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AN UNDETERMINED TIME-DEPENDENT COEFFICIENT IN A FRACTIONAL DIFFUSION EQUATION

Zhidong Zhang

Department of Mathematics, Texas A&M University
College Station, TX 77843-3368, USA

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Abstract. In this work, we consider a FDE (fractional diffusion equation)

$$^{C}D_t^{\alpha}u(x,t) - a(t)\mathcal{L}u(x,t) = F(x,t)$$

with a time-dependent diffusion coefficient $a(t)$. This is an extension of [13], which deals with this FDE in one-dimensional space. For the direct problem, given an $a(t)$, we establish the existence, uniqueness and some regularity properties with a more general domain Ω and right-hand side $F(x,t)$. For the inverse problem–recovering $a(t)$, we introduce an operator K one of whose fixed points is $a(t)$ and show its monotonicity, uniqueness and existence of its fixed points. With these properties, a reconstruction algorithm for $a(t)$ is created and some numerical results are provided to illustrate the theories.

1. **Introduction.** This paper considers the fractional diffusion equation (FDE) with a continuous and positive coefficient function $a(t)$:

$$\begin{aligned} {}^{C}D_t^{\alpha}u(x,t) - a(t)\mathcal{L}u(x,t) &= F(x,t), && x\in\Omega,\ t\in(0,T];\\ u(x,t) &= 0, && (x,t)\in\partial\Omega\times(0,T];\\ u(x,0) &= u_0(x), && x\in\Omega, \end{aligned} \tag{1}$$

where Ω is a bounded and smooth subset of $R^n, n = 1,2,3$, $-\mathcal{L}$ is a symmetric uniformly elliptic operator defined as

$$-\mathcal{L}u = -\sum_{i,j=1}^{n}(a^{ij}(x)u_{x_i})_{x_j} + c(x)u$$

with conditions

$$a^{ij}, c \in C^2(\overline{\Omega})\ (i,j = 1,\dots,n),\ \partial\Omega \text{ is } C^3, \tag{2}$$

and $^{C}D_t^{\alpha}$ is the left-sided Djrbashian-Caputo α-th order derivative with respect to time t. The definition for $^{C}D_t^{\alpha}$ is

$$^{C}D_t^{\alpha}u(x,t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1}\frac{d^n}{d\tau^n}u(x,\tau)\mathrm{d}\tau$$

with Gamma function $\Gamma(\cdot)$ and the nearest integer n with $\alpha \le n$. In this paper, we are assuming a subdiffusion process, i.e. $\alpha\in(0,1)$. This simplifies the definition of $^{C}D_t^{\alpha}$ as

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$$^{C}D_t^{\alpha}u(x,t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha}\frac{d}{d\tau}u(x,\tau)\mathrm{d}\tau.$$

This work is an extension of [13] from a simple space domain Ω to $\mathbb{R}^n$, considers the more general analysis for the direct problem and contains an existence argument for the inverse problem of recovering $a(t)$.

This paper consists of two parts; the direct problem and the inverse problem. For the direct problem, we build the spectral representation of the weak solution $u(x,t;a)$. The notation $u(x,t;a)$ is used for displaying the dependence of the solution u on the diffusivity $a(t)$. Then the existence, uniqueness and regularity results are proved with several assumptions on the coefficient function $a(t)$. Unlike [13], the right hand side function $F(x,t)$ is not of the form $f(x)g(t)$, so that the proof of regularity is more delicate. For the inverse problem, we use the single point flux data

$$a(t)\frac{\partial u}{\partial \overrightarrow{\mathbf{n}}}(x_0,t;a)=g(t),\ x_0\in\partial\Omega$$

to recover the coefficient $a(t)$ (We choose the data $a(t)\frac{\partial u}{\partial \overrightarrow{\mathbf{n}}}(x_0,t;a)=g(t)$ instead of the classical flux $\frac{\partial u}{\partial \overrightarrow{\mathbf{n}}}(x_0,t;a)$ because in practice, $a(t)\frac{\partial u}{\partial \overrightarrow{\mathbf{n}}}(x_0,t;a)$ is usually measured as the flux). For the reconstruction, we only consider to recover a continuous and positive $a(t)$ to match the assumptions set in the direct problem. Acting a flux data, we introduce an operator K one of whose fixed points is the coefficient $a(t)$. Using the weak maximum principle [7], we establish the monotonicity and uniqueness of the fixed points of operator K, and the proof of uniqueness leads to a numerical reconstruction algorithm. Since we consider a multidimensional domain Ω here, the Sobolev Embedding Theorem yields that we need to add the condition (2) on the operator $-\mathcal{L}$ to ensure the C^1-regularity of the series representation of u. Then the operator K is well-defined, where the proofs can be seen in section 4. This is a significant difference from [13]. Furthermore, an existence argument of the fixed points of K is included by this paper, which [13] does not contain.

The rest of this paper follows the following structure. In section 2, we collect some preliminary results about fractional calculus and the eigensystem of $-\mathcal{L}$. The direct problem is discussed in section 3, i.e. we establish the existence, uniqueness and some regularity results of the weak solution for FDE (1). Then section 4 deals with the inverse problem of recovering $a(t)$. Specifically, an operator K is introduced at the beginning of this section, then its monotonicity and uniqueness of its fixed points give an algorithm to recover the coefficient $a(t)$. In particular, the existence argument of the fixed points of K is included by this section. In section 5, some numerical results are presented to illustrate the theoretical basis.

2. Preliminary material.

2.1. Mittag-Leffler function.

In this part, we describe the Mittag-Leffler function which plays an important role in fractional diffusion equations. This is a two-parameter function defined as

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\alpha+\beta)},\ z\in\mathbb{C}.$$

It generalizes the natural exponential function in the sense that $E_{1,1}(z)=e^z$. We list some important properties of the Mittag-Leffler function for future use.

Lemma 2.1. *Let $0 < \alpha < 2$ and $\beta \in \mathbb{R}$ be arbitrary, and $\frac{\alpha\pi}{2} < \mu < \min(\pi, \alpha\pi)$. Then there exists a constant $C = C(\alpha, \beta, \mu) > 0$ such that*

$$|E_{\alpha,\beta}(z)| \leq \frac{C}{1+|z|}, \quad \mu \leq |\arg(z)| \leq \pi.$$

Proof. This proof can be found in [5]. □

Lemma 2.2. *For $\lambda > 0,\ \alpha > 0$ and $n \in \mathbb{N}^+$, we have*

$$\frac{d^n}{dt^n} E_{\alpha,1}(-\lambda t^\alpha) = -\lambda t^{\alpha-n} E_{\alpha,\alpha-n+1}(-\lambda t^\alpha),\ t > 0.$$

In particular, if we set $n = 1$, then there holds

$$\frac{d}{dt} E_{\alpha,1}(-\lambda t^\alpha) = -\lambda t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha),\ t > 0.$$

Proof. This is [10, Lemma 3.2]. □

Lemma 2.3. *If $0 < \alpha < 1$ and $z > 0$, then $E_{\alpha,\alpha}(-z) \geq 0$.*

Proof. This proof can be found in [8, 9, 12]. □

Lemma 2.4. *For $0 < \alpha < 1$, $E_{\alpha,1}(-t^\alpha)$ is completely monotonic, that is,*

$$(-1)^n \frac{d^n}{dt^n} E_{\alpha,1}(-t^\alpha) \geq 0,\ \textit{for } t > 0 \textit{ and } n = 0, 1, 2, \cdots.$$

Proof. See [2]. □

2.2. **Fractional calculus.** In this part, we collect some results of fractional calculus. The next lemma states the extremal principle of ${}^C D_t^\alpha$.

Lemma 2.5. *Fix $0 < \alpha < 1$ and given $f(t) \in C[0,T]$ with ${}^C D_t^\alpha f \in C[0,T]$. If f attains its maximum (minimum) over the interval $[0,T]$ at the point $t = t_0,\ t_0 \in (0,T]$, then ${}^C D_{t_0}^\alpha f \geq (\leq) 0$.*

Proof. Even though the conditions are different from the ones of [7, Theorem 1], the maximum case can be proved following the proof of [7, Theorem 1]. For the minimum case, we only need to set $\overline{f} = -f$. □

The following lemma about the composition between ${}^C D_t^\alpha$ and the fractional integral I_t^α is presented in [11].

Lemma 2.6. *Define the Riemann-Liouville α-th order integral I_t^α as*

$$I_t^\alpha u = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} u(\tau) \mathrm{d}\tau.$$

For $0 < \alpha < 1$, $u(t), {}^C D_t^\alpha u \in C[0,T]$, we have

$$({}^C D_t^\alpha \circ I_t^\alpha u)(t) = u(t), \quad (I_t^\alpha \circ {}^C D_t^\alpha u)(t) = u(t) - u(0),\ t \in [0,T].$$

2.3. **Eigensystem of $-\mathcal{L}$.** Since $-\mathcal{L}$ is a symmetric uniformly elliptic operator, we denote the eigensystem of $-\mathcal{L}$ by $\{(\lambda_n, \phi_n) : n \in \mathbb{N}^+\}$. Then we have $0 < \lambda_1 \le \lambda_2 \le \cdots$ where finite multiplicity is possible, $\lambda_n \to \infty$ and $\{\phi_n : n \in \mathbb{N}^+\} \subset H^2(\Omega) \cap H_0^1(\Omega)$ forms an orthonormal basis of $L^2(\Omega)$.

Moreover, with the condition (2), for each $n \in \mathbb{N}^+$, it holds that $\phi_n \in H^3(\Omega)$ [1]. Then by the Sobolev Embedding Theorem, we have $\phi_n \in C^1(\overline{\Omega})$ and $\frac{\partial \phi_n}{\partial \overrightarrow{\mathbf{n}}}(x_0)$ is well-defined for each $n \in \mathbb{N}^+$. Hence, without loss of generality, we can suppose

$$\frac{\partial \phi_n}{\partial \overrightarrow{\mathbf{n}}}(x_0) \ge 0, \text{ for each } n \in \mathbb{N}^+. \tag{3}$$

Otherwise, if $\frac{\partial \phi_k}{\partial \overrightarrow{\mathbf{n}}}(x_0) < 0$ for some $k \in \mathbb{N}^+$, we can replace ϕ_k by $-\phi_k$. $-\phi_k$ satisfies all the properties we need, such as it is an eigenfunction of $-\mathcal{L}$ corresponding to the eigenvalue λ_k, composes an orthonormal basis of $L^2(\Omega)$ together with $\{\phi_n : n \in \mathbb{N}^+, n \ne k\}$ and $\frac{\partial(-\phi_k)}{\partial \overrightarrow{\mathbf{n}}}(x_0) \ge 0$. The assumption (3) will be used in Section 4.

3. **Direct problem–existence, uniqueness and regularity.** Throughout this section, we suppose $a(t)$, $u_0(x)$ and $F(x,t)$ satisfy the following assumptions:

Assumption 3.1.

(a) $a(t) \in C^+[0,T] := \{\psi \in C[0,T] : \psi(t) > 0,\ t \in [0,T]\}$;
(b) $F(x,t) \in C([0,T]; L^2(\Omega))$;
(c) $u_0(x) \in H_0^1(\Omega)$.

3.1. **Spectral representation.**

Definition 3.2. We call $u(x,t;a)$ a weak solution of FDE (1) in $L^2(\Omega)$ corresponding to the coefficient $a(t)$ if $u(\cdot,t;a) \in H_0^1(\Omega)$ for $t \in (0,T]$ and for any $\psi(x) \in H^2(\Omega) \cap H_0^1(\Omega)$, it holds

$$\begin{aligned} &({}^C D_t^\alpha u(x,t;a), \psi(x)) - (a(t)\mathcal{L}u(x,t;a), \psi(x)) = (F(x,t), \psi(x)),\ t \in (0,T];\\ &(u(x,0;a), \psi(x)) = (u_0(x), \psi(x)), \end{aligned}$$

where $(\cdot,\cdot)$ is the inner product in $L^2(\Omega)$.

With the above definition, we give a spectral representation for the weak solution in the following lemma.

Lemma 3.3. *Define $b_n := (u_0(x), \phi_n(x)), F_n(t) = (F(x,t), \phi_n(x)),\ n \in \mathbb{N}^+$. The spectral representation of the weak solution of FDE (1) is*

$$u(x,t;a) = \sum_{n=1}^{\infty} u_n(t;a)\phi_n(x),\ (x,t) \in \Omega \times [0,T], \tag{4}$$

where $u_n(t;a)$ satisfies the fractional ODE

$${}^C D_t^\alpha u_n(t;a) + \lambda_n a(t) u_n(t;a) = F_n(t),\ u_n(0;a) = b_n,\ n \in \mathbb{N}^+. \tag{5}$$

Proof. For each $n \in \mathbb{N}^+$, multiplying $\phi_n(x)$ on both sides of FDE (1) and integrating it on x over Ω allow us to deduce that

$${}^C D_t^\alpha (u(x,t;a), \phi_n(x)) + \lambda_n a(t)(u(x,t;a), \phi_n(x)) = F_n(t), \tag{6}$$

where $(-\mathcal{L}u(x,t;a), \phi_n(x)) = (u(x,t;a), -\mathcal{L}\phi_n(x)) = \lambda_n(u(x,t;a), \phi_n(x))$ follows from the symmetricity of $-\mathcal{L}$. Set $u_n(t;a) = (u(x,t;a), \phi_n(x))$ and define $u(x,t;a) = \sum_{n=1}^{\infty} u_n(t;a)\phi_n(x)$. Then (6) and the completeness of $\{\phi_n(x) : n \in \mathbb{N}^+\}$ lead to the desired result. □

3.2. **Existence and uniqueness.** In order to show the existence and uniqueness of the weak solution (4), we state the following lemma [5, Theorem 3.25].

