \int_s^A w(t)f(u(t))\,dt\,ds. \tag{14}$$

Differentiating (14) gives

$$p(x)(L_1u)'(x) = \int_x^A w(t)f(u(t))\,dt \tag{15}$$

Since $u \in \mathbf{K} \cap \partial\Omega_{j,2}$, $u$ satisfies the condition (10) on $[A/q, A]$ and thus $f(u(t))$ is bounded below by $kb_j$. Then (15) gives

$$p(x)(L_1u)'(x) \ge kb_j \int_x^A w(t)\,dt$$

for any $x \in [A/q, A]$. In particular, observe that for $x = A/q$,

$$p(A/q)(L_1u)'(A/q) \ge kb_j \int_{A/q}^A w(t)\,dt.$$

By Lemma 2, $p(L_1u)'$ is nonincreasing and thus implies that $p(x)(L_1u)'(x) \ge kb_j \int_{A/q}^A w(t)dt$ for all $x \in [0, A/q]$. Furthermore, this last inequality must be strict for some $x \in [0, A/q]$. For otherwise, $p(L_1u)'(x)$ is constant on $[0, A/q]$ and so $w(x)f(u(x)) = (p(L_1u)')'(x) \equiv 0$ on $[0, A/q]$; however, this cannot occur since by assumption $f(u(A/q)) = f(b_j) \ge kb_j > 0$ and $f$ is continuous. Now, integration from 0 to any $x \in [0, A/q]$ yields

$$(L_1u)(x) = \int_0^x (L_1u)'(s)\,ds \ge kb_j \int_0^x \frac{1}{p(s)}\,ds \left(\int_{A/q}^A w(t)\,dt\right),$$

and specifically for $x = A/q$,

$$(L_1u)(A/q) > kb_j \int_0^{A/q} \frac{1}{p(s)}\,ds \int_{A/q}^A w(t)\,dt. \tag{16}$$

Thus $(L_1u)(x) \geq kb_j\beta_1(x)$ on the interval $[0, A/q]$, and this inequality is strict at $A/q$. Now, consider the case $x \in [A/q, A]$. Using (14), (16) and $f(u(t)) \geq kb_j$ for $A/q \leq t \leq A$, we get

$$(L_1u)(x) > kb_j \int_0^{A/q} \frac{1}{p(s)}\, ds \int_{A/q}^{A} w(t)\, dt + kb_j \int_{A/q}^{x} \frac{1}{p(s)} \int_s^A w(t)\, dt\, ds = kb_j\beta_1(x)$$

for $x \in [A/q, A]$, which proves the claim for $n = 1$. A simple induction argument completes the proof.

**Corollary 3.** *If $k = (\beta_n(A/q))^{-1}$, then $L_nu(A/q) > b_j$ for any $u \in \mathbf{K} \cap \partial\Omega_{j,2}$.*

3. **Main Results.** Corollaries 1-3 indicate that the necessary values of the growth condition constants $K$, $k$ vary with the value of $n$; therefore, for each $n \in \mathbb{N}$, choose $K_n = (\alpha_n(A))^{-1}$ and $k_n = (\beta_n(A/q))^{-1}$.

**Theorem 2.** *Suppose there exist an integer $m > 0$ and real numbers*

$$0 < a_0 < b_1 < \frac{a_1}{\sigma} < \frac{b_2}{\sigma} < \frac{a_2}{\sigma^2} < \cdots < \frac{b_m}{\sigma^{m-1}} < \frac{a_m}{\sigma^m}$$

*such that $f\colon \mathbb{R} \to [0, \infty)$ is continuous and satisfies*

$$\begin{array}{lll} f(u) \leq K_na_0,\ f(u) \not\equiv K_na_0, & 0 \leq u \leq a_0; & \\ f(u) \geq k_nb_j, & b_j \leq u \leq \sigma b_j, & j = 1, 2, \cdots, m; \\ f(u) \leq K_na_j, & 0 \leq u \leq a_j, & j = 1, 2, \cdots, m. \end{array}$$

*Then the boundary value problem (1)-(2) has at least $2m$ positive solutions $y_1, y_2, \cdots, y_{2m}$ satisfying*

$$\begin{gathered} y_1(A) > a_0, \\ y_{2j-1}(A/q) < b_j < y_{2j}(A/q), \qquad j = 1, 2, \cdots, m, \\ y_{2j}(A) < a_j < y_{2j+1}(A), \quad j = 1, 2, \cdots, m-1, \\ y_{2m}(A) < a_m. \end{gathered}$$

