DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Supplement Volume 2005

Website: http://AIMsciences.org

pp. **672–681**

CONTROLLABILITY TO TRAJECTORIES FOR SEMILINEAR THERMOELASTIC PLATES

Maria Grazia Naso

Department of Mathematics, University of Brescia
Via Valotti, 9. 25133 Brescia, Italy

Abstract. The controllability to trajectories for semilinear thermoelastic plates by a control source acting only on the heat equation of the system is considered. The method we use combines the analyticity of the associated semigroup of the linearized problem and Kakutani fixed point theorem.

1. **Introduction.** Let Ω be a bounded, open, connected subset of $\mathbb{R}^2$, with smooth boundary $\partial\Omega$. Let $T > 0$ and set $Q_T = (0,T)\times\Omega$, $\Sigma_T = (0,T)\times\partial\Omega$. We consider a model which describes small vibrations of a semilinear, homogeneous, elastically and thermally isotropic Euler-Bernoulli plate, under the influence of a control function $f \in L^2(Q_T)$. The system with hinged mechanical and Dirichlet thermal boundary conditions we are going to study is the following one

$$\begin{cases} u_{tt} + \Delta^2 u + \Delta\theta = g_1(u_t) & \text{in } Q_T,\\ \theta_t - \Delta\theta - \Delta u_t = g_2(\theta) + f & \text{in } Q_T,\\ u = 0,\ \Delta u = 0,\ \theta = 0 & \text{on } \Sigma_T,\\ u(0) = u_0,\ u_t(0) = u_1,\ \theta(0) = \theta_0 & \text{in } \Omega. \end{cases} \tag{1}$$

Here, u is the *vertical deflection* of the plate and θ is the *variation of temperature* of the plate with respect to its reference temperature; $g_i : \mathbb{R} \to \mathbb{R}$, $i = 1, 2$, are C^1 functions. The subscript $\cdot_t$ denotes time derivative; u_0, u_1, θ_0 are initial data taken in a suitable space.

The problem we will consider is the controllability to trajectories of the previous model. This notion of controllability can be described as follows. The PDE system (1) is said to be *controllable to trajectories* at time $T > 0$ if for any initial data (u_0, u_1, θ_0) in a suitable space $\mathcal{H}$, there exists a control function $f \in L^2(Q_T)$ such that the corresponding solution (u, u_t, θ) of (1) is defined on $[0, T]$ and satisfies

$$u(T) = \widehat{u}(T), \quad u_t(T) = \widehat{u}_t(T), \quad \theta(T) = \widehat{\theta}(T), \quad \text{a.e. in } \Omega, \tag{2}$$

where $\left[\widehat{u}, \widehat{u}_t, \widehat{\theta}\right]$ is a solution of (1) defined on $[0, T]$ associated to given initial data $\left[\widehat{u}_0, \widehat{u}_1, \widehat{\theta}_0\right]$ in the same space $\mathcal{H}$ and a given function $\widehat{f} \in L^2(Q_T)$.

Moreover, the PDE system (1) is said to be *locally-controllable to trajectories* at time $T > 0$ if there exists a positive constant ρ such that for any initial data

2000 *Mathematics Subject Classification.* Primary: 35B37, 74F05, 93B05; Secondary: 65K10, 65M70.

Key words and phrases. Controllability to trajectories, thermo-elastic plate, Kakutani fixed point theorem.

$(u_0, u_1, \theta_0) \in \mathcal{H}$ satisfying $\left\| [u_0, u_1, \theta_0] - \left[\widehat{u}_0, \widehat{u}_1, \widehat{\theta}_0\right] \right\|_{\mathcal{H}} < \rho$, the solution $\left[\widehat{u}, \widehat{u}_t, \widehat{\theta}\right]$ of system (1) satisfies (2).

1.1. **The control problem.** Given a Hilbert space $\mathcal{H}$, we denote by $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and $\| \cdot \|_{\mathcal{H}}$ the inner product and the norm on $\mathcal{H}$, respectively. In particular, $\| \cdot \|_\infty$ is the norm on $L^\infty(Q_T)$. We use the notation $\boldsymbol{\mathcal{H}} = (\mathcal{H})^2$. When $\mathcal{H} = L^2(\Omega)$ the subscripts are omitted. Let us introduce the operator A on $H_0 = L^2(\Omega)$ defined as $A = -\Delta$ with domain $H_2 = \mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega)$. For $r = 0, 1, 2$, we introduce the Hilbert spaces $H_r = \mathcal{D}(A^{r/2})$, endowed with the inner products $\langle \phi_1, \phi_2 \rangle_{H_r} = \langle A^{r/2}\phi_1, A^{r/2}\phi_2 \rangle$. Moreover, $H_2 \Subset H_1 \Subset H_0$. Finally, we define the product Hilbert spaces $\mathcal{H}_r = H_{r+2} \times H_r \times H_r$, with $r = 0, 1, 2$. As in [22], introduce the following spaces

$$W_2^{2,1}(Q_T) = \left\{ \zeta \in L^2(Q_T) : \ D_t^r D_x^s \zeta \in L^2(Q_T), \ 2r + s \leq 2 \right\},$$

and

$$\mathcal{W} = \left\{ (u, u_t, \theta) : \ Au, u_t, \theta \in W_2^{2,1}(Q_T) \right\}.$$