Lemma 3.4. *For the Cauchy-type problem*

$$ {}^C D_t^\alpha y = f(y,t),\ \ y(0) = c_0, $$

if for any continuous $y(t)$, $f(y,t) \in C[0,T]$, $\exists A > 0$ *which is independent of* $y \in C[0,T]$ *and* $t \in [0,T]$ *s.t.* $|f(t,y_1) - f(t,y_2)| \le A|y_1 - y_2|$, *then there exists a unique solution* $y(t)$ *for the Cauchy-type problem, which satisfies* ${}^C D_t^\alpha y \in C[0,T]$.

The theorem of existence and uniqueness for $u(x,t;a)$ follows from Lemma 3.4.

Theorem 3.5 (Existence and Uniqueness)**.** *Suppose Assumption 3.1 holds. Under Definition 3.2, there exists a unique weak solution* $u(x,t;a)$ *of FDE* (1) *with the spectral representation* (4) *and for each* $n \in \mathbb{N}^+$, $u_n(t;a) \in C[0,T]$ *is the unique solution of the fractional ODE* (5) *with* ${}^C D_t^\alpha u_n(t;a) \in C[0,T]$.

Proof. From the spectral representation (4), it suffices to show the existence and uniqueness of $u_n(t;a), n \in \mathbb{N}^+$. Fix $n \in \mathbb{N}^+$, Assumption 3.1 (a) and (b) yield that the fractional ODE (5) satisfies the conditions of Lemma 3.4. Hence the existence and uniqueness for $u_n(t;a)$ hold. □

3.3. **Sign of $u_n(t;a)$.** In this part, we state two properties of $u_n(t;a)$ which play important roles in building the regularity of $u(x,t;a)$.

Lemma 3.6. *Given* $h \in C^+[0,T]$, $f \in C[0,T]$ *with* ${}^C D_t^\alpha f \in C[0,T]$, *if* $f(0) \le (\ge) 0$ *and* ${}^C D_t^\alpha f + h(t)f(t) \le (\ge) 0$, *then* $f \le (\ge) 0$ *on* $[0,T]$.

Proof. Since $f(t) \in C[0,T]$, $f(t)$ attains its maximum over $[0,T]$ at some point $t_0 \in [0,T]$. If $t_0 = 0$, then $f(t) \le f(0) \le 0$. If $t_0 \in (0,T]$, with Lemma 2.5, we have ${}^C D_t^\alpha f(t_0) \ge 0$, which yields $h(t_0)f(t_0) \le 0$, i.e. $f(t_0) \le 0$ due to $h > 0$ on $[0,T]$. The definition of t_0 assures $f \le 0$.

For the case of "≥ 0", let $\overline{f}(t) = -f(t)$, then the above proof gives $\overline{f} \le 0$, i.e. $f \ge 0$. □

The following corollary, which concerns the sign of $u_n(t;a)$, follows from Lemma 3.6 directly.

Corollary 1. *Set* $u_n(t;a)$ *be the unique solution of the fractional ODE* (5)*. Then* ${}^C D_t^\alpha u_n(t;a) + \lambda_n a(t) u_n(t;a) \le (\ge) 0$ *on* $[0,T]$ *and* $u_n(0;a) \le (\ge) 0$ *imply* $u_n(t;a) \le (\ge) 0$ *on* $[0,T]$, $n \in \mathbb{N}^+$.

Proof. Assumption 3.1 gives that $\lambda_n a(t) \in C^+[0,T]$. Then the proof is completed by applying Lemma 3.6 to the fractional ODE (5). □

3.4. **Regularity.** In this part, we establish the regularity of $u(x,t;a)$. To this end, we split FDE (1) into

$$ \begin{aligned} {}^C D_t^\alpha u(x,t) - a(t)\mathcal{L}u(x,t) &= F(x,t), && x \in \Omega,\ t \in (0,T]; \\ u(x,t) &= 0, && (x,t) \in \partial\Omega \times (0,T]; \\ u(x,0) &= 0, && x \in \Omega, \end{aligned} \tag{7} $$

and

$$ \begin{aligned} {}^C D_t^\alpha u(x,t) - a(t)\mathcal{L}u(x,t) &= 0, && x \in \Omega,\ t \in (0,T]; \\ u(x,t) &= 0, && (x,t) \in \partial\Omega \times (0,T]; \\ u(x,0) &= u_0(x), && x \in \Omega. \end{aligned} \tag{8} $$

Denote the weak solutions of FDEs (7) and (8) by $u^r(x,t;a)$ and $u^i(x,t;a)$, respectively ("r" and "i" denote the initials of "right-hand side" and "initial condition"). The following lemma about $u^r(x,t;a)$ and $u^i(x,t;a)$ follows from Lemma 3.3 and Theorem 3.5.

Lemma 3.7. *Suppose Assumption 3.1 holds. Then $u^r(x,t;a)$ and $u^i(x,t;a)$ are the unique solutions for FDEs (7) and (8), respectively, with the spectral representations as*

$$u^r(x,t;a)=\sum_{n=1}^{\infty}u_n^r(t;a)\phi_n(x),\ \ u^i(x,t;a)=\sum_{n=1}^{\infty}u_n^i(t;a)\phi_n(x), \tag{9}$$

where $u_n^r(t;a),\ u_n^i(t;a)$ satisfy the following fractional ODEs

$${}^CD_t^\alpha u_n^r(t;a)+\lambda_n a(t)u_n^r(t;a)=F_n(t),\ u_n^r(0;a)=0,\ n\in\mathbb{N}^+; \tag{10}$$

$${}^CD_t^\alpha u_n^i(t;a)+\lambda_n a(t)u_n^i(t;a)=0,\ u_n^i(0;a)=b_n,\ n\in\mathbb{N}^+. \tag{11}$$

Moreover, Theorem 3.5 ensures the weak solution $u(x,t;a)$ of FDE (1) can be written as $u(x,t;a)=u^r(x,t;a)+u^i(x,t;a)$, i.e. $u_n(t;a)=u_n^r(t;a)+u_n^i(t;a),\ n\in\mathbb{N}^+$.

3.4.1. *Regularity of u^r.* For each $n\in\mathbb{N}^+$, define

$$F_n^+(t)=\begin{cases}F_n(t), & \text{if } F_n(t)\ge 0;\\ 0, & \text{if } F_n(t)<0,\end{cases}\quad F_n^-(t)=\begin{cases}F_n(t), & \text{if } F_n(t)<0;\\ 0, & \text{if } F_n(t)\ge 0.\end{cases} \tag{12}$$

It is obvious that $F_n=F_n^++F_n^-$, the supports of F_n^+ and F_n^- are disjoint and $F_n^+,F_n^-\in C[0,T]$ which follows from $F_n\in C[0,T]$. Split $u_n^r(t;a)$ as $u_n^r(t;a)=u_n^{r,+}(t;a)+u_n^{r,-}(t;a)$, where $u_n^{r,+}(t;a),\ u_n^{r,-}(t;a)$ satisfy

$${}^CD_t^\alpha u_n^{r,+}(t;a)+\lambda_n a(t)u_n^{r,+}(t;a)=F_n^+(t),\ u_n^{r,+}(0;a)=0,\ n\in\mathbb{N}^+; \tag{13}$$

$${}^CD_t^\alpha u_n^{r,-}(t;a)+\lambda_n a(t)u_n^{r,-}(t;a)=F_n^-(t),\ u_n^{r,-}(0;a)=0,\ n\in\mathbb{N}^+, \tag{14}$$

respectively. The existence and uniqueness of $u_n^{r,+}(t;a)$ and $u_n^{r,-}(t;a)$ hold due to Lemma 3.4 and we can write

$$u^r(x,t;a)=u^{r,+}(x,t;a)+u^{r,-}(x,t;a), \tag{15}$$

where

$$u^{r,+}(x,t;a)=\sum_{n=1}^{\infty}u_n^{r,+}(t;a)\phi_n(x),\ \ u^{r,-}(x,t;a)=\sum_{n=1}^{\infty}u_n^{r,-}(t;a)\phi_n(x). \tag{16}$$

Then we state some properties of $u_n^{r,+}(t;a)$ and $u_n^{r,-}(t;a)$.

Lemma 3.8. *For any $n\in\mathbb{N}^+$, $u_n^{r,+}(t;a)\ge 0$ and $u_n^{r,-}(t;a)\le 0$ on $[0,T]$.*

Proof. This proof follows from Corollary 1 directly. □

Lemma 3.9. *Given $a_1(t),a_2(t)\in C^+[0,T]$ with $a_1(t)\le a_2(t)$ on $[0,T]$, we have*

$$0\le u_n^{r,+}(t;a_2)\le u_n^{r,+}(t;a_1),\ u_n^{r,-}(t;a_1)\le u_n^{r,-}(t;a_2)\le 0,\ t\in[0,T],\ n\in\mathbb{N}^+.$$

Proof. Pick $n\in\mathbb{N}^+$, $u_n^{r,+}(t;a_1)$ and $u_n^{r,+}(t;a_2)$ satisfy the following system:

$$\begin{cases}{}^CD_t^\alpha u_n^{r,+}(t;a_1)+\lambda_n a_1(t)u_n^{r,+}(t;a_1)=F_n^+(t);\\ {}^CD_t^\alpha u_n^{r,+}(t;a_2)+\lambda_n a_2(t)u_n^{r,+}(t;a_2)=F_n^+(t);\\ u_n^{r,+}(0;a_1)=u_n^{r,+}(0;a_2)=0,\end{cases}$$

which leads to

$$ {}^C D_t^\alpha w + \lambda_n a_1(t) w(t) = \lambda_n u_n^{r,+}(t;a_2)(a_2(t) - a_1(t)) \ge 0,\ w(0) = 0, $$

where $w(t) = u_n^{r,+}(t;a_1) - u_n^{r,+}(t;a_2)$ and the last inequality follows from Lemma 3.8 and $a_1 \le a_2$. Hence, Corollary 1 shows that $w(t) \ge 0$, i.e. $u_n^{r,+}(t;a_2) \le u_n^{r,+}(t;a_1)$ and Lemma 3.8 gives $0 \le u_n^{r,+}(t;a_2) \le u_n^{r,+}(t;a_1),\ t \in [0,T]$.

Similarly, we have $u_n^{r,-}(t;a_1) \le u_n^{r,-}(t;a_2) \le 0,\ t \in [0,T]$, completing the proof. □

Assumption 3.1 (a) implies there exists constants q_a, Q_a s.t.

$$ 0 < q_a < a(t) < Q_a \text{ on } [0,T]. \tag{17} $$

From Lemma 3.9, we obtain

$$ |u_n^{r,+}(t;a)| \le |u_n^{r,+}(t;q_a)|,\ |u_n^{r,-}(t;a)| \le |u_n^{r,-}(t;q_a)| \text{ on } t \in [0,T],\ n \in \mathbb{N}^+, \tag{18} $$

where $u_n^{r,+}(t;q_a), u_n^{r,-}(t;q_a)$ are the unique solutions of fractional ODEs (13) and (14) respectively with $a(t) \equiv q_a$ on $[0,T]$. The next two lemmas concern the regularity of $u^{r,+}(x,t;a)$ and ${}^C D_t^\alpha u^{r,+}(x,t;a)$, respectively.