Proof: It must be verified that the sequence of operators $\{L_n\}$ and the open sets $\Omega_{0,1}$, $\{\Omega_{j,1}\}$, $\{\Omega_{j,2}\}$ satisfy the hypotheses of Theorem 1. Clearly $0 \in \Omega_{0,1}$ and $0 \in \Omega_{j,1}$, $0 \in \Omega_{j,2}$ for all $j = 1, 2, 3, \ldots, m$. It is also apparent that $\overline{\Omega}_{j-1,1} \subset \Omega_{j,2}$ and that $\overline{\Omega}_{j,2} \subset \Omega_{j,1}$ for each $j$. Lemma 1 shows that each operator in the sequence $\{L_n\}$ is completely continuous on $\mathbf{B}$, and Lemma 3 implies that each $L_n$ maps the cone $\mathbf{K}$ into itself; it immediately follows that each $L_n$ will map the intersection of $\mathbf{K}$ with $\overline{\Omega}_{j,1} - \Omega_{j,2}$ and $\overline{\Omega}_{j,2} - \Omega_{j-1,1}$ into $\mathbf{K}$ for each $j = 1, 2, 3, \ldots, m$. Now, by Corollary 1, for any $u \in \mathbf{K} \cap \partial\Omega_{0,1}$, $L_nu(A) < a_0 = u(A)$, which implies that $(L_nu - u)(A) < 0$; thus, recalling that any element of $\mathbf{K}$ must be nonnegative, $L_nu - u \notin \mathbf{K}$ and hence $u \not\leq L_nu$. By Corollary 3, for each $j = 1, 2, \cdots, m$, given any $u \in \mathbf{K} \cap \partial\Omega_{j,2}$, $L_nu(A/q) > b_j = u(A/q)$, so $(u - L_nu)(A/q) < 0$. It then follows in this case that $u - L_nu \notin \mathbf{K}$ and $L_nu \not\leq u$. Also, for any $j = 1, 2, \ldots, m$, Corollary 2 shows that if $u \in \mathbf{K} \cap \partial\Omega_{j,1}$, $L_nu(A) < a_j = u(A)$, implying that $(L_nu - u)(A) < 0$ and $L_nu - u \notin \mathbf{K}$. Hence $u \not\leq L_nu$ here as well. Now, using Theorem 1, $L_n$ must have a fixed point $y_1 \in \Omega_{1,2} - \overline{\Omega}_{0,1}$. But since $L_n(u) - u \notin \mathbf{K}$ for $u \in \mathbf{K} \cap \partial\Omega_{1,1}$, there must be a second fixed point $y_2 \in \Omega_{1,1} - \overline{\Omega}_{1,2}$. Similarly, for each $j$ from 2 to $m$, Theorem 1 guarantees the existence of a fixed point $y_{2j-1} \in \Omega_{j,2} - \overline{\Omega}_{j-1,1}$ of the operator $L_n$ and another fixed point $y_{2j} \in \Omega_{j,1} - \overline{\Omega}_{j,2}$. For each fixed point, $L_ny_i = y_i$; it follows that $T_ny_i = f(y_i)$ and therefore each $y_i$ is a solution to the boundary value problem

(1)- (2). Finally, for each $j$, the function $y_{2j-1} \in \Omega_{j,2} - \overline{\Omega}_{j-1,1}$, so by definition of the sets its value at $A$ must be bounded below by $a_{j-1}$, and its value at $A/q$ is bounded above by $b_j$. Furthermore, the function $y_{2j} \in \Omega_{j,1} - \overline{\Omega}_{j,2}$, so it is bounded below at $A/q$ by $b_j$ and bounded above at $A$ by $a_j$.

**Theorem 3.** *In addition to the hypotheses of Theorem 2, if there exists a real number $b_0$ such that $0 < \sigma b_0 < a_0$ and the function $f$ satisfies*

$$f(u) \geq k_n b_0, \qquad b_0 \leq u \leq \sigma b_0,$$

*then the boundary value problem (1)-(2) has at least $2m+1$ positive solutions $y_0, y_1, \cdots, y_{2m}$ satisfying*

$$\begin{gathered} y_0(A/q) > b_0, \\ y_{2j-1}(A/q) < b_j < y_{2j}(A/q), \quad j = 1, 2, \cdots, m, \\ y_{2j}(A) < a_j < y_{2j+1}(A), \quad j = 0, 1, \cdots, m-1, \\ y_{2m}(A) < a_m. \end{gathered}$$

Proof: From Theorem 2, the boundary value problem has $2m$ positive solutions $y_1$, $y_2, \cdots, y_{2m}$ satisfying the stated conditions. Now, define another open set by

$$\Omega_{0,2} = \{u \in \mathbf{B} : \|u\| < qb_0\} \cap \{u \in \mathbf{B} : |u(A/q)| < b_0\}.$$

It is easily verified that $0 \in \Omega_{0,2}$ and $\overline{\Omega}_{0,2} \subset \Omega_{0,1}$. Also, by the same methods used in Lemma 6 and Corollary 3, for any $u \in \mathbf{K} \cap \partial\Omega_{0,2}$, $L_n u(A/q) > b_0 = u(A/q)$. So $(u - L_n u)(A/q) < 0$ and thus $u - L_n u \notin \mathbf{K}$; this implies that $L_n u \nleq u$. It has already been shown that $u \nleq L_n u$ for $u \in \mathbf{K} \cap \partial\Omega_{0,1}$. Hence, by Theorem 1, the operator $L_n$ must have a fixed point $y_0 \in \Omega_{0,1} - \overline{\Omega}_{0,2}$, which must satisfy $y_0(A/q) > b_0$ and $y_0(A) < a_0$. By differentiation, $y_0$ is also solution of (1)-(2), completing the proof.