Setting $v = u_t$ and $\boldsymbol{z}(t) = \begin{bmatrix} u(t), & v(t), & \theta(t) \end{bmatrix}^\top$, $\boldsymbol{z}_0 = \begin{bmatrix} u_0, & u_1, & \theta_0 \end{bmatrix}^\top \in \mathcal{H}_0$, problem (1) can be rewritten as an abstract evolution equation in the Hilbert space $\mathcal{H}_0$ of the form

$$\begin{cases} \dfrac{d}{dt}\boldsymbol{z}(t) = L\,\boldsymbol{z}(t) + N\,\boldsymbol{z}(t) + B\,f(t) & \text{in } Q_T, \\ \boldsymbol{z}(0) = \boldsymbol{z}_0 & \text{in } \Omega, \\ u(t) = 0, \ Au(t) = 0, \ \theta(t) = 0 & \text{on } \Sigma_T \end{cases} \tag{3}$$

where the linear operator $L : \mathcal{D}(L) \to \mathcal{H}_0$ is defined as

$$L = \begin{bmatrix} 0 & I & 0 \\ -A^2 & 0 & A \\ 0 & -A & -A \end{bmatrix} \tag{4}$$

with domain $\mathcal{D}(L) = \{\boldsymbol{z} \in \mathcal{H}_0 : \ Au, v, \theta \in H_2\}$, whereas operator N denotes

$$N\boldsymbol{z} = \begin{bmatrix} 0, & g_1(u_t), & g_2(\theta) \end{bmatrix}^\top. \tag{5}$$

The control operator $B : H_0 \to \mathcal{H}_0$ is defined as $B\,f = \begin{bmatrix} 0, & 0, & f \end{bmatrix}^\top$.

In our paper we analyse the *controllability to trajectories* at time $T > 0$ for the semilinear thermoelastic system (3). This problem can be formulated in the following way. System (3) is said to be controllable to trajectories at time $T > 0$ if, for any initial data $\boldsymbol{z}_0 \in \mathcal{H}_0$ and any globally defined bounded trajectory $\widehat{\boldsymbol{z}}$ (corresponding to the data $\widehat{\boldsymbol{z}}_0 \in \mathcal{H}_0$ and $\widehat{f} \in L^2(Q_T)$), there exists a control $f \in L^2(Q_T)$ such that the corresponding solution of (3) is also globally defined in $[0, T]$ and satisfies $\boldsymbol{z}(T) = \widehat{\boldsymbol{z}}(T)$ in Ω. When system (3) is linear, this is equivalent to prove that system (3) is *null controllable* at time $T > 0$. That is, if for any initial data $\boldsymbol{z}_0 \in \mathcal{H}_0$, there exists a control function $f \in L^2(Q_T)$ such that, for the corresponding solution $\boldsymbol{z}$ to (3), condition $\boldsymbol{z}(T) = \mathbf{0}$ in Ω holds.

1.2. **Literature.** Many efforts have been devoted to studying the controllability of thermoelastic plates, under varying boundary conditions, and with different choices of control on the boundary or in the control domain.

In $(1)_1$ the plate component does not contain rotational inertia term which, otherwise, would give hyperbolic characteristics. Then, underlying dynamics of the thermoelastic model, the linearized system, associated to (1), is governed by analytic semigroups (cf. [1, 12, 24]). The controllability problem for linear thermoelastic plates has been intensively considered by several authors (see for instance [6, 8, 9, 14, 20, 23, 26] and the references therein). In particular, in [11] null controllability is obtained by one control function (heat source) acting only in an open subset of the domain. For global and local controllability results for semilinear parabolic systems we refer the reader to the following papers [2, 3, 4, 7, 10, 16, 13, 17, 18, 19, 21] where the controllability of semilinear heat equations, phase-field models, reaction-diffusion systems, and thin von Kármán plates is studied.

1.3. **Plan of the paper.** We briefly sketch the plan of the paper. The main results are stated in Section 2 and proved in Section 3 via Kakutani fixed point theorem. Section 4 shows that the linearized system associated to our problem (1) is null controllable at time T.

2. **The main result.** In the sequel we assume for simplicity that $g_i : \mathbb{R} \to \mathbb{R}$, $i = 1, 2$, satisfy

$$g_i \in C^1(\mathbb{R}) \quad \text{and} \quad g_i(0) = 0. \tag{6}$$

Now we are ready to formulate the main result of this paper.

Theorem 1. *Assume that g_i, $i = 1, 2$, verify* (6). *Let $T > 0$ and let $\widehat{\boldsymbol{z}}$ be a globally defined and bounded solution of* (3) *associated with the data $\widehat{\boldsymbol{z}}_0 \in \mathcal{H}_0$ and $\widehat{f} \in L^2(Q_T)$.*

(a) *Controllability to trajectories:*
Let

$$|g_i(s)| \leq |s|^{\alpha_i + 1}, \quad 0 \leq \alpha_i < 1/5, \quad i = 1, 2, \quad \forall s \in \mathbb{R}. \tag{7}$$

Then, for any $\boldsymbol{z}_0 \in \mathcal{H}_0$ with $\boldsymbol{z}_0 - \widehat{\boldsymbol{z}}_0 \in \mathcal{H}_1$, there exists a control function $f \in L^2(Q_T)$ such that $\boldsymbol{z}$, solution of (3) *with $\boldsymbol{z} - \widehat{\boldsymbol{z}} \in L^2(0, T; \mathcal{H}_1) \cap \mathcal{W}$, satisfies*

$$\boldsymbol{z}(T) = \widehat{\boldsymbol{z}}(T), \quad \textit{a.e in } \Omega.$$

(b) *Local controllability to trajectories:*
Let ρ be a positive real number such that for $\boldsymbol{z}_0 - \widehat{\boldsymbol{z}}_0 \in \mathcal{H}_1$ $\|\boldsymbol{z}_0 - \widehat{\boldsymbol{z}}_0\|_{\mathcal{H}_1} \leq \rho$ holds. Then, there exists $f - \widehat{f} \in L^2(Q_T)$ such that $\boldsymbol{z}$, solution of (3) *with $\boldsymbol{z} - \widehat{\boldsymbol{z}} \in L^2(0, T; \mathcal{H}_1) \cap \mathcal{W}$, satisfies*

$$\boldsymbol{z}(T) = \widehat{\boldsymbol{z}}(T), \quad \textit{a.e in } \Omega.$$