Lemma 3.10.

$$ \|u^{r,+}\|_{L^2(0,T;H^2(\Omega))} \le C\|F\|_{L^2([0,T]\times\Omega)}. $$

Proof. Calculating $\|u^{r,+}(x,t;a)\|^2_{L^2(0,T;H^2(\Omega))}$ directly yields

$$ \begin{aligned} &\|u^{r,+}(x,t;a)\|^2_{L^2(0,T;H^2(\Omega))} \\ &= \int_0^T \|u^{r,+}(x,t;a)\|^2_{H^2(\Omega))} \mathrm{d}t \\ &\le \int_0^T C\|(-\mathcal{L}u^{r,+})(x,t;a)\|^2_{L^2(\Omega)} \mathrm{d}t \\ &= C\int_0^T \|\sum_{n=1}^\infty \lambda_n u_n^{r,+}(t;a)\phi_n(x)\|^2_{L^2(\Omega)} \mathrm{d}t \\ &= C\int_0^T \sum_{n=1}^\infty \lambda_n^2 |u_n^{r,+}(t;a)|^2 \mathrm{d}t \le C\int_0^T \sum_{n=1}^\infty \lambda_n^2 |u_n^{r,+}(t;q_a)|^2 \mathrm{d}t, \end{aligned} $$

where the last inequality is obtained from (18). By the Monotone Convergence Theorem, we have

$$ \begin{aligned} \|u^{r,+}(x,t;a)\|^2_{L^2(0,T;H^2(\Omega))} &\le C\int_0^T \sum_{n=1}^\infty \lambda_n^2 |u_n^{r,+}(t;q_a)|^2 \mathrm{d}t \\ &= C\sum_{n=1}^\infty \int_0^T |\lambda_n u_n^{r,+}(t;q_a)|^2 \mathrm{d}t. \end{aligned} \tag{19} $$

For each $n \in \mathbb{N}^+$, [10] gives the explicit representation of $u_n^{r,+}(t;q_a)$

$$ u_n^{r,+}(t;q_a) = \int_0^t F_n^+(\tau)(t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n q_a (t-\tau)^\alpha)\mathrm{d}\tau, $$

which together with Young's inequality leads to

$$ \begin{aligned} \int_0^T |\lambda_n u_n^{r,+}(t;q_a)|^2 \mathrm{d}t &= \|F_n^+(t) * (\lambda_n t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n q_a t^\alpha))\|^2_{L^2[0,T]} \\ &\le \|F_n^+\|^2_{L^2[0,T]} \|\lambda_n t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n q_a t^\alpha)\|^2_{L^1[0,T]}. \end{aligned} $$

Lemmas 2.2, 2.3 and 2.4 give the bound of $\|\lambda_n t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n q_a t^\alpha)\|_{L^1[0,T]}$

$$\begin{aligned}\|\lambda_n t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n q_a t^\alpha)\|_{L^1[0,T]} &= \int_0^T \left|\lambda_n \tau^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n q_a \tau^\alpha)\right|\mathrm{d}\tau\\ &= \int_0^T \lambda_n \tau^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n q_a \tau^\alpha)\mathrm{d}\tau\\ &= -q_a^{-1}\int_0^T \frac{d}{d\tau}E_{\alpha,1}(-\lambda_n q_a\tau^\alpha)\mathrm{d}\tau\\ &= q_a^{-1}(1-E_{\alpha,1}(-\lambda_n q_a T^\alpha)) \le q_a^{-1};\end{aligned}$$

while the definition (12) provides the bound of $\|F_n^+\|_{L^2[0,T]}$ as $\|F_n^+\|_{L^2[0,T]} \le \|F_n\|_{L^2[0,T]}$. Consequently, it holds $\int_0^T |\lambda_n u_n^{r,+}(t;q_a)|^2\mathrm{d}t \le q_a^{-2}\|F_n\|^2_{L^2[0,T]},\ n\in\mathbb{N}^+$, i.e.

$$\sum_{n=1}^\infty \int_0^T |\lambda_n u_n^{r,+}(t;q_a)|^2\mathrm{d}t \le q_a^{-2}\sum_{n=1}^\infty \|F_n\|^2_{L^2[0,T]},$$

which together with (19) and the completeness of $\{\phi_n(x): n\in\mathbb{N}^+\}$ in $L^2(\Omega)$ gives

$$\begin{aligned}\|u^{r,+}(x,t;a)\|^2_{L^2(0,T;H^2(\Omega))} &\le C\sum_{n=1}^\infty \int_0^T |\lambda_n u_n^{r,+}(t;q_a)|^2\mathrm{d}t\\ &\le C\sum_{n=1}^\infty \|F_n\|^2_{L^2[0,T]} = C\|F\|^2_{L^2([0,T]\times\Omega)},\end{aligned}$$

where the constant C only depends on $a(t)$. This completes the proof. □

Lemma 3.11.

$$\|{}^C D_t^\alpha u^{r,+}\|_{L^2([0,T]\times\Omega)} \le C\|F\|_{L^2([0,T]\times\Omega)}.$$

Proof. (13), (16), definition (12) and the Monotone Convergence Theorem give

$$\begin{aligned}&\|{}^C D_t^\alpha u^{r,+}\|^2_{L^2([0,T]\times\Omega)}\\ &= \int_0^T \|\sum_{n=1}^\infty {}^C D_t^\alpha u_n^{r,+}(\cdot;a)\phi_n(x)\|^2_{L^2(\Omega)}\mathrm{d}t = \sum_{n=1}^\infty \int_0^T |{}^C D_t^\alpha u_n^{r,+}(\cdot;a)|^2\mathrm{d}t\\ &\le \sum_{n=1}^\infty \int_0^T \left(2|\lambda_n a(t)u_n^{r,+}(t;a)|^2 + 2|F_n^+(t)|^2\right)\mathrm{d}t\\ &\le 2\sum_{n=1}^\infty \int_0^T |\lambda_n a(t)u_n^{r,+}(t;a)|^2\mathrm{d}t + 2\sum_{n=1}^\infty \int_0^T |F_n(t)|^2\mathrm{d}t.\end{aligned} \tag{20}$$

The estimate of $\sum\limits_{n=1}^\infty \int_0^T |\lambda_n a(t)u_n^{r,+}(t;a)|^2\mathrm{d}t$ follows from (17), (18) and the proof of Lemma 3.10

$$\sum_{n=1}^\infty \int_0^T |\lambda_n a(t)u_n^{r,+}(t;a)|^2\mathrm{d}t \le Q_a\sum_{n=1}^\infty \int_0^T |\lambda_n u_n^{r,+}(t;q_a)|^2\mathrm{d}t \le C\|F\|^2_{L^2([0,T]\times\Omega)};$$

while the completeness of $\{\phi_n(x) : n \in \mathbb{N}^+\}$ gives $\sum\limits_{n=1}^\infty \int_0^T |F_n(t)|^2\mathrm{d}t = \|F\|^2_{L^2([0,T]\times\Omega)}$. Hence, (20) develops $\|{}^C D_t^\alpha u^{r,+}\|^2_{L^2([0,T]\times\Omega)} \le C\|F\|^2_{L^2([0,T]\times\Omega)}$, which implies the indicated conclusion. □

The following corollary follows immediately from the proofs of Lemmas 3.10 and 3.11.

Corollary 2.

$$\|u^{r,-}\|_{L^2(0,T;H^2(\Omega))} \le C\|F\|_{L^2([0,T]\times\Omega)},\ \|{}^C D_t^\alpha u^{r,-}\|_{L^2([0,T]\times\Omega)} \le C\|F\|_{L^2([0,T]\times\Omega)}.$$

From Lemmas 3.10, 3.11, Corollary 2 and (15), we are able to deduce the regularity for $u^r(x,t;a)$ and ${}^C D_t^\alpha u^r(x,t;a)$.

Lemma 3.12 (Regularity of u^r)**.**

$$\|u^r\|_{L^2(0,T;H^2(\Omega))} + \|{}^C D_t^\alpha u^r\|_{L^2([0,T]\times\Omega)} \le C\|F\|_{L^2([0,T]\times\Omega)}.$$

Proof. (15) gives $u^r(x,t;a) = u^{r,+}(x,t;a) + u^{r,-}(x,t;a)$, which leads to

$$\begin{aligned}
&\|u^r\|_{L^2(0,T;H^2(\Omega))} + \|{}^C D_t^\alpha u^r\|_{L^2([0,T]\times\Omega)}\\
&\le \|u^{r,+}\|_{L^2(0,T;H^2(\Omega))} + \|u^{r,-}\|_{L^2(0,T;H^2(\Omega))}\\
&\quad + \|{}^C D_t^\alpha u^{r,+}\|_{L^2([0,T]\times\Omega)} + \|{}^C D_t^\alpha u^{r,-}\|_{L^2([0,T]\times\Omega)}\\
&\le C\|F\|_{L^2([0,T]\times\Omega)}.
\end{aligned}$$

□

If we impose a higher regularity on F, we can obtain the regularity estimate of $\|u^r\|_{C([0,T];H^2(\Omega))}$.

Corollary 3. *Under Assumption 3.1, if* $F \in C^\theta([0,T];L^2(\Omega)),\ 0<\theta<1$, *then*

$$\|u^r\|_{C([0,T];H^2(\Omega))} + \|{}^C D_t^\alpha u^r\|_{C([0,T];L^2(\Omega))} \le C\|F\|_{C^\theta([0,T];L^2(\Omega))},$$

where C *depends on* Ω, $-\mathcal{L}$ *and* $a(t)$.

Proof. For each $t\in[0,T]$, we have

$$\begin{aligned}
&\|u^{r,+}(x,t;a)\|^2_{H^2(\Omega)}\\
&\le C\|-\mathcal{L}u^{r,+}\|^2_{L^2(\Omega)} \le C\sum_{n=1}^\infty |\lambda_n u_n^{r,+}(t;a)|^2\\
&\le C\sum_{n=1}^\infty \left|\lambda_n \int_0^t F_n^+(\tau)(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n q_a(t-\tau)^\alpha)\mathrm{d}\tau\right|^2\\
&\le C\sum_{n=1}^\infty \left|\lambda_n \int_0^t |F_n^+(\tau)-F_n^+(t)|(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n q_a(t-\tau)^\alpha)\mathrm{d}\tau\right|^2\\
&\quad + C\sum_{n=1}^\infty \left|F_n^+(t)\int_0^t \lambda_n(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n q_a(t-\tau)^\alpha)\mathrm{d}\tau\right|^2.
\end{aligned}$$

The definition of $F_n^+(t)$ yields that $|F_n^+(\tau)-F_n^+(t)| \le |F_n(\tau)-F_n(t)|$; Lemma 2.2 gives

$$0 < \int_0^t \lambda_n(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n q_a(t-\tau)^\alpha)\mathrm{d}\tau = q_a^{-1}(1-E_{\alpha,1}(-\lambda_n q_a t^\alpha)) < q_a^{-1}.$$

Hence,

$$\begin{aligned}
&\|u^{r,+}(x,t;a)\|^2_{H^2(\Omega)}\\
&\le C\sum_{n=1}^\infty \left|\lambda_n \int_0^t |F_n(\tau)-F_n(t)|(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n q_a(t-\tau)^\alpha)\mathrm{d}\tau\right|^2
\end{aligned}$$

$$+ C\sum_{n=1}^{\infty} |F_n(t)|^2 .$$

By [10, Lemma 3.4], we have

$$\|u^{r,+}(x,t;a)\|^2_{H^2(\Omega)} \le C\|F\|^2_{C^\theta([0,T];L^2(\Omega))} + C\|F(\cdot,t)\|^2_{L^2(\Omega)},\ t\in[0,T],$$

which gives

$$\|u^{r,+}\|_{C([0,T];H^2(\Omega))} \le C\|F\|_{C^\theta([0,T];L^2(\Omega))},$$

and the constant C depends on Ω, $-\mathcal{L}$ and $a(t)$. Similarly, we can show $\|u^{r,-}\|_{C([0,T];H^2(\Omega))} \le C\|F\|_{C^\theta([0,T];L^2(\Omega))}$.

For ${}^C D_t^\alpha u^r$, by (10), we have ${}^C D_t^\alpha u^{r,+} = \sum_{n=1}^{\infty}[-\lambda_n a(t) u_n^{r,+}(t;a) + F_n^+(t)]\phi_n(x)$. Then for each $t\in[0,T]$,

$$\begin{aligned}\|{}^C D_t^\alpha u^{r,+}\|^2_{L^2(\Omega)} &\le C\sum_{n=1}^{\infty} Q_a^2|\lambda_n u_n^{r,+}(t;a)|^2 + C\sum_{n=1}^{\infty}|F_n(t)|^2\\ &\le C\sum_{n=1}^{\infty}|\lambda_n u_n^{r,+}(t;a)|^2 + C\|F(\cdot,t)\|^2_{L^2(\Omega)}.\end{aligned}$$

From the above proof for $\|u^{r,+}\|^2_{H^2(\Omega)}$, it holds

$$\|{}^C D_t^\alpha u^{r,+}\|^2_{L^2(\Omega)} \le C\|F\|^2_{C^\theta([0,T];L^2(\Omega))} + C\|F(\cdot,t)\|^2_{L^2(\Omega)},\ t\in[0,T],$$

which gives

$$\|{}^C D_t^\alpha u^{r,+}\|_{C([0,T];L^2(\Omega))} \le C\|F\|_{C^\theta([0,T];L^2(\Omega))}.$$

Analogously, we can show $\|{}^C D_t^\alpha u^{r,-}\|_{C([0,T];L^2(\Omega))} \le C\|F\|_{C^\theta([0,T];L^2(\Omega))}$.

The estimates of $u^{r,+}$, $u^{r,-}$, ${}^C D_t^\alpha u^{r,+}$ and ${}^C D_t^\alpha u^{r,-}$ yield the desired result and complete this proof. ☐

3.4.2. *Regularity of u^i.* In this part we consider the regularity of u^i. Just as in the regularity results for u^r, we first state two lemmas which concern the positivity and monotonicity of u^i, respectively.