It should be noted that the values of the constants $K_n$ and $k_n$ are tedious to compute explicitly for large values of $n$. For each $n \in \mathbb{N}$, it would be natural to attempt to determine a value for the number $q$ such that the constant $k_n$ is minimized. This value of $q$ would of course vary with the functions $w, p$. For the case $w \equiv p \equiv 1$, $A = 1/2$ and $n = 1$, it is straightforward to verify by hand that the minimum occurs at $q = 2$, and then that $\sigma = 2$, $K_1 = 8$, and $k_1 = 16$. Further, calculations using MAPLE have shown that this minimal value occurs at $q = 2$ for

| $n$ | $K_n$ | $k_n$ | $H_n$ | $h_n$ | $C_n$ |
|---|---|---|---|---|---|
| 1 | 8 | 16 | 8 | 16 | 2 |
| 2 | 76.8 | 153.6 | 36 | 1219 | 23.28 |
| 3 | 755.4 | 1511 | 216 | $4.255 \times 10^4$ | 135.5 |
| 4 | 7452 | $1.491 \times 10^4$ | 1296 | $1.486 \times 10^6$ | 787.9 |
| 5 | $7.355 \times 10^4$ | $1.472 \times 10^5$ | 7776 | $5.185 \times 10^7$ | 4584 |
| 6 | $7.259 \times 10^5$ | $1.452 \times 10^6$ | $4.665 \times 10^4$ | $1.810 \times 10^9$ | $2.667 \times 10^4$ |
| 7 | $7.164 \times 10^6$ | $1.433 \times 10^7$ | $2.799 \times 10^5$ | $6.317 \times 10^{10}$ | $1.552 \times 10^5$ |
| 8 | $7.071 \times 10^7$ | $1.415 \times 10^8$ | $1.679 \times 10^6$ | $2.206 \times 10^{12}$ | $9.028 \times 10^5$ |
| 9 | $6.978 \times 10^8$ | $1.396 \times 10^9$ | $1.007 \times 10^7$ | $7.700 \times 10^{13}$ | $5.253 \times 10^6$ |
| 10 | $6.887 \times 10^9$ | $1.378 \times 10^{10}$ | $6.046 \times 10^7$ | $2.688 \times 10^{15}$ | $3.056 \times 10^7$ |

TABLE 1. Values for bounding constants when $w \equiv p \equiv 1$, $A = 1/2$ and $q = 2$.

| $n$ | $K_n$ | $k_n$ |
|---|---|---|
| 1 | 7.071 | 10.67 |
| 2 | 54.54 | 81.41 |
| 3 | 422.9 | 631.0 |
| 4 | 3281 | 4894 |
| 5 | $2.545 \times 10^4$ | $3.797 \times 10^4$ |
| 6 | $1.974 \times 10^5$ | $2.945 \times 10^5$ |
| 7 | $1.531 \times 10^6$ | $2.285 \times 10^6$ |
| 8 | $1.187 \times 10^7$ | $1.772 \times 10^7$ |
| 9 | $9.214 \times 10^7$ | $1.375 \times 10^8$ |
| 10 | $7.147 \times 10^8$ | $1.067 \times 10^9$ |

TABLE 2. Bounding constants for $w(x) = x$, $p(x) = \sqrt{x}$, $A = 1/2$ and $q = 2$.

$n \leq 5$ (the tenth-order problem). It is suspected that $q = 2$ gives a minimal $k_n$ for all $n \in \mathbb{N}$. The second and third columns of Table 1 give values (computed using MAPLE) of $K_n$ and $k_n$ using $q = 2$ for several values of $n$. Henderson and Thompson [5] have different bounding constants $H_n, h_n$ corresponding to our $K_n, k_n$. Moreover, they replace our $\sigma = 2$ with a number $C_n$ which depends on $n$. They do not compute values of these constants (except when $n = 1$) but provide rough upper bounds for $H_n$ and $C_n$ and a rough lower bound for $h_n$. These bounds are presented in the fourth, fifth, and sixth columns of Table 1.

In closing, an example of the conclusions of Theorems 2 and 3 is considered for the case $w(x) = x$, $p(x) = \sqrt{x}$. For comparison with the case already considered, we use $q = 2$ and $A = 1/2$. However, a careful analysis would be required to verify that this choice of $q$ is optimal; in fact, this choice is likely not optimal for the particular problem. The values of the constants $K_n$, $k_n$ for several values of $n$, which are computed using MAPLE, are contained in Table 2.

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