3. **Proof of Theorem 1.** For any fixed $\widehat{\boldsymbol{z}}_0 \in \mathcal{H}_0$ and $\widehat{f} \in L^2(Q_T)$ (resp., $\bar{\boldsymbol{z}}_0 \in \mathcal{H}_0$ and $\bar{f} \in L^2(Q_T)$), let $\widehat{\boldsymbol{z}}$ (resp., $\bar{\boldsymbol{z}}$) be the corresponding solution of system (3). Denoting by $\boldsymbol{z} = \bar{\boldsymbol{z}} - \widehat{\boldsymbol{z}}$, we find

$$\begin{cases} u_{tt} + A^2 u - A\theta = g_1(u_t + \widehat{u}_t) - g_1(\widehat{u}_t) & \text{in } Q_T, \\ \theta_t + A\theta + Au_t = g_2(\theta + \widehat{\theta}) - g_2(\widehat{\theta}) + f & \text{in } Q_T, \\ u = 0, \ Au = 0, \ \theta = 0 & \text{on } \Sigma_T, \\ u(0) = u_0, \ u_t(0) = u_1, \ \theta(0) = \theta_0 & \text{in } \Omega, \end{cases}$$

where $f = \bar{f} - \widehat{f}$. For sake of simplicity, in the sequel we assume that $\widehat{\boldsymbol{z}}_0 = \boldsymbol{0}$, $\widehat{f} = 0$, so that $\widehat{\boldsymbol{z}}(t) = \boldsymbol{0}$.

Let us introduce the functions

$$b_i(s) = \begin{cases} \dfrac{g_i(s)}{s} & \text{if } s \neq 0, \\ g_i'(0) & \text{if } s = 0, \end{cases}$$

for $i = 1, 2$. Hence, the linearized version of the previous system can be written as

$$\begin{cases} u_{tt} + A^2 u - A\theta = b_1\, u_t & \text{in } Q_T, \\ \theta_t + A\theta + Au_t = b_2\, \theta + f & \text{in } Q_T, \\ u = 0,\ Au = 0,\ \theta = 0 & \text{on } \Sigma_T, \\ u(0) = u_0,\ u_t(0) = u_1,\ \theta(0) = \theta_0 & \text{in } \Omega, \end{cases} \tag{8}$$

where $b_i \in L^\infty(Q_T)$, $i = 1, 2$.

In the sequel, we will denote by $c > 0$ a generic constant, which may vary even within the same formula, but which is independent of $\boldsymbol{z}_0$, T and b_i, $i = 1, 2$.

In the following lemma we can prove that for each $\boldsymbol{z}_0 \in \mathcal{H}_0$, the null controllability problem associated to (8) has at least one solution $\boldsymbol{z}$.

Lemma 1. *Let $\boldsymbol{z}_0 \in \mathcal{H}_0$. For any $T > 0$, there exists $f \in L^2(Q_T)$ such that the corresponding solution $\boldsymbol{z} = [u, u_t, \theta]^\top$ of (8) satisfies $\boldsymbol{z}(T) = \boldsymbol{0}$, a.e. in Ω, and*

$$\|f\|^2_{L^2(Q_T)} \leq C(T; b_1, b_2) \|\boldsymbol{z}_0\|^2_{\mathcal{H}_0},$$

where

$$C(T; b_1, b_2) = c\, e^{2(\|b_1\|_\infty + \|b_2\|_\infty)T} \Big\{ \Big[1 + \|A^{-1}\|^2_{\mathcal{L}(H_0)} \|b_1\|^2_\infty + \|A^{-1}\|^2_{\mathcal{L}(H_0)} \|b_2\|^2_\infty \Big] T^{-1} + \Big[\|A^{-1}\|^2_{\mathcal{L}(H_0)} + \|A^{-2}\|^2_{\mathcal{L}(H_0)} \|b_2\|^2_\infty \Big] T^{-3} + \|A^{-2}\|^2_{\mathcal{L}(H_0)} T^{-5} \Big\}.$$

provided that $b_i \in L^\infty(Q_T)$, $i = 1, 2$.

Section 4 is devoted to the proof of this Lemma.

For any fixed $R > 0$, we set $\mathcal{K}_R = \big\{ (\nu, \eta) \in \boldsymbol{L}^\infty(Q_T) :\ \|(\nu, \eta)\|_{\boldsymbol{L}^\infty(Q_T)} \leq R \big\}$, which is closed and convex in $\boldsymbol{L}^\infty(Q_T)$. For each $(\nu, \eta) \in \mathcal{K}_R$ we denote by $\Gamma(\nu, \eta) \subset \boldsymbol{L}^2(Q_T)$ the set of the component (u_t, θ) of all solutions $\boldsymbol{z} = [u, u_t, \theta]^\top$ to system (8) with $b_1 = b_1(\nu)$, $b_2 = b_2(\eta)$ such that $\boldsymbol{z}(T) = \boldsymbol{0}$, a.e. in Ω, and $\|f\|^2_{L^2(Q_T)} \leq C(T; b_1, b_2)\, \|\boldsymbol{z}_0\|^2_{\mathcal{H}_0}$ hold. The set is a nonempty closed convex subset of $\boldsymbol{L}^2(Q_T)$. Let us prove that for $R > 0$ sufficiently large, we have that $\Gamma(\mathcal{K}_R) \subset \mathcal{K}_R$. Multiplying the first two equations in (8) by Au_t and $A\theta$ respectively, we find

$$\frac{1}{2} \frac{d}{dt} \|\boldsymbol{z}(t)\|^2_{\mathcal{H}_1} + \|A\theta(t)\|^2 = \langle b_1(t)\, u_t(t), Au_t(t) \rangle + \langle b_2(t)\, \theta(t), A\theta(t) \rangle + \langle f(t), A\theta(t) \rangle.$$