Lemma 3.13. *With the representation (9) and the fractional ODE (11), for each $n\in\mathbb{N}^+$, $b_n \le (\ge) 0$ implies that $u_n^i(t;a) \le (\ge) 0$ on $[0,T]$.*

Proof. This is a directly result of Corollary 1. ☐

Lemma 3.14. *Given $a_1, a_2 \in C^+[0,T]$ with $a_1 \le a_2$ on $[0,T]$, for each $n\in\mathbb{N}^+$, we have*

$$\begin{cases} 0 \le u_n^i(t;a_2) \le u_n^i(t;a_1), & \text{if } b_n \ge 0;\\ u_n^i(t;a_1) \le u_n^i(t;a_2) \le 0, & \text{if } b_n \le 0.\end{cases}$$

Proof. Fix $n\in\mathbb{N}^+$, from the fractional ODE (11), the functions $u_n^i(t;a_1)$ and $u_n^i(t;a_2)$ satisfy the following system

$$\begin{cases} {}^C D_t^\alpha u_n^i(t;a_1) + \lambda_n a_1(t) u_n^i(t;a_1) = 0;\\ {}^C D_t^\alpha u_n^i(t;a_2) + \lambda_n a_2(t) u_n^i(t;a_2) = 0;\\ u_n^i(0;a_1) = u_n^i(0;a_2) = b_n.\end{cases}$$

This gives

$$(21)\qquad {}^C D_t^\alpha w + \lambda_n a_1(t) w(t) = \lambda_n u_n^i(t;a_2)(a_2(t) - a_1(t)),\ w(0) = 0,$$

where $w(t) = u_n^i(t;a_1) - u_n^i(t;a_2)$.

If $b_n \geq 0$, Corollary 1 shows that $u_n^i(t;a_1), u_n^i(t;a_2) \geq 0$. Also, Lemma 3.13 and $a_1 \leq a_2$ ensures the right side of (21) is nonnegative, which together with Corollary 1 implies $w \geq 0$, i.e. $0 \leq u_n^i(t;a_2) \leq u_n^i(t;a_1)$. The similar argument yields $u_n^i(t;a_1) \leq u_n^i(t;a_2) \leq 0$ for the case $b_n \leq 0$. □

Lemma 3.15 (Regularity for u^i)**.**

$$\|u^i\|_{L^2(0,T;H^2(\Omega))} + \|{}^C D_t^\alpha u^i\|_{L^2([0,T]\times\Omega)} \leq CT^{\frac{1-\alpha}{2}} \|u_0\|_{H^1(\Omega)}.$$

Proof. Given $t \in [0,T]$, the direct calculation and Lemma 3.14 yield that

$$\begin{aligned}\|u^i(x,t;a)\|^2_{H^2(\Omega)} &\leq C\| -\mathcal{L}u^i(x,t;a)\|^2_{L^2(\Omega)} = C\|\sum_{n=1}^\infty \lambda_n u_n^i(t;a)\phi_n(x)\|^2_{L^2(\Omega)}\\ &= C\sum_{n=1}^\infty |\lambda_n u_n^i(t;a)|^2 \leq C\sum_{n=1}^\infty |\lambda_n u_n^i(t;q_a)|^2.\end{aligned}$$

Recall that [10] established the representation as $u_n^i(t;q_a) = b_n E_{\alpha,1}(-\lambda_n q_a t^\alpha),\ n \in \mathbb{N}^+$. Hence, by Lemma 2.1,

(22)

$$\begin{aligned}\|u^i(x,t;a)\|^2_{H^2(\Omega)} &\leq C\| -\mathcal{L}u^i(x,t;a)\|^2_{L^2(\Omega)} \leq C\sum_{n=1}^\infty |\lambda_n b_n E_{\alpha,1}(-\lambda_n q_a t^\alpha)|^2\\ &\leq C\sum_{n=1}^\infty |\frac{1}{1+\lambda_n q_a t^\alpha}|^2 \lambda_n^2 b_n^2 = C\sum_{n=1}^\infty |\frac{(\lambda_n q_a t^\alpha)^{\frac12}}{1+\lambda_n q_a t^\alpha}|^2 t^{-\alpha} q_a^{-1} \lambda_n b_n^2\\ &\leq Ct^{-\alpha}\sum_{n=1}^\infty ((-\mathcal{L})^{\frac12} u_0, \phi_n)^2 \leq Ct^{-\alpha}\|u_0\|^2_{H^1(\Omega)},\end{aligned}$$

which leads to $\|u^i\|^2_{L^2(0,T;H^2(\Omega))} \leq C\int_0^T t^{-\alpha}\|u_0\|^2_{H^1(\Omega)}\mathrm{d}t = CT^{1-\alpha}\|u_0\|^2_{H^1(\Omega)}$, i.e.

(23) $$\|u^i\|_{L^2(0,T;H^2(\Omega))} \leq CT^{\frac{1-\alpha}{2}}\|u_0\|_{H^1(\Omega)}.$$

For the estimate of ${}^C D_t^\alpha u^i(x,t;a)$, (9) and (11) yield

$${}^C D_t^\alpha u^i(x,t;a) = \sum_{n=1}^\infty {}^C D_t^\alpha u_n^i(t;a)\phi_n(x) = -\sum_{n=1}^\infty \lambda_n a(t) u_n^i(t;a)\phi_n(x),$$

which together with (17) gives

$$\begin{aligned}\|{}^C D_t^\alpha u^i(x,t;a)\|^2_{L^2(\Omega)} &\leq Q_a^2 \sum_{n=1}^\infty |\lambda_n u_n^i(t;a)|^2\\ &= Q_a^2\| -\mathcal{L}u^i(x,t;a)\|^2_{L^2(\Omega)} \leq Ct^{-\alpha}\|u_0\|^2_{H^1(\Omega)},\ t\in[0,T],\end{aligned}$$

where the last inequality follows from (22). This result implies that

$$\|{}^C D_t^\alpha u^i(x,t;a)\|^2_{L^2([0,T]\times\Omega)} = \int_0^T \|{}^C D_t^\alpha u^i(x,t;a)\|^2_{L^2(\Omega)}\mathrm{d}t \leq CT^{1-\alpha}\|u_0\|^2_{H^1(\Omega)},$$

i.e. $\|{}^C D_t^\alpha u^i\|_{L^2([0,T]\times\Omega)} \leq CT^{\frac{1-\alpha}{2}}\|u_0\|_{H^1(\Omega)}$, which together with (23) completes the proof. □

Moreover, with a stronger condition on u_0, such as assuming $u_0 \in H^2(\Omega)\cap H_0^1(\Omega)$, we can deduce the C-regularity estimate of u^i.

Corollary 4. *With Assumption 3.1 and $u_0 \in H^2(\Omega)\cap H_0^1(\Omega)$, then*

$$\|u^i\|_{C([0,T];H^2(\Omega))} + \|{}^C D_t^\alpha u^i\|_{C([0,T];L^2(\Omega))} \le C\|u_0\|_{H^2(\Omega)}.$$

Proof. Lemma 2.1 yields that

$$\sum_{n=1}^{\infty} |\lambda_n b_n E_{\alpha,1}(-\lambda_n q_a t^\alpha)|^2$$
$$\le C\sum_{n=1}^{\infty} |\lambda_n b_n|^2 = C\|-\mathcal{L}u_0\|^2_{L^2(\Omega)} \le C\|u_0\|^2_{H^2(\Omega)},\ t\in[0,T];$$

meanwhile, the following estimates have been shown in the proof of Theorem 3.15

$$\begin{cases}\|u^i(x,t;a)\|^2_{H^2(\Omega)} \le C\|-\mathcal{L}u^i(x,t;a)\|^2_{L^2(\Omega)} \le C\sum_{n=1}^\infty |\lambda_n b_n E_{\alpha,1}(-\lambda_n q_a t^\alpha)|^2,\\ \|{}^C D_t^\alpha u^i(x,t;a)\|^2_{L^2(\Omega)} \le Q_a^2\sum_{n=1}^\infty |\lambda_n u_n^i(t;a)|^2 = C\|-\mathcal{L}u^i(x,t;a)\|^2_{L^2(\Omega)}.\end{cases}$$

Hence, it holds that

$$\|u^i(x,t;a)\|_{H^2(\Omega)} + \|{}^C D_t^\alpha u^i(x,t;a)\|_{L^2(\Omega)} \le C\|u_0\|_{H^2(\Omega)},\ t\in[0,T],$$

which leads to the claimed result. □

3.5. **Main theorem for the direct problem.** The main theorem for the direct problem follows from Theorem 3.5, Lemmas 3.12 and 3.15, Corollaries 3 and 4, and the relation $u(x,t;a) = u^r(x,t;a) + u^i(x,t;a)$.

Theorem 3.16 (Main theorem for the direct problem)**.** *Let Assumption 3.1 be valid, then under Definition 3.2, there exists a unique weak solution $u(x,t;a)$ of FDE (1) with the spectral representation (4) and the following regularity estimates:*

$$\|u\|_{L^2(0,T;H^2(\Omega))} + \|{}^C D_t^\alpha u\|_{L^2([0,T]\times\Omega)} \le C(\|F\|_{L^2([0,T]\times\Omega)} + T^{\frac{1-\alpha}{2}}\|u_0\|_{H^1(\Omega)}).$$

Moreover, if the conditions $u_0 \in H^2(\Omega)\cap H_0^1(\Omega)$ and $F \in C^\theta([0,T];L^2(\Omega))$, $0 < \theta < 1$ are added, we have:

$$\|u\|_{C([0,T];H^2(\Omega))} + \|{}^C D_t^\alpha u\|_{C([0,T];L^2(\Omega))} \le C(\|F\|_{C^\theta([0,T];L^2(\Omega))} + \|u_0\|_{H^2(\Omega)}).$$

4. **Inverse problem–reconstruction of the diffusion coefficient $a(t)$.** In this section, we discuss how to recover the coefficient $a(t)$ through the output flux data

$$a(t)\frac{\partial u}{\partial \overrightarrow{\mathbf{n}}}(x_0,t;a) = g(t),\ x_0\in\partial\Omega.$$

All cross the inverse problem work, the operator $-\mathcal{L}$ is assumed to satisfy the condition (2), then the expression $\frac{\partial \phi_n}{\partial \overrightarrow{\mathbf{n}}}(x_0)$ makes sense. We only consider this reconstruction in the space $C^+[0,T]$, which can be regarded as the admissible set for $a(t)$. To this end, we introduce an operator K, which will be shown to have a fixed point consisting of the desired coefficient $a(t)$.

4.1. **Operator K.** The operator K is defined as

$$K\psi(t) := \frac{g(t)}{\frac{\partial u}{\partial \overrightarrow{\mathbf{n}}}(x_0,t;\psi)} = \frac{g(t)}{\sum\limits_{n=1}^{\infty} u_n(t;\psi)\frac{\partial \phi_n}{\partial \overrightarrow{\mathbf{n}}}(x_0)},\ t\in[0,T]$$

with domain

$$\mathcal{D}(K) := \{\psi \in C^+[0,T] : \psi(t) \ge g(t)\Big[\frac{\partial u_0}{\partial \overrightarrow{\mathbf{n}}}(x_0) + I_t^\alpha[\frac{\partial F}{\partial \overrightarrow{\mathbf{n}}}(x_0,t)]\Big]^{-1},\ t\in[0,T]\}.$$

To analyze K, we make the following assumptions.

Assumption 4.1. *u_0, F and g should satisfy the following restrictions:*

(a) $u_0 \in H^3(\Omega) \cap H_0^1(\Omega)$ *with* $b_n := (u_0, \phi_n) \geq 0,\ n \in \mathbb{N}^+$;
(b) $\exists \theta \in (0,1)$ *s.t.* $F(x,t) \in C^\theta([0,T]; H^3(\Omega) \cap H_0^1(\Omega))$ *with* $F_n(t) := (F(\cdot,t), \phi_n)$ ≥ 0 *on* $[0,T]$ *for each* $n \in \mathbb{N}^+$;
(c) $\exists N \in \mathbb{N}^+$ *s.t.* $\frac{\partial \phi_N}{\partial \vec{\mathbf{n}}}(x_0) > 0$, $b_N > 0$ *and* $F_N(t) > 0$ *on* $[0,T]$;
(d) $g \in C^+[0,T]$.

The next remark shows that the equality in the definition of K is valid.

Remark 1. Given $\psi \in C^+[0,T]$ and for each $t \in [0,T]$, by the proofs of Corollaries 3 and 4, we have

$$
\begin{aligned}
&\|u^{r,+}(x,t;\psi)\|_{H^3(\Omega)}^2 \\
&\leq C\|(-\mathcal{L})^{3/2}u^{r,+}\|_{L^2(\Omega)}^2 \leq C\sum_{n=1}^\infty |\lambda_n^{3/2}u_n^{r,+}(t;\psi)|^2 \\
&\leq C\sum_{n=1}^\infty \left|\lambda_n^{3/2}\int_0^t F_n^+(\tau)(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n q_\psi (t-\tau)^\alpha)\mathrm{d}\tau\right|^2 \\
&\leq C\sum_{n=1}^\infty \left|\lambda_n \int_0^t \lambda_n^{1/2}|F_n^+(\tau) - F_n^+(t)|(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n q_\psi (t-\tau)^\alpha)\mathrm{d}\tau\right|^2 \\
&\quad + C\sum_{n=1}^\infty \left|\lambda_n^{1/2}F_n^+(t)(1 - E_{\alpha,1}(-\lambda_n q_\psi t^\alpha))\right|^2 \\
&\leq C\sum_{n=1}^\infty \left|\lambda_n \int_0^t \lambda_n^{1/2}|F_n(\tau) - F_n(t)|(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n q_\psi (t-\tau)^\alpha)\mathrm{d}\tau\right|^2 \\
&\quad + C\sum_{n=1}^\infty \left|\lambda_n^{1/2}F_n(t)(1 - E_{\alpha,1}(-\lambda_n q_\psi t^\alpha))\right|^2 \\
&\leq C\|(-\mathcal{L})^{1/2}F\|_{C^\theta([0,T];L^2(\Omega))}^2 + C\|(-\mathcal{L})^{1/2}F(\cdot,t)\|_{L^2(\Omega)}^2 \\
&\leq C\|F\|_{C^\theta([0,T];H^1(\Omega))}^2 + C\|F(\cdot,t)\|_{H^1(\Omega)}^2
\end{aligned}
$$

and

$$\|u^{r,-}(x,t;\psi)\|_{H^3(\Omega)}^2 \leq C\|F\|_{C^\theta([0,T];H^1(\Omega))}^2 + C\|F(\cdot,t)\|_{H^1(\Omega)}^2,$$

which give $\|u^r\|_{C([0,T];H^3(\Omega))} \leq C\|F\|_{C^\theta([0,T];H^1(\Omega))}$;

$$
\begin{aligned}
\|u^i(x,t;\psi)\|_{H^3(\Omega)}^2 &\leq C\|(-\mathcal{L})^{3/2}u^i\|_{L^2(\Omega)}^2 \leq C\|\sum_{n=1}^\infty \lambda_n^{3/2}u_n^i(t;\psi)\phi_n(x)\|_{L^2(\Omega)}^2 \\
&\leq C\sum_{n=1}^\infty |\lambda_n^{3/2}b_n E_{\alpha,1}(-\lambda_n q_\psi t^\alpha)|^2 \leq C\sum_{n=1}^\infty |\lambda_n^{3/2}b_n|^2 \\
&= C\|(-\mathcal{L})^{3/2}u_0\|_{L^2(\Omega)}^2 \leq C\|u_0\|_{H^3(\Omega)}^2,
\end{aligned}
$$

which gives $\|u^i\|_{C([0,T];H^3(\Omega))} \leq C\|u_0\|_{H^3(\Omega)}$. Combining the above two results yields that

$$\|u\|_{C([0,T];H^3(\Omega))} \leq C(\|F\|_{C^\theta([0,T];H^1(\Omega))} + \|u_0\|_{H^3(\Omega)}) < \infty,$$

which means for each $t \in [0,T]$, $\|u\|_{H^3(\Omega)} < \infty$. Recall that $\Omega \subset R^n, n = 1,2,3$, then the Sobolev Embedding Theorem gives

$$u(x,t;\psi) = \sum_{n=1}^{\infty} u_n(t;\psi)\phi_n(x) \in C^1(\overline{\Omega}) \text{ for each } t \in [0,T].$$

Hence, $\sum_{n=1}^{\infty} u_n(t;\psi)\frac{\partial \phi_n}{\partial \overrightarrow{\mathbf{n}}}(x_0)$ is well-defined and

$$\frac{\partial u}{\partial \overrightarrow{\mathbf{n}}}(x_0,t;\psi) = \sum_{n=1}^{\infty} u_n(t;\psi)\frac{\partial \phi_n}{\partial \overrightarrow{\mathbf{n}}}(x_0), \quad t \in [0,T].$$

The following two remarks will explain the reasonableness and reason for Assumption 4.1.

Remark 2. For the inverse problem, the right-hand side function $F(x,t)$ and the initial condition $u_0(x)$ are input data, which, at least in some circumstance, can be assumed to be controlled. Even though Assumption 4.1 (a), (b) and (c) appear restrictive, it is not hard to construct functions that satisfy them. For example, in (a) if $u_0 = c\phi_k$ for some $c > 0$, then Assumption 4.1 (a) will be satisfied. This will also be true if $u_0 = \sum_{k=1}^{M} c_k\phi_k$ with all $c_k > 0$. Similarly, (b) is satisfied if $F(x,t)$ is also a linear combination of $\{\phi_n : n \in \mathbb{N}^+\}$ with positive coefficients. For (c), by the completeness of $\{\phi_n : n \in \mathbb{N}^+\}$ in $L^2(\Omega)$, there should exist $N \in \mathbb{N}^+$ s.t. $\frac{\partial \phi_N}{\partial \overrightarrow{\mathbf{n}}}(x_0) > 0$. Otherwise, for each $\psi \in H^3(\Omega) \subset L^2(\Omega)$, $\frac{\partial \psi}{\partial \overrightarrow{\mathbf{n}}}(x_0) = 0$ and obviously it is incorrect. Then for this N, we only need to set the coefficients of u_0 and F upon ϕ_N be strictly positive.

The output flux data $g(t)$, it is not under our control. However, if there exists $a \in C^+[0,T]$ s.t. $a(t)\frac{\partial u}{\partial \overrightarrow{\mathbf{n}}}(x_0,t;a) = g(t)$, Assumption 4.1 (a), (b) and Corollary 1 yield that $u_n(t;a) \geq 0$; (3) gives $\frac{\partial \phi_n}{\partial \overrightarrow{\mathbf{n}}}(x_0) \geq 0, n \in \mathbb{N}^+$; Assumption 4.1 (c) ensures $\frac{\partial \phi_N}{\partial \overrightarrow{\mathbf{n}}}(x_0) > 0$ and $u_N(t;a) > 0$ on $[0,T]$, where the proof can be seen in Lemma 4.2. Consequently,

$$\frac{\partial u}{\partial \overrightarrow{\mathbf{n}}}(x_0,t;a) = \sum_{n=1}^{\infty} u_n(t;a)\frac{\partial \phi_n}{\partial \overrightarrow{\mathbf{n}}}(x_0) \geq u_N(t;a)\frac{\partial \phi_N}{\partial \overrightarrow{\mathbf{n}}}(x_0) > 0, \ t \in [0,T].$$

This together with $a \in C^+[0,T]$ gives that $g > 0$. The continuity of g follows from the ones of a and $u_n(t;a)$, $n \in \mathbb{N}^+$, which are derived from the admissible set $C^+[0,T]$ and Theorem 3.5, respectively. Therefore, Assumption 4.1 (d) is reasonable and can be attained.

Remark 3. The well-definedness of the domain $\mathcal{D}(K)$ is guaranteed by Assumption 4.1 (a), (b), (c) and (d) in the sense that the H^3-regularity of u_0, F and the Sobolev Embedding Theorem support that $\frac{\partial u_0}{\partial \overrightarrow{\mathbf{n}}}(x_0)$ and $\frac{\partial F}{\partial \overrightarrow{\mathbf{n}}}(x_0,t)$ are well defined, and the dominator of the lower bound of $\mathcal{D}(K)$

$$\frac{\partial u_0}{\partial \overrightarrow{\mathbf{n}}}(x_0) + I_t^\alpha[\frac{\partial F}{\partial \overrightarrow{\mathbf{n}}}(x_0,t)] = \sum_{n=1}^{\infty}(b_n + I_t^\alpha F_n)\frac{\partial \phi_n}{\partial \overrightarrow{\mathbf{n}}}(x_0) \geq (b_N + I_t^\alpha F_N)\frac{\partial \phi_N}{\partial \overrightarrow{\mathbf{n}}}(x_0) > 0$$

on $[0,T]$. Recall that the numerator $g > 0$, so that the lower bound $g(t)\Big[\frac{\partial u_0}{\partial \overrightarrow{\mathbf{n}}}(x_0) + I_t^\alpha[\frac{\partial F}{\partial \overrightarrow{\mathbf{n}}}(x_0,t)]\Big]^{-1} > 0$, which gives that $\mathcal{D}(K)$ is a subspace of $C^+[0,T]$. Also, $F(x,t) \in C^\theta([0,T]; H^3(\Omega) \cap H_0^1(\Omega))$ yields that $F_N(t)$ is continuous on $[0,T]$, so is $(b_N + I_t^\alpha F_N)\frac{\partial \phi_N}{\partial \overrightarrow{\mathbf{n}}}(x_0)$. Then $\exists C > 0$ s.t. $(b_N + I_t^\alpha F_N)\frac{\partial \phi_N}{\partial \overrightarrow{\mathbf{n}}}(x_0) > C > 0$, which leads to the dominator

$$\frac{\partial u_0}{\partial \overrightarrow{\mathbf{n}}}(x_0) + I_t^\alpha[\frac{\partial F}{\partial \overrightarrow{\mathbf{n}}}(x_0, t)] > C > 0 \text{ on } [0, T].$$

The strict positivity of the dominator avoids $\mathcal{D}(K)$ degenerating to an empty set.

In order to show the well-definedness of K, Assumption 4.1 (a), (b) and (c) will be used. Furthermore, Assumption 4.1 (a) and (b) are crucial to build the monotonicity of operator K; meanwhile, Assumption 4.1 (c) is stated for the uniqueness of fixed points of K.

For the operator K, we have the following lemmas.

Lemma 4.2. *The operator K is well-defined.*

Proof. For each $\psi \in \mathcal{D}(K)$, Theorem 3.5 ensures that there exists a unique $u_n(t;\psi)$ for $n \in \mathbb{N}^+$, which implies the existence and uniqueness of $K\psi$.

Then it is suffice to show the dominator $\sum_{n=1}^{\infty} u_n(t;\psi)\frac{\partial \phi_n}{\partial \overrightarrow{\mathbf{n}}}(x_0) > 0$ on $[0, T]$. With (5), Lemma 1 and Assumption 4.1 (a) and (b), we have $u_n(t;\psi) \geq 0$ on $[0, T]$, which together with $\frac{\partial \phi_n}{\partial \overrightarrow{\mathbf{n}}}(x_0) \geq 0$ gives $\sum_{n=1}^{\infty} u_n(t;\psi)\frac{\partial \phi_n}{\partial \overrightarrow{\mathbf{n}}}(x_0) \geq u_N(t;\psi)\frac{\partial \phi_N}{\partial \overrightarrow{\mathbf{n}}}(x_0)$. Due to the assumption $\frac{\partial \phi_N}{\partial \overrightarrow{\mathbf{n}}}(x_0) > 0$, we claim that $u_N(t;\psi) > 0$. Assume not, i.e. $\exists t_0 \in [0, T]$ s.t. $u_N(t_0;\psi) \leq 0$. The result $u_N(t;\psi) \geq 0$ yields that $u_N(t_0;\psi) = 0$ so that $u_N(t;\psi)$ attains its minimum at $t = t_0$. $u_N(0;\psi) = b_N > 0$ implies $t_0 \neq 0$, i.e. $t_0 \in (0, T]$. Then Lemma 2.5, $u_N(t_0;\psi) = 0$ and the ODE (5) show that ${}^C D_t^\alpha u_N(t_0;\psi) = F_N(t_0) \leq 0$, which contradicts with Assumption 4.1 (c) and confirms the claim. Hence,

$$\sum_{n=1}^{\infty} u_n(t;\psi)\frac{\partial \phi_n}{\partial \overrightarrow{\mathbf{n}}}(x_0) \geq u_N(t;\psi)\frac{\partial \phi_N}{\partial \overrightarrow{\mathbf{n}}}(x_0) > 0,$$

which completes the proof. □

Lemma 4.3. *K maps $\mathcal{D}(K)$ into $\mathcal{D}(K)$.*

Proof. Given $\psi \in \mathcal{D}(K)$. The continuity of $K\psi$ follows from the continuity of $u_n(t;\psi)$ for each $n \in \mathbb{N}^+$ and the continuity of g, which are established by Theorem 3.5 and Assumption 4.1 (d) respectively.