Introducing the following functional

$$\mathcal{K}(t) = \langle \theta(t), Au_t(t) \rangle + \frac{1}{2} \langle u_t(t), A^2 u(t) \rangle,$$

there holds

$$\frac{d}{dt} \mathcal{K}(t) + \frac{1}{2} \|A^2 u(t)\|^2 + \frac{1}{2} \|Au_t(t)\|^2 = \|A\theta\|^2 - \langle A\theta(t), Au_t(t) \rangle + \langle f(t), Au_t(t) \rangle + - \frac{1}{2} \langle A\theta(t), A^2 u(t) \rangle + \langle A\theta(t), b_1(t) u_t(t) \rangle + \frac{1}{2} \langle b_1(t) u_t(t), A^2 u(t) \rangle + \langle b_2(t) \theta(t), Au_t(t) \rangle.$$

For any $\varepsilon > 0$, we consider

$$\begin{aligned}
&\frac{d}{dt}\left[\frac{1}{2}\|\boldsymbol{z}(t)\|^2_{\mathcal{H}_1} + \varepsilon\mathcal{K}(t)\right] + \frac{\varepsilon}{2}\|A^2u(t)\|^2 + \frac{\varepsilon}{2}\|Au_t(t)\|^2 + \|A\theta(t)\|^2 \\
&\quad = \langle b_1(t)\, u_t(t), Au_t(t)\rangle + \langle b_2(t)\, \theta(t), A\theta(t)\rangle + \langle f(t), A\theta(t)\rangle + \varepsilon\|A\theta\|^2 + \\
&\quad - \varepsilon\langle A\theta(t), Au_t(t)\rangle + \varepsilon\langle b_2(t)\theta(t), Au_t(t)\rangle + \varepsilon\langle f(t), Au_t(t)\rangle + \\
&\quad - \frac{\varepsilon}{2}\langle A\theta(t), A^2u(t)\rangle + \varepsilon\langle A\theta(t), b_1(t)u_t(t)\rangle + \frac{\varepsilon}{2}\langle b_1(t)u_t(t), A^2u(t)\rangle.
\end{aligned}$$

By Young and Cauchy inequalities we have

$$\begin{aligned}
&\frac{d}{dt}\left[\frac{1}{2}\|\boldsymbol{z}(t)\|^2_{\mathcal{H}_1} + \varepsilon\mathcal{K}(t)\right] + \frac{\varepsilon}{8}\|A^2u(t)\|^2 + \frac{\varepsilon}{8}\|Au_t(t)\|^2 + (1-7\varepsilon)\,\|A\theta(t)\|^2 \leq \\
&\quad + \left(\frac{7}{12}\varepsilon + \frac{8}{3\varepsilon}\right)\|b_1(t)\|^2_{L^\infty(\Omega)}\|u_t(t)\|^2 + \left(\frac{8}{3}\varepsilon + \frac{1}{4\varepsilon}\right)\|b_2(t)\|^2_{L^\infty(\Omega)}\|\theta(t)\|^2 \\
&\quad + \left(\frac{8}{3}\varepsilon + \frac{1}{4\varepsilon}\right)\|f(t)\|^2. \qquad (9)
\end{aligned}$$

Notice that for any $0 < \varepsilon < 2/\sqrt{5}$ there exists a positive constant c such that $c\|\boldsymbol{z}(t)\|^2_{\mathcal{H}_1} \leq \|\boldsymbol{z}(t)\|^2_{\mathcal{H}_1} + 2\varepsilon\mathcal{K}(t)$. Moreover there holds $\|\boldsymbol{z}(t)\|^2_{\mathcal{H}_1} + 2\varepsilon\mathcal{K}(t) \leq (1 + 3\varepsilon/2)\|\boldsymbol{z}(t)\|^2_{\mathcal{H}_1}$. Integrating (9) with respect to $[0,t] \subseteq [0,T]$, for $0 < \varepsilon < 8/57$ we get

$$\begin{aligned}
&\frac{c}{2}\|\boldsymbol{z}(t)\|^2_{\mathcal{H}_1} + \frac{\varepsilon}{8}\int_0^t \left[\|A^2u(\tau)\|^2 + \|Au_t(\tau)\|^2 + \|A\theta(\tau)\|^2\right] d\tau \leq \\
&\qquad \left(1 + \frac{3}{2}\varepsilon\right)\frac{1}{2}\|\boldsymbol{z}_0\|^2_{\mathcal{H}_1} + \left(\frac{8}{3}\varepsilon + \frac{1}{4\varepsilon}\right)\int_0^T \|f(\tau)\|^2\, d\tau \\
&\qquad + c\left[\left(\frac{7}{12}\varepsilon + \frac{8}{3\varepsilon}\right)\|b_1\|^2_\infty + \left(\frac{8}{3}\varepsilon + \frac{1}{4\varepsilon}\right)\|b_2\|^2_\infty\right]\int_0^t \|\boldsymbol{z}(\tau)\|^2_{\mathcal{H}_1}\, d\tau. \qquad (10)
\end{aligned}$$