For each $n \in \mathbb{N}^+$, (5) ensures $u_n(t;\psi)$ satisfies

$${}^C D_t^\alpha u_n(t;\psi) + \lambda_n \psi(t) u_n(t;\psi) = F_n(t), \; u_n(0;\psi) = b_n.$$

Taking I_t^α on both sides of the above ODE and using Lemma 2.6 yield that

$$u_n(t;\psi) + \lambda_n I_t^\alpha[\psi(t) u_n(t;\psi)] = I_t^\alpha F_n + b_n.$$

From the proof of Lemma 4.2, we have $u_n(t;\psi) \geq 0$ on $[0, T]$, which together with $\lambda_n > 0$, the positivity of ψ and the definition of I_t^α yields that $\lambda_n I_t^\alpha[\psi(t) u_n(t;\psi)] \geq 0$. Since $u_n(t;\psi) \geq 0$ and $\lambda_n I_t^\alpha[\psi(t) u_n(t;\psi)] \geq 0$, we deduce that $0 \leq u_n(t;\psi) \leq I_t^\alpha F_n + b_n$ on $[0, T]$. Hence, with $\frac{\partial \phi_n}{\partial \overrightarrow{\mathbf{n}}}(x_0) \geq 0$ and the smoothness assumptions $u_0 \in H^3(\Omega) \cap H_0^1(\Omega)$, $F \in C^\theta([0, T]; H^3(\Omega) \cap H_0^1(\Omega))$ stated in Assumption 4.1 (a) and (b) respectively, the following inequality holds

$$\sum_{n=1}^{\infty} u_n(t;\psi)\frac{\partial \phi_n}{\partial \overrightarrow{\mathbf{n}}}(x_0) \leq \sum_{n=1}^{\infty}(I_t^\alpha F_n + b_n)\frac{\partial \phi_n}{\partial \overrightarrow{\mathbf{n}}}(x_0) = \frac{\partial u_0}{\partial \overrightarrow{\mathbf{n}}}(x_0) + I_t^\alpha[\frac{\partial F}{\partial \overrightarrow{\mathbf{n}}}(x_0, t)],$$

which together with $g > 0$ yields that

$$K\psi(t) = \frac{g(t)}{\sum_{n=1}^{\infty} u_n(t;\psi)\frac{\partial \phi_n}{\partial \overrightarrow{\mathbf{n}}}(x_0)} \ge g(t)\Big[\frac{\partial u_0}{\partial \overrightarrow{\mathbf{n}}}(x_0) + I_t^\alpha[\frac{\partial F}{\partial \overrightarrow{\mathbf{n}}}(x_0,t)]\Big]^{-1} > 0,\ t \in [0,T],$$

where the last inequality follows from Remark 3. The above result and the continuity of $K\psi$ lead to $K\psi \in \mathcal{D}(K)$, which is the expected result. □

4.2. **Monotonicity.** In this part, we show the monotonicity of the operator K.

Theorem 4.4 (Monotonicity)**.** *Given $a_1, a_2 \in \mathcal{D}(K)$ with $a_1 \le a_2$, then $Ka_1 \le Ka_2$ on $[0,T]$.*

Proof. Pick $n \in \mathbb{N}^+$, due to (5), $u_n(t;a_1)$ and $u_n(t;a_2)$ satisfy

$$\begin{cases} {}^C D_t^\alpha u_n(t;a_1) + \lambda_n a_1(t) u_n(t;a_1) = F_n(t),\ u_n(0;a_1) = b_n; \\ {}^C D_t^\alpha u_n(t;a_2) + \lambda_n a_2(t) u_n(t;a_2) = F_n(t),\ u_n(0;a_2) = b_n, \end{cases}$$

which together with $a_1 \le a_2$ and Lemma 3.6 yields

$$(24)\qquad {}^C D_t^\alpha w + \lambda_n a_1(t) w(t) = \lambda_n u_n(t;a_2)(a_2(t) - a_1(t)) \ge 0,\ w(0) = 0,$$

where $w(t) = u_n(t;a_1) - u_n(t;a_2)$. Applying Lemma 3.6 to the above ODE yields that $w \ge 0$, i.e. $u_n(t;a_1) \ge u_n(t;a_2) \ge 0$, which together with assumption (3) leads to

$$\sum_{n=1}^{\infty} u_n(t;a_1)\frac{\partial \phi_n}{\partial \overrightarrow{\mathbf{n}}}(x_0) \ge \sum_{n=1}^{\infty} u_n(t;a_2)\frac{\partial \phi_n}{\partial \overrightarrow{\mathbf{n}}}(x_0) > 0,\ t \in [0,T].$$

Therefore, with the condition $g > 0$ stated in Assumption 4.1 (d),

$$Ka_1(t) = \frac{g(t)}{\sum_{n=1}^{\infty} u_n(t;a_1)\frac{\partial \phi_n}{\partial \overrightarrow{\mathbf{n}}}(x_0)} \le \frac{g(t)}{\sum_{n=1}^{\infty} u_n(t;a_2)\frac{\partial \phi_n}{\partial \overrightarrow{\mathbf{n}}}(x_0)} = Ka_2(t),\ t \in [0,T],$$

which completes this proof. □

4.3. **Uniqueness.** In order to show the uniqueness, we state two lemmas.

Lemma 4.5. *If $a_1, a_2 \in \mathcal{D}(K)$ are both fixed points of K with $a_1 \le a_2$, then $a_1 \equiv a_2$.*

Proof. Pick a fixed point $a(t)$, then

$$a(t)\sum_{n=1}^{\infty} u_n(t;a)\frac{\partial \phi_n}{\partial \overrightarrow{\mathbf{n}}}(x_0) = \sum_{n=1}^{\infty} a(t) u_n(t;a)\frac{\partial \phi_n}{\partial \overrightarrow{\mathbf{n}}}(x_0) = g(t),$$

which gives

$$(25)\qquad \sum_{n=1}^{\infty} I_t^\alpha[a(t)u_n(t;a)]\frac{\partial \phi_n}{\partial \overrightarrow{\mathbf{n}}}(x_0) = I_t^\alpha g$$

by taking I_t^α on both sides. Similarly, taking I_t^α on the both sides of (5) and applying Lemma 2.6 yield that

$$I_t^\alpha[a(t)u_n(t;a)] = \lambda_n^{-1} I_t^\alpha F_n + \lambda_n^{-1} b_n - \lambda_n^{-1} u_n(t;a),\ n \in \mathbb{N}^+,$$

which together with (25) generates

$$(26)\qquad \sum_{n=1}^{\infty} \lambda_n^{-1} u_n(t;a)\frac{\partial \phi_n}{\partial \overrightarrow{\mathbf{n}}}(x_0) = \sum_{n=1}^{\infty} \lambda_n^{-1}(I_t^\alpha F_n + b_n)\frac{\partial \phi_n}{\partial \overrightarrow{\mathbf{n}}}(x_0) - I_t^\alpha g.$$

In (26), the convergence of the two series in $C[0,T]$ is supported by Assumption 4.1, Remark 1 and the fact that $0<\lambda_1\le\lambda_2\le\cdots$.

Given two fixed points a_1, a_2 with $a_1\le a_2$, then a_1 and a_2 should satisfy (26) simultaneously, which gives

$$\sum_{n=1}^{\infty}\lambda_n^{-1}\frac{\partial\phi_n}{\partial\overrightarrow{\mathbf{n}}}(x_0)(u_n(t;a_1)-u_n(t;a_2))=0. \tag{27}$$

In the proof of Theorem 4.4, we have shown that $u_n(t;a_1)\ge u_n(t;a_2)\ge 0$. Also recall that $\lambda_n^{-1}\frac{\partial\phi_n}{\partial\overrightarrow{\mathbf{n}}}(x_0)\ge 0,\ n\in\mathbb{N}^+$, then $\lambda_n^{-1}\frac{\partial\phi_n}{\partial\overrightarrow{\mathbf{n}}}(x_0)(u_n(t;a_1)-u_n(t;a_2))\ge 0$ on $[0,T]$ for $n\in\mathbb{N}^+$. Hence, (27) implies that

$$\lambda_n^{-1}\frac{\partial\phi_n}{\partial\overrightarrow{\mathbf{n}}}(x_0)(u_n(t;a_1)-u_n(t;a_2))=0,\ t\in[0,T],\ n\in\mathbb{N}^+.$$

Let $n=N$, $\lambda_N^{-1}\frac{\partial\phi_N}{\partial\overrightarrow{\mathbf{n}}}(x_0)>0$ gives $u_N(t;a_1)\equiv u_N(t;a_2)$ on $[0,T]$. Set $w(t)=u_N(t;a_1)-u_N(t;a_2)=0$. Then (24) yields that

$$0={}^CD_t^\alpha w+\lambda_N a_1(t)w(t)=\lambda_N u_N(t;a_2)(a_2(t)-a_1(t)),$$

i.e. $u_N(t;a_2)(a_2(t)-a_1(t))\equiv 0$ on $[0,T]$; while the proof of Lemma 4.2 yields that $u_N(t;a_2)>0$. Hence, we have $a_1=a_2$ on $[0,T]$, which completes the proof. □

Before showing uniqueness, we introduce a successive iteration procedure which will generate a sequence converging to a fixed point if it exists. Set

$$\overline{a}_0(t)=g(t)\Big[\frac{\partial u_0}{\partial\overrightarrow{\mathbf{n}}}(x_0)+I_t^\alpha[\frac{\partial F}{\partial\overrightarrow{\mathbf{n}}}(x_0,t)]\Big]^{-1},\ \overline{a}_{n+1}=K\overline{a}_n,\ n\in\mathbb{N}.$$

Then this iteration reproduces a sequence $\{\overline{a}_n:n\in\mathbb{N}\}$ which is contained by $\mathcal{D}(K)$ due to Lemma 4.3.

Lemma 4.6. *If there exists a fixed point $a(t)\in\mathcal{D}(K)$ of operator K, then the sequence $\{\overline{a}_n:n\in\mathbb{N}\}$ will converge to $a(t)$.*

Proof. $\overline{a}_0$ is the lower bound of $\mathcal{D}(K)$ and $\{\overline{a}_n:n\in\mathbb{N}\}\subset\mathcal{D}(K)$ yield that $\overline{a}_0\le\overline{a}_1$. Using Theorem 4.4, we have $\overline{a}_1=K\overline{a}_0\le K\overline{a}_1=\overline{a}_2$, i.e. $\overline{a}_1\le\overline{a}_2$. The same argument gives $\overline{a}_2=K\overline{a}_1\le K\overline{a}_2=\overline{a}_3$. Continue this process, we can deduce $\overline{a}_0\le\overline{a}_1\le\overline{a}_2\le\dots$, which means $\{\overline{a}_n:n\in\mathbb{N}\}$ is increasing. Since the results that $\overline{a}_0$ is the lower bound of $\mathcal{D}(K)$ and $a(t)\in\mathcal{D}(K)$, it holds $\overline{a}_0\le a$. Applying Theorem 4.4 to this inequality, we obtain $\overline{a}_1=K\overline{a}_0\le Ka=a$, i.e. $\overline{a}_1\le a$. This argument generates $\overline{a}_n\le a,\ n\in\mathbb{N}$, which means $a(t)$ is an upper bound of $\{\overline{a}_n:n\in\mathbb{N}\}$.

We have proved $\{\overline{a}_n:n\in\mathbb{N}\}$ is an increasing sequence in $\mathcal{D}(K)$ with an upper bound $a(t)$, which leads to $\{\overline{a}_n:n\in\mathbb{N}\}$ is convergent in $\mathcal{D}(K)$ and the limit is smaller than $a(t)$. Denote the limit of $\{\overline{a}_n:n\in\mathbb{N}\}$ by $\overline{a}$. We have $\overline{a}\in\mathcal{D}(K)$, $\overline{a}\le a$ and $\overline{a}$ is a fixed point of K in $\mathcal{D}(K)$. Hence, Lemma 4.5 yields $\overline{a}=a$, which is the desired result. □

Now, we are able to prove the uniqueness of fixed points of K.

Theorem 4.7 (Uniqueness)**.** *There is at most one fixed point of K in $\mathcal{D}(K)$.*

Proof. Let $a_1,a_2\in\mathcal{D}(K)$ be both fixed points of K. Lemma 4.6 implies that $\overline{a}_n\to a_1$ and $\overline{a}_n\to a_2$, which leads to $a_1=a_2$ and completes this proof. □

4.4. **Existence.** Assumption 4.1 is not sufficient to deduce the existence of the fixed points of K since $\mathcal{D}(K)$ has no upper bound so that an increasing sequence in $\mathcal{D}(K)$ may not be convergent. In this part, we discuss the existence of fixed points, by providing some extra conditions.

Assumption 4.8. *Additional assumptions on u_0, F and g:*

(a) $-\mathcal{L}u_0 \in H^3(\Omega) \cap H_0^1(\Omega)$;

(b) $F(x,t) = -\mathcal{L}u_0(x) \cdot f(t)$ *s.t.* $f \in C^\theta[0,T], 0 < \theta < 1$ *and* $f(t) \geq g(t)\big[\frac{\partial u_0}{\partial \overrightarrow{\mathbf{n}}}(x_0)\big]^{-1}$ *on* $[0,T]$.

Remark 4. Assumption 4.8 is set up to make sure that $F(x,t) = -\mathcal{L}u_0(x) \cdot f(t) \in C^\theta([0,T]; H^3(\Omega) \cap H_0^1(\Omega))$, so that $F(x,t)$ also satisfies Assumption 4.1.

Fix u_0 and f, if the measured data g does not satisfy Assumption 4.8 (*b*), then we can modify u_0 by increasing the value of u_0 in a very small neighborhood of the point x_0 so that the value of $\frac{\partial u_0}{\partial \overrightarrow{\mathbf{n}}}(x_0)$ becomes larger. Meanwhile, since u_0 is changed in a small domain, the coefficients $\{b_n : n \in \mathbb{N}^+\}$ only vary slightly, so do $u_n(t;a)$ and $u(x,t;a)$. Hence, $\frac{\partial u}{\partial \overrightarrow{\mathbf{n}}}(x_0,t;a)$ and $g(t)$ will not appear a significant change that can violate Assumption 4.8 (*b*).