By Gronwall lemma, for any $t \in [0,T]$, we find

$$\|\boldsymbol{z}(t)\|^2_{\mathcal{H}_1} \leq \left[c\|\boldsymbol{z}_0\|^2_{\mathcal{H}_1} + 2\left(\frac{8}{3}\varepsilon + \frac{1}{4\varepsilon}\right)\int_0^T \|f(\tau)\|^2\, d\tau\right] e^{c\, b_\infty\, t}, \qquad (11)$$

where $b_\infty = \left(\frac{7}{12}\varepsilon + \frac{8}{3\varepsilon}\right)\|b_1\|^2_\infty + \left(\frac{8}{3}\varepsilon + \frac{1}{4\varepsilon}\right)\|b_2\|^2_\infty$. From (10)-(11), we obtain

$$\int_0^T \|A\theta(\tau)\|^2\, d\tau \leq c\left[\|\boldsymbol{z}_0\|^2_{\mathcal{H}_1} + \left(\frac{8}{3}\varepsilon + \frac{1}{4\varepsilon}\right)\int_0^T \|f(\tau)\|^2\, d\tau\right] e^{c\, b_\infty\, T}.$$

This implies that $A\theta \in L^2(Q_T)$. Analogously from (10)-(11), we have that $Au_t \in L^2(Q_T)$. Then, from $(8)_2$ we find $\theta_t \in L^2(Q_T)$. Recalling that, since $\Omega \subset \mathbb{R}^2$, the following embedding [22] $W_2^{2,1}(Q_T) \subset L^\infty(Q_T)$ is verified, we obtain $\theta \in L^\infty(Q_T)$. Analogously, since $Au_t \in L^2(Q_T)$ and $u_{tt} \in L^2(Q_T)$, we get $u_t \in L^\infty(Q_T)$. From Lemma 1 the inequality

$$\|u_t\|^2_{L^\infty(Q_T)} + \|\theta\|^2_{L^\infty(Q_T)} \leq c\,[1 + C(T; b_1, b_2)]\, e^{c\, b_\infty\, T}\|\boldsymbol{z}_0\|^2_{\mathcal{H}_1},$$

holds. From (7) it follows that $|b_i(s)| \leq |s|^{\alpha_i}$, $i = 1,2$. Then, setting $\alpha = \max\{\alpha_1, \alpha_2\}$, we have

$$[1 + C(T; b_1, b_2)]\, e^{c\, b_\infty\, T} \leq c\, e^{cR^{2\alpha}T + cR^{\alpha}T}\left[\left(1 + c\, R^{2\alpha}\right)T^{-1} + \left(1 + R^{2\alpha}\right)T^{-3} + T^{-5}\right].$$

Choosing $T = R^{-2\alpha}$, for R sufficiently large we require that $R^{10\alpha} \le c\, R^2$, and this holds for $0 \le \alpha < 1/5$. Hence the size order of the non linear functions must be smaller than $6/5$. Then $\Gamma(K_R) \subset K_R$ provided (7) with $\alpha_i \le 1/5$, $i = 1, 2$.

For any $(\nu, \eta) \in K_R$, $\Gamma(\nu, \eta)$ is a closed and convex subset of $\boldsymbol{L}^2(Q_T)$; $\Gamma(\nu, \eta)$ is relatively compact in $\boldsymbol{L}^2(Q_T)$; Γ is upper semi-continuous in $\boldsymbol{L}^2(Q_T)$. Applying Kakutani fixed-point theorem (cf. [5]) in the space $L^2(Q_T)$, we find that there is at least one $(\nu, \eta) \in K_R$ such that $(\nu, \eta) \in \Gamma(\nu, \eta)$.

The second claim of Theorem 1 is found by choosing $\boldsymbol{z}_0$ such that

$$\|\boldsymbol{z}_0\|^2_{\mathcal{H}_1} \le \frac{R}{c\,[1 + C(T; b_1, b_2)]} e^{-c\, b_\infty\, T}.$$

4. **Proof of Lemma 1.** Problem (8) can be rewritten as an abstract evolution equation in $\mathcal{H}_0$ of the form

$$\begin{cases} \boldsymbol{z}_t = L\,\boldsymbol{z} + \widetilde{N}\,\boldsymbol{z} + B\,f & \text{in } Q_T, \\ \boldsymbol{z}(0) = \boldsymbol{z}_0 & \text{in } \Omega, \\ z_1 = 0,\ Az_1 = 0,\ z_3 = 0 & \text{on } \Sigma_T, \end{cases} \tag{12}$$

where operator $\widetilde{N}$ is defined as $\widetilde{N}\,\boldsymbol{z} = \begin{bmatrix} 0 & b_1\, u_t & b_2\, \theta \end{bmatrix}^\top$. Given $\boldsymbol{\zeta}^T = [\zeta_1^T,\ \zeta_2^T,\ \zeta_3^T]^\top \in \mathcal{H}_0$, we denote by $\boldsymbol{\zeta}(t) = [\zeta_1(t),\ \zeta_2(t),\ \zeta_3(t)]^\top$ the solution of the following adjoint system with respect to (12):

$$\begin{cases} \boldsymbol{\zeta}_t = -L^*\,\boldsymbol{\zeta} - \widetilde{N}^*\,\boldsymbol{\zeta} & \text{in } Q_T, \\ \boldsymbol{\zeta}(T) = \boldsymbol{\zeta}^T & \text{in } \Omega, \\ \zeta_1 = 0,\ A\zeta_1 = 0,\ \zeta_3 = 0 & \text{on } \Sigma_T, \end{cases} \tag{13}$$

where

$$L^* = \begin{bmatrix} 0 & -I & 0 \\ A^2 & 0 & -A \\ 0 & A & -A \end{bmatrix} \quad \text{and} \quad \widetilde{N}^*\,\boldsymbol{\zeta} = \begin{bmatrix} 0 & b_1\,\zeta_2 & b_2\,\zeta_3 \end{bmatrix}^\top.$$