Define the subspace $\mathcal{D}(K)'$ of $\mathcal{D}(K)$ as

$$\mathcal{D}(K)' := \Big\{\psi \in C^+[0,T] : g(t)\Big[\frac{\partial u_0}{\partial \overrightarrow{\mathbf{n}}}(x_0) + I_t^\alpha\big[\frac{\partial F}{\partial \overrightarrow{\mathbf{n}}}(x_0,t)\big]\Big]^{-1} \leq \psi(t) \leq g(t)\Big[\frac{\partial u_0}{\partial \overrightarrow{\mathbf{n}}}(x_0)\Big]^{-1},\ t \in [0,T]\Big\}.$$

We have proved the lower bound of $\mathcal{D}(K)'$ is positive in Remark 3 and clearly the upper bound of $\mathcal{D}(K)'$ is larger than the lower bound. Consequently, $\mathcal{D}(K)'$ is well-defined.

The next lemma concerns the range of K with domain $\mathcal{D}(K)'$.

Lemma 4.9. *With Assumptions 4.1 and 4.8, K maps $\mathcal{D}(K)'$ into $\mathcal{D}(K)'$.*

Proof. Given $\psi \in \mathcal{D}(K)'$, we have proved $K\psi \in C^+[0,T]$ and

$$K\psi(t) \geq g(t)\Big[\frac{\partial u_0}{\partial \overrightarrow{\mathbf{n}}}(x_0) + I_t^\alpha\big[\frac{\partial F}{\partial \overrightarrow{\mathbf{n}}}(x_0,t)\big]\Big]^{-1},\ t \in [0,T]$$

in the proof of Lemma 4.3, so that it is sufficient to show $K\psi \leq g(t)\big[\frac{\partial u_0}{\partial \overrightarrow{\mathbf{n}}}(x_0)\big]^{-1}$ on $[0,T]$.

For each $n \in \mathbb{N}^+$, let $w_n(t;\psi) = u_n(t;\psi) - b_n$, (5) yields the following ODE by direct calculation

$${}^C D_t^\alpha w_n(t;\psi) + \lambda_n \psi(t) w_n(t;\psi) = \lambda_n b_n (f(t) - \psi(t)) \geq 0,\ w_n(0,\psi) = 0,$$

where $\lambda_n b_n(f(t) - \psi(t)) \geq 0$ follows from the fact $\psi(t) \leq g(t)\big[\frac{\partial u_0}{\partial \overrightarrow{\mathbf{n}}}(x_0)\big]^{-1}$ and Assumption 4.8 (*b*). Applying Corollary 1 to the above ODE gives $w_n(t;\psi) \geq 0$, i.e. $u_n(t;\psi) \geq b_n \geq 0$ on $[0,T]$. Hence,

$$K\psi(t) = \frac{g(t)}{\sum_{n=1}^\infty u_n(t;\psi)\frac{\partial \phi_n}{\partial \overrightarrow{\mathbf{n}}}(x_0)} \leq \frac{g(t)}{\sum_{n=1}^\infty b_n \frac{\partial \phi_n}{\partial \overrightarrow{\mathbf{n}}}(x_0)} = g(t)\big[\frac{\partial u_0}{\partial \overrightarrow{\mathbf{n}}}(x_0)\big]^{-1}$$

and this proof is complete. □

The existence conclusion is derived from Lemmas 4.6 and 4.9.

Theorem 4.10 (Existence)**.** *Suppose Assumptions 4.1 and 4.8 be valid, then there exists a fixed point of K in $\mathcal{D}(K)'$.*

Proof. Lemma 4.6 yields the sequence $\{\overline{a}_n : n \in \mathbb{N}\}$ is increasing, while Lemma 4.9 gives $\{\overline{a}_n : n \in \mathbb{N}\} \subset \mathcal{D}(K)'$. Then $\{\overline{a}_n : n \in \mathbb{N}\}$ is an increasing sequence with an upper bound $g(t)\big[\frac{\partial u_0}{\partial \overrightarrow{\mathbf{n}}}(x_0)\big]^{-1}$, which implies the convergence of $\{\overline{a}_n : n \in \mathbb{N}\}$. Denote the limit by $\overline{a}$, clearly $\overline{a}$ is a fixed point of K. Also, the closedness of $\mathcal{D}(K)'$ yields that $\overline{a} \in \mathcal{D}(K)'$. Therefore, $\overline{a}$ is a fixed point of K in $\mathcal{D}(K)'$, which confirms the existence. □

4.5. **Main theorem for the inverse problem and reconstruction algorithm.** Lemma 4.6, Theorems 4.7 and 4.10 allow us to deduce the main theorem for this inverse problem.

Theorem 4.11 (Main theorem for the inverse problem)**.** *Suppose Assumption 4.1 holds.*

(a) If there exists a fixed point of K in $\mathcal{D}(K)$, then it is unique and coincides with the limit of $\{\overline{a}_n : n \in \mathbb{N}\}$;
(b) If Assumption 4.8 is also valid, then there exists a unique fixed point of K in $\mathcal{D}(K)'$, which is the limit of $\{\overline{a}_n : n \in \mathbb{N}\}$.

The following reconstruction algorithm for $a(t)$ is based on Theorem 4.11.

TABLE 1. Numerical Algorithm

Iteration algorithm to recover the coefficient $a(t)$
1: Set up the right-hand side function $F(x,t)$ and the initial condition $u_0(x)$, then measure the output flux data $g(t)$. F, u_0 and g should satisfy Assumption 4.1;
2: Set the initial guess as $\overline{a}_0(t) = g(t)\Big[\frac{\partial u_0}{\partial \overrightarrow{\mathbf{n}}}(x_0) + I_t^\alpha\big[\frac{\partial F}{\partial \overrightarrow{\mathbf{n}}}(x_0,t)\big]\Big]^{-1}$;
3: **for k = 1,...,N do**
4: Using the L1 time-stepping [3] to compute $u(x,t;\overline{a}_{k-1})$, which is the weak solution of FDE (1) with coefficient function $\overline{a}_{k-1}$;
5: Update the coefficient $\overline{a}_{k-1}$ by $\overline{a}_k = K\overline{a}_{k-1}$;
6: Check stopping criterion $\lVert\overline{a}_k - \overline{a}_{k-1}\rVert_{L^2[0,T]} \le \epsilon_0$ for some $\epsilon_0 > 0$;
7: **end for**
8: **output** the approximate coefficient function $\overline{a}_N$.

5. Numerical results for inverse problem.

5.1. **L1 time-stepping of ${}^C D_t^\alpha$.** The fourth step of Algorithm 1 includes solving the direct problem of FDE (1) numerically. To this end, we choose L1 time stepping [3, 6] to discretize the term ${}^C D_t^\alpha u(x,t)$:

$$
\begin{aligned}
{}^C D_t^\alpha u(x,t_N) &= \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \frac{\partial u(x,s)}{\partial s}(t_N - s)^{-\alpha}\, ds \\
&\approx \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{N-1} \frac{u(x,t_{j+1}) - u(x,t_j)}{\tau} \int_{t_j}^{t_{j+1}} (t_N - s)^{-\alpha} ds \\
&= \sum_{j=0}^{N-1} b_j \frac{u(x,t_{N-j}) - u(x,t_{N-j-1})}{\tau^\alpha}
\end{aligned}
$$

$$= \tau^{-\alpha}[b_0 u(x,t_N) - b_{N-1}u(x,t_0) + \sum_{j=1}^{N-1}(b_j - b_{j-1})u(x,t_{N-j})],$$

where

$$b_j = ((j+1)^{1-\alpha} - j^{1-\alpha})/\Gamma(2-\alpha),\ j = 0, 1, \ldots, N-1.$$

5.2. **Numerical results for noise free data.** In this part, we set $\Omega = (0,1),\ x_0 = 0,\ T = 1,\ \mathcal{L}u = u_{xx}$, pick $u_0(x) = -\sin\pi x,\ F(x,t) = -(t+1)\sin\pi x$ and consider the following two coefficients:

(a1) smooth coefficient: $a(t) = \sin 5\pi t + 1.3$;
(a2) nonsmooth coefficient ("smile" function):

$$a(t) = [0.8\sin 3\pi t + 1.5]\chi_{[0,1/3]} + [-0.5\sin(3\pi t - \pi) + 0.6]\chi_{(1/3,2/3)} + [0.8\sin(3\pi t - 2\pi) + 1.5]\chi_{[2/3,1]}.$$

In experiment (a1), the exact coefficient we pick is a smooth function. Figure 1 shows the initial guess and the first three iterations, while Figure 2 presents the exact and approximate coefficients. From these two figures, we observe that $\{\overline{a}_n : n \in \mathbb{N}\}$ converges to $a(t)$ monotonically, which illustrates Theorems 4.4 and 4.11. Moreover, the L^2 error of the approximation in Figure 2 is $\|a - \overline{a}_N\|_{L^2[0,T]} = 1.04\times 10^{-6}$, which implies us the L^2 error of this approximation may be bounded by the stopping criterion number ϵ_0. This guess is confirmed by Figure 4 and can be expressed as

$$\|a - \overline{a}_N\|_{L^2[0,T]} = O(\epsilon_0).$$

Several attempts of experiment (a1) for different $\alpha \in (0,1)$ are taken to find the dependence of the convergence rate of Algorithm 1 on the fractional order α, which is shown in Figure 3. This figure shows the amounts of iterations required, i.e. N, corresponding to different α, which imply that restricted $\alpha \in (0,1)$, the larger α is, the faster the convergence rate of Algorithm 1 is. This phenomenon is explained in [4] by a property of the Mittag-Leffler function; for $\alpha \in (0,1)$, the larger α is, the faster the decay rate of $E_{\alpha,1}(-z)$ is as $z \to \infty$.

The definition of $\mathcal{D}(K)$ restricts the coefficient $a(t)$ in the space $C^+[0,T]$, however, the results of experiment (a2) indicate that Algorithm 1 still works for nonsmooth $a(t)$, which means the numerical restriction on $a(t)$ can possibly be extended from $a(t) \in C^+[0,T]$ to $a(t) \in L^\infty[0,T]$. For discontinuous $a(t)$, Figures 5 and 6 explain that Theorems 4.4 and 4.11 still hold, while Figures 3 and 4 illustrate the similar conclusions as the larger α is, the faster the convergence rate of Algorithm 1 is, and

$$\|a - \overline{a}_N\|_{L^2[0,T]} = O(\epsilon_0).$$

5.3. **Numerical results for noisy data.** In this subsection, we will consider data polluted by noise. Set g be the exact data and denote the noisy data by g_δ with relative noise level δ, i.e. $\|(g - g_\delta)/g\|_{L^\infty[0,T]} \le \delta$. Then the perturbed operator K_δ is

$$K_\delta\psi(t) = \frac{g_\delta(t)}{\sum_{n=1}^{\infty} u_n(t;\psi)\frac{\partial\phi_n}{\partial\overrightarrow{\mathbf{n}}}(x_0)}$$

with domain

$$\mathcal{D}(K_\delta) := \{\psi \in C^+[0,T] : g_\delta(t)\Big[\frac{\partial u_0}{\partial\overrightarrow{\mathbf{n}}}(x_0) + I_t^\alpha[\frac{\partial F}{\partial\overrightarrow{\mathbf{n}}}(x_0,t)]\Big]^{-1} \le \psi(t),\ t \in [0,T]\}.$$

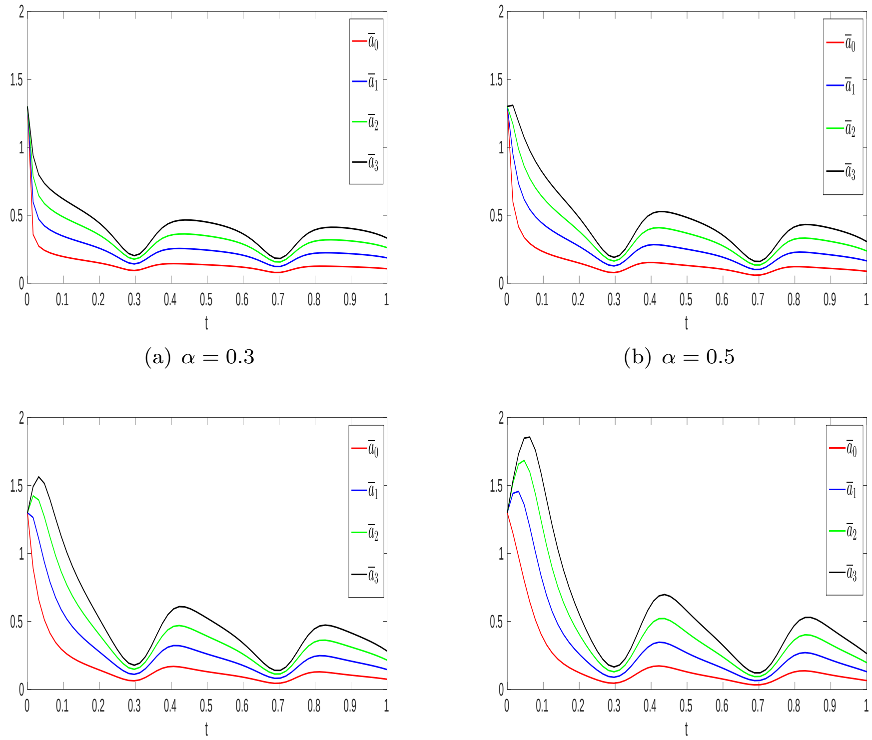


FIGURE 1. Experiment (a1): the initial guess and first three iterations

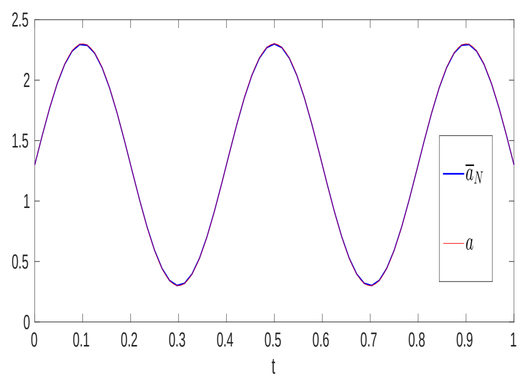


FIGURE 2. Experiment (a1): the exact and approximate coefficients for $\alpha = 0.9$ and $\epsilon_0 = 10^{-6}$