By simple calculation $D(L^*) = D(L)$ holds. Moreover, putting $\boldsymbol{\xi}(x,t) = \boldsymbol{\zeta}(x, T-t)$, system (13) is equivalent to find $\boldsymbol{\xi} = [\xi_1,\ \xi_2,\ \xi_3]^\top = [\varphi,\ \varphi_t,\ w]^\top \in D(L)$:

$$\begin{cases} \varphi_{tt} + A^2\varphi - Aw = b_1\,\varphi_t & \text{in } Q_T \\ w_t + Aw + A\varphi_t = b_2\,w & \text{in } Q_T \\ \varphi = 0,\ A\varphi = 0,\ w = 0 & \text{on } \Sigma_T \\ \varphi(0) = \zeta_1^T,\ \varphi_t(0) = -\zeta_2^T,\ w(0) = \zeta_3^T & \text{in } \Omega. \end{cases} \tag{14}$$

For any solution of (14), we introduce the energy function

$$\mathcal{E}(t) = \frac{1}{2}\left[\|A\varphi(t)\|^2 + \|\varphi_t(t)\|^2 + \|w(t)\|^2\right]. \tag{15}$$

We can show now the following inequality.

Lemma 2. *For any $T > 0$ and any solution of* (14)*, there holds*

$$e^{-2(\|b_1\|_\infty + \|b_2\|_\infty)T}\mathcal{E}(T) \le \mathcal{E}(t), \quad t \in [0,T], \quad \textit{provided } b_i \in L^\infty(Q_T),\ i = 1, 2.$$

Proof. Introducing $\phi(t) = e^{-(\|b_1\|_\infty + \|b_2\|_\infty)t}\,\varphi_t(t)$ and $\psi(t) = e^{-(\|b_1\|_\infty + \|b_2\|_\infty)t}\,w(t)$, by standard calculation our conclusion follows. $\square$

Let $\big[\varphi(t), \quad \varphi_t(t), \quad w(t)\big]^\top$ be a solution of system (14) corresponding to an initial data $\big[\varphi_0, \quad \varphi_1, \quad w_0\big]^\top \in D(L^*) = D(L)$. Denoting by $h(t) = t^4(T-t)^4$, $t \in (0,T)$, we introduce the following function

$$\mathcal{G}(T) = \int_0^T h(t)\,\mathcal{E}(t)\,dt, \tag{16}$$

and we set

$$\mathcal{F}(t) = \frac{1}{2}\langle \varphi_t(t), h(t)\,\varphi(t)\rangle + \delta\langle A^{-1}\varphi_t(t), h(t)\,w(t)\rangle + \frac{1}{2}\langle w(t), h'(t)\,A^{-1}\varphi(t)\rangle, \tag{17}$$

where δ is a positive constant that will be chosen later.

Remark 1. The main technical ingredient for the proof of Lemma 1 is the introduction of the multiplier $A^{-1}\varphi_t(t)$ in the definition of the functional $\mathcal{F}(t)$. This multiplier has been proven in the past to be intrinsic to obtaining stability and controllability properties of linear thermoelastic systems (see [1, 6, 11] and references therein).

Proof of Lemma 1. Recalling (1) and (15)-(17), we find

$$\begin{aligned}
\frac{d}{dt}\mathcal{F}(t) &= -h(t)\,\mathcal{E}(t) + h(t)\langle\left(\frac{1}{2}-\delta\right)A\varphi(t) - \delta\,\varphi_t(t), w(t)\rangle + \frac{1}{2}\,h(t)\langle b_1(t)\,\varphi_t, \varphi\rangle \\
&+ \frac{1}{2}\,h'(t)\langle(1+2\delta)A^{-1}\varphi_t(t) - \varphi(t), w(t)\rangle + \frac{1}{2}h''(t)\langle A^{-1}\varphi(t), w(t)\rangle \\
&+ \left(\delta + \frac{1}{2}\right)h(t)\|w(t)\|^2 - (\delta - 1)\,h(t)\|\varphi_t(t)\|^2 + \delta\,h(t)\langle b_1(t)\,\varphi_t(t), A^{-1}w(t)\rangle \\
&+ \frac{1}{2}\,h'(t)\langle A^{-1}\varphi(t), b_2(t)\,w(t)\rangle + \delta\,h(t)\langle A^{-1}\varphi_t(t), b_2(t)\,w(t)\rangle.
\end{aligned} \tag{18}$$

By an integration on $[0,T]$, we find

$$\begin{aligned}
\frac{1}{2}\,\mathcal{G}(T) &\le \underbrace{\int_0^T h(t)\langle\left(\frac{1}{2}-\delta\right)A\varphi(t) - \delta\,\varphi_t(t), w(t)\rangle dt}_{=I_1} \\
&+ \underbrace{\frac{1}{2}\int_0^T h'(t)\langle(1+2\delta)A^{-1}\varphi_t(t) - \varphi(t), w(t)\rangle dt}_{=I_2} + \underbrace{\frac{1}{2}\int_0^T h''(t)\langle A^{-1}\varphi(t), w(t)\rangle dt}_{=I_3} \\
&+ \left(\delta + \frac{1}{2}\right)\int_0^T h(t)\|w(t)\|^2 dt - \big(\delta - 1 - \|b_1\|_\infty^2\big)\int_0^T h(t)\|\varphi_t(t)\|^2 dt \\
&+ \underbrace{\frac{1}{2}\int_0^T h'(t)\langle A^{-1}\varphi(t), b_2(t)\,w(t)\rangle dt}_{=I_4} + \underbrace{\delta\int_0^T h(t)\langle A^{-1}\varphi_t(t), b_2(t)\,w(t)\rangle dt}_{=I_5} \\
&+ \underbrace{\delta\int_0^T h(t)\langle b_1(t)\,\varphi_t(t), A^{-1}w(t)\rangle dt}_{=I_6}.
\end{aligned} \tag{19}$$

By Young inequality and recalling that $\max_{t\in(0,T)} h(t) = \dfrac{T^8}{256}$, we obtain

$$I_1 \le \varepsilon\,\mathcal{G}(T) + \frac{5}{8\varepsilon}\int_0^T h(t)\|w(t)\|^2 dt \le \varepsilon\,\mathcal{G}(T) + \frac{5}{8\varepsilon}\,\frac{T^8}{256}\int_0^T \|w(t)\|^2 dt$$

for any $\varepsilon > 0$. Considering the second term, we obtain

$$\begin{aligned} I_2 &\le \frac{1}{2}\int_0^T 3\,h'(t)\|A^{-1}\varphi_t(t)\|\|w(t)\|\,dt + \frac{1}{2}\int_0^T h'(t)\|\varphi(t)\|\|w(t)\|\,dt \\ &\le \varepsilon\,\mathcal{G}(T) + \frac{5}{4\varepsilon}\|A^{-1}\|^2_{\mathcal{L}(H_0)}\int_0^T \frac{[h'(t)]^2}{h(t)}\,\|w(t)\|^2 dt. \end{aligned}$$

Since $\dfrac{[h'(t)]^2}{h(t)} = 16\,t^2(T-t)^2(T-2t)^2 \le \dfrac{4}{27}\,T^6$, $t \in (0,T)$, the previous inequality becomes

$$I_2 \le \varepsilon\,\mathcal{G}(T) + \frac{c}{\varepsilon}\|A^{-1}\|^2_{\mathcal{L}(H_0)}\,T^6\int_0^T \|w(t)\|^2 dt,$$

for any $\varepsilon > 0$. The third term can be estimate as

$$I_3 \le \varepsilon\,\mathcal{G}(T) + \frac{1}{8\varepsilon}\,\|A^{-2}\|^2_{\mathcal{L}(H_0)}\int_0^T \frac{[h''(t)]^2}{h(t)}\|w(t)\|^2 dt$$

for any $\varepsilon > 0$. Since $\dfrac{[h''(t)]^2}{h(t)} = 16(14\,t^2 - 14\,T\,t + 3T^2)^2 \le 144\,T^4$, $t \in (0,T)$, we have

$$I_3 \le \varepsilon\,\mathcal{G}(T) + \frac{c}{\varepsilon}\,\|A^{-2}\|^2_{\mathcal{L}(H_0)}\,T^4\int_0^T \|w(t)\|^2 dt.$$

The others terms are estimated as

$$\begin{aligned} I_4 &\le \varepsilon\,\mathcal{G}(T) + \frac{c}{\varepsilon}\,\|A^{-2}\|^2_{\mathcal{L}(H_0)}\|b_2\|^2_\infty\,T^6\int_0^T \|w(t)\|^2 dt, \\ I_5 &\le \varepsilon\,\mathcal{G}(T) + \frac{c}{\varepsilon}\,\|A^{-1}\|^2_{\mathcal{L}(H_0)}\,\|b_2\|^2_\infty\,T^8\int_0^T \|w(t)\|^2 dt, \\ I_6 &\le \varepsilon\,\mathcal{G}(T) + \frac{c}{\varepsilon}\,\|A^{-1}\|^2_{\mathcal{L}(H_0)}\|b_1\|^2_\infty\,T^8\int_0^T \|w(t)\|^2 dt. \end{aligned}$$

Choosing $\delta > 1 + \|b_1\|^2_\infty$, and substituting the previous estimates into (19), for any $T > 0$, we have

$$\begin{aligned} (1-6\,\varepsilon)\mathcal{G}(T) \le \frac{c}{\varepsilon}\Big\{&\Big[1 + \|A^{-1}\|^2_{\mathcal{L}(H_0)}\|b_1\|^2_\infty + \|A^{-1}\|^2_{\mathcal{L}(H_0)}\|b_2\|^2_\infty\Big]\,T^8 \\ &+ \Big[\|A^{-1}\|^2_{\mathcal{L}(H_0)} + \|A^{-2}\|^2_{\mathcal{L}(H_0)}\|b_2\|^2_\infty\Big]\,T^6 + \|A^{-2}\|^2_{\mathcal{L}(H_0)}T^4\Big\}\int_0^T \|w(t)\|^2 dt. \end{aligned}$$

Recalling that $\displaystyle\int_0^T h(t)\,dt = \frac{T^9}{630}$, choosing $0 < \varepsilon < 1/6$ and from Lemma 2, there holds

$$\mathcal{E}(T) \le C(T; b_1, b_2)\int_0^T \|w(t)\|^2 dt, \qquad (20)$$

where

$$C(T;b_1,b_2) = \frac{c}{\varepsilon}\, e^{2(\|b_1\|_\infty+\|b_2\|_\infty)T} \Big\{ \Big[1 + \|A^{-1}\|^2_{\mathcal{L}(H_0)}\|b_1\|^2_\infty + \|A^{-1}\|^2_{\mathcal{L}(H_0)}\|b_2\|^2_\infty\Big] T^{-1} + \Big[\|A^{-1}\|^2_{\mathcal{L}(H_0)} + \|A^{-2}\|^2_{\mathcal{L}(H_0)}\|b_2\|^2_\infty\Big] T^{-3} + \|A^{-2}\|^2_{\mathcal{L}(H_0)} T^{-5}\Big\}.$$

We introduce the following functional

$$J_k(f) = \frac{1}{2}\,\|f\|^2_{L^2(Q_T)} + \frac{k}{2}\|\boldsymbol{z}(T;\boldsymbol{z}_0,f)\|^2_{\mathcal{H}_0},$$

where $\boldsymbol{z}(T;\boldsymbol{z}_0,f)$ is the associated solution of (12). We consider also the dual functional (see [3, 15, 20])

$$J_k^*(\boldsymbol{\zeta}^T) = \frac{1}{2}\,\|\zeta_3\|^2_{L^2(Q_T)} + \langle \boldsymbol{\zeta}(0), \boldsymbol{z}_0\rangle_{\mathcal{H}_0} + \frac{1}{2k}\,\|\boldsymbol{\zeta}^T\|^2_{\mathcal{H}_0}.$$

The minimization problems $\min_{f\in L^2(Q_T)} J_k(f)$ and $\min_{\ T\in\mathcal{H}_0} J_k^*(\boldsymbol{\zeta}^T)$ have both one solution $(f_k, \boldsymbol{\zeta}_k^T)$ and, by application of the duality theorem of Fenchel-Rockafeller [15, 25], we obtain (see [20]) $f_k = \zeta_{k3}$ and $\boldsymbol{\zeta}_k^T = -k\,\boldsymbol{z}(T;\boldsymbol{z}_0,f_k)$. By setting $\boldsymbol{z}_k(T) = \boldsymbol{z}(T;\boldsymbol{z}_0,f_k)$, we find $\|\boldsymbol{z}_k(T)\|_{\mathcal{H}_0} = \dfrac{1}{k}\|\boldsymbol{\zeta}_k^T\|_{\mathcal{H}_0}$. By considering (13) with $\boldsymbol{\zeta}^T \equiv \mathbf{0}$, we find that $\boldsymbol{\zeta}(t) \equiv \mathbf{0}$, for any $t \in [0,T]$. This implies that the dual functional J_k^* evaluated for $\boldsymbol{\zeta}^T \equiv \mathbf{0}$ becomes $J_k^*(\mathbf{0}) = 0$, and there holds $J_k^*(\boldsymbol{\zeta}_k^T) \le J_k^*(\mathbf{0}) = 0$ for any $k > 0$. Then, we have $\|\zeta_{k3}\|^2_{L^2(Q_T)} \le 2\|\boldsymbol{\zeta}_k(0)\|_{\mathcal{H}_0}\,\|\boldsymbol{z}_0\|_{\mathcal{H}_0}$. Recalling (20), for any solution of system (14), there holds

$$\|\boldsymbol{\zeta}_k(0)\|^2_{\mathcal{H}_0} \quad \le \quad C(T;b_1,b_2)\|\zeta_{k3}\|^2_{L^2(Q_T)}.$$

Hence, we find $\|f_k\|_{L^2(Q_T)} = \|\zeta_{k3}\|_{L^2(Q_T)} \le 2\,C(T;b_1,b_2)^{1/2}\,\|\boldsymbol{z}_0\|_{\mathcal{H}_0}$. By this inequality, it follows, at least for a subsequence, that, for $\varepsilon \to 0$, $\zeta_{3k_n} \rightharpoonup f$ in $L^2(Q_T)$ and $\boldsymbol{z}_{k_{n_m}} \rightharpoonup \boldsymbol{z}$ in $\mathcal{H}_0$, a.e. $t \in (0,T)$. Then, f satisfies the same previous estimate and it is chosen as the control function. Recalling that $\|\boldsymbol{z}(T)\|_{\mathcal{H}_0} \le \liminf_{k\to\infty} \|\boldsymbol{z}_k(T)\|_{\mathcal{H}_0} = 0$, we find $\boldsymbol{z}(T) = \mathbf{0}$. □

REFERENCES

[1] F. Ammar Khodja and A. Benabdallah, *Sufficient conditions for uniform stabilization of second order equations by dynamical controllers*, Dynam. Contin. Discrete Impuls. Systems, **7** (2000), no. 2, 207–222.

[2] F. Ammar Khodja, A. Benabdallah, and C. Dupaix, *Null-controllability of some reaction-diffusion systems with one control force*, preprint, 2004.

[3] F. Ammar Khodja, A. Benabdallah, C. Dupaix, and I. Kostin, *Controllability to the trajectories of phase-field models by one control force*, SIAM J. Control Optim., **42** (2004), no. 5, 1661–1680.

[4] S. Anita and V. Barbu, *Null controllability of nonlinear covective heat equation*, ESAIM, Control Optim. Calculus Variations, **5** (2000), 157–173.

[5] J.-P. Aubin and A. Cellina, *Differential inclusions*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 264, Springer-Verlag, Berlin, 1984.

[6] G. Avalos, *Exact controllability of a thermoelastic system with control in the thermal component only*, Differential Integral Equations, **13** (2000), no. 4-6, 613–630.

[7] G. Avalos, *Concerning the exact null controllability of a nonlinear thermoelastic plate*, preprint, 2004.

[8] G. Avalos and I. Lasiecka, *Mechanical and thermal null controllability of thermoelastic plates and singularity of the associated minimal energy function*, Control Cybernet., (2003), no. 32, 473–490.

[9] G. Avalos and I. Lasiecka *The null controllability of thermoelastic plates and singularity of the associated minimal energy function*, J. Math. Anal. Appl., (2004), no. 1, 34–61.

[10] V. Barbu, *Exact controllability of the superlinear heat equation*, Appl. Math. Optim., **42** (2000), 73–89.

[11] A. Benabdallah and M.G. Naso, *Null controllability of a thermoelastic plate*, Abstr. Appl. Anal., **7** (2002), no. 11, 585–599.

[12] S.K. Chang and R. Triggiani, *Spectral analysis of thermo-eslatic plates with rotational forces*, Optimal control (Gainesville, FL, 1997), Kluwer Acad. Publ., 1998, pp