Also, the sequence $\{\overline{a}_{\delta,n} : n \in \mathbb{N}\}$ can be obtained from the iteration

$$\overline{a}_{\delta,0} = g_\delta \Big[\frac{\partial u_0}{\partial \overrightarrow{\mathbf{n}}}(x_0) + I_t^\alpha \big[\frac{\partial F}{\partial \overrightarrow{\mathbf{n}}}(x_0,t)\big]\Big]^{-1},\ \overline{a}_{\delta,n+1} = K_\delta \overline{a}_{\delta,n},\ n \in \mathbb{N}.$$

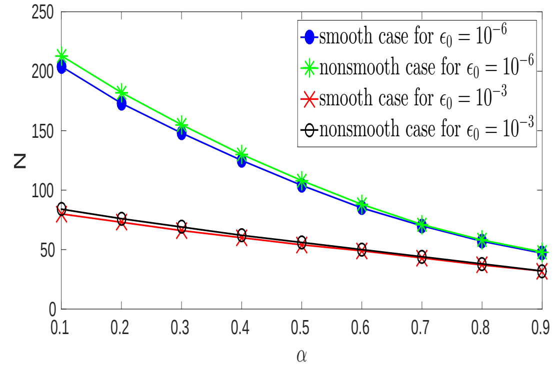


FIGURE 3. the amounts of iterations N for different α

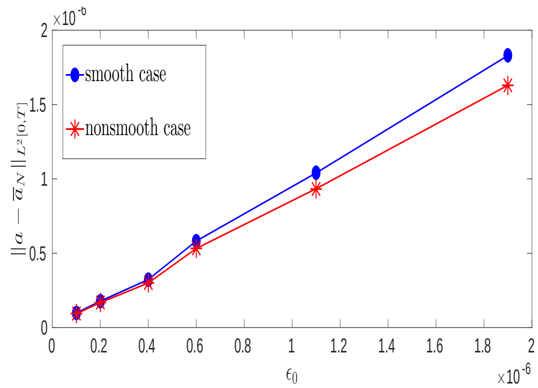


FIGURE 4. $\|a - \overline{a}_N\|_{L^2[0,T]}$ for different ϵ_0 under $\alpha = 0.9$

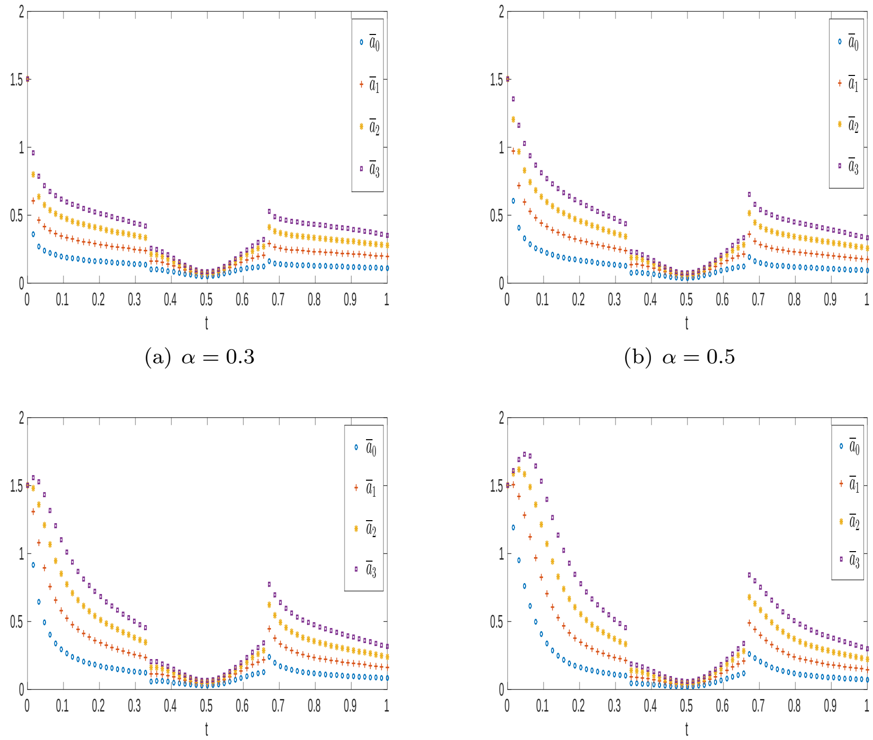
Since δ is a small positive number and g is a strictly positive function, we can assume g_δ is still positive, which means Theorem 4.11 still holds for K_δ. Hence, if there exists a fixed point $a_\delta \in \mathcal{D}(K_\delta)$, the sequence $\{\overline{a}_{\delta,n} : n \in \mathbb{N}\}$ will converge to a_δ monotonically and we denote the limit by $\overline{a}_\delta$. Algorithm 1 is still able to be used to recover $\overline{a}_\delta$ after a slightly modification$-$replacing g and K by g_δ and K_δ, respectively.

We take the experiments (a1) and (a2) with noise level $\delta > 0$. Figures 7 and 8 present the exact and approximate coefficients under $\delta = 3\%$ for experiments (a1) and (a2) respectively. From figures 7 and 8, we observe that the smaller $|a(t)|$ is, the better the approximation is. This can be explained by δ means the relatively noise level, i.e. we pick $g_\delta = (1 + \zeta\delta)g$ in the codes, where ζ follows a uniform distribution on $[-1, 1]$. Figure 9 illustrates that

$$\|a - \overline{a}_{\delta,N}\|_{L^2[0,T]}/\|a\|_{L^2[0,T]} = O(\delta),$$

showing the domination of the noise level δ in relatively L^2 error with the reason that $\epsilon_0 \ll \delta$.

5.4. **Numerical results in two dimensional case.** In this part, the numerical experiments on a two dimensional domain will be considered. We set $\alpha = 0.9$, $\epsilon_0 = 10^{-6}$, $\Omega = (0,1)^2$, $x_0 = (0, 1/2)$, $T = 1$, $\mathcal{L}u = \triangle u$, choose $u_0(x, y) = -\sin[\pi xy(1-x)(1-y)]$, $F(x, y) = -(t + 1)\cdot \sin[\pi xy(1-x)(1-y)]$, and consider



(a) $\alpha = 0.3$ (b) $\alpha = 0.5$

(c) $\alpha = 0.7$ (d) $\alpha = 0.9$

FIGURE 5. Experiment (a2): the initial guess and first three iterations

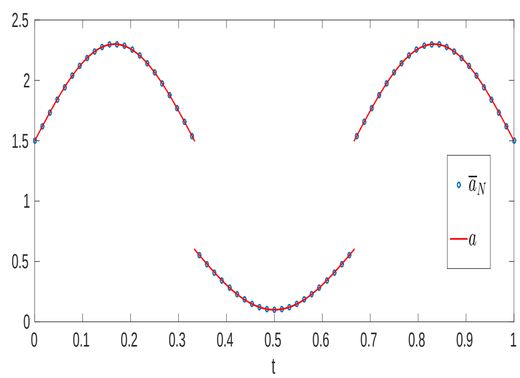


FIGURE 6. Experiment (a2): the exact and approximate coefficients for $\alpha = 0.9$ and $\epsilon_0 = 10^{-6}$

experiments (a1) and (a2). Figures 10 and 11 confirm the theoretical conclusions in section 4.

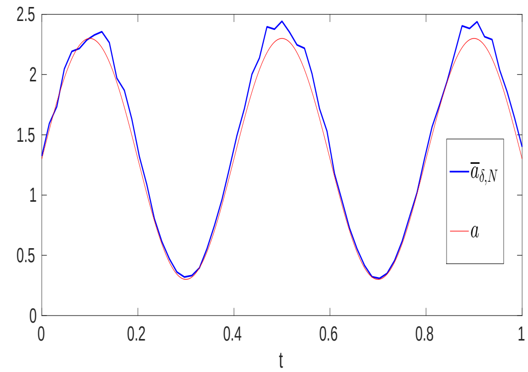


FIGURE 7. Experiment (a1): the exact and approximate coefficients with $\alpha = 0.9$, $\epsilon_0 = 10^{-6}$ and $\delta = 3\%$

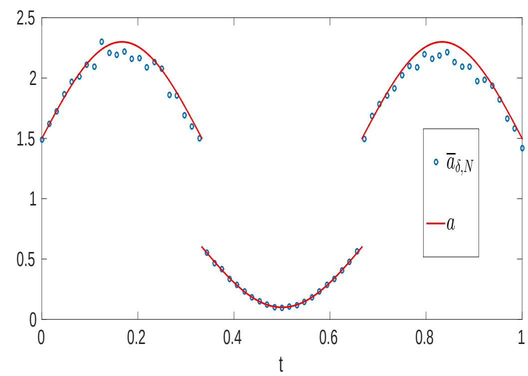


FIGURE 8. Experiment (a2): the exact and approximate coefficients with $\alpha = 0.9$, $\epsilon_0 = 10^{-6}$ and $\delta = 3\%$

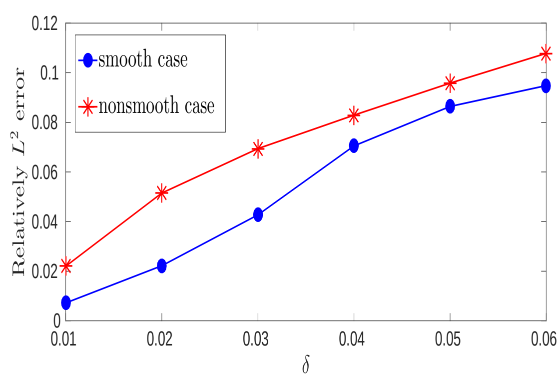
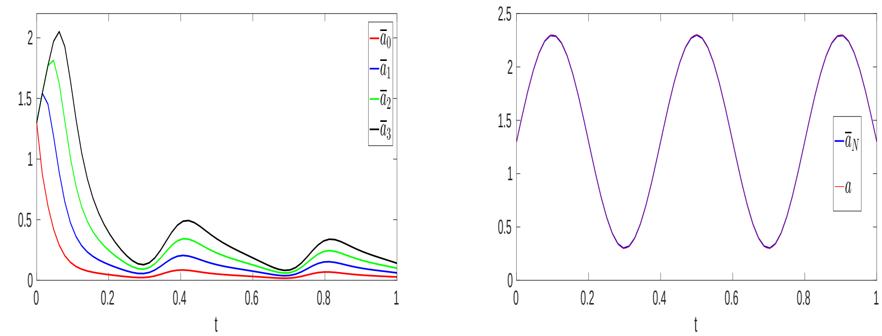
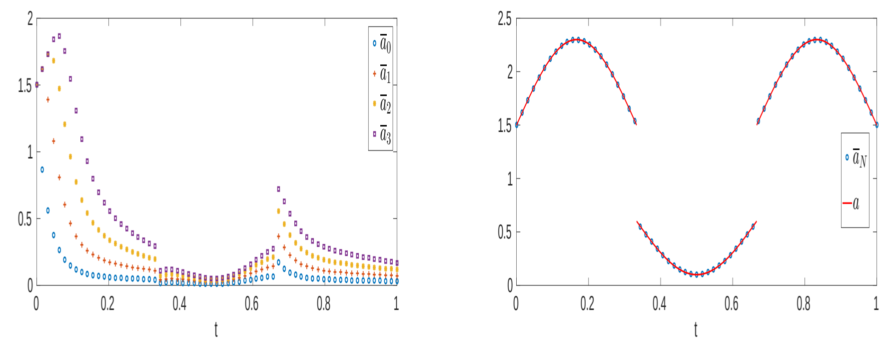


FIGURE 9. $\|a - \overline{a}_{\delta,N}\|_{L^2[0,T]} / \|a\|_{L^2[0,T]}$ for different δ under $\alpha = 0.9$ and $\epsilon_0 = 10^{-6}$



(a) Initial guess and first three iterations (b) Exact and approximate coefficients

FIGURE 10. Experiment (a1) in two dimensional case



(a) Initial guess and first three iterations (b) Exact and approximate coefficients

FIGURE 11. Experiment (a2) in two dimensional case

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E-mail address: zhidong@math.tamu.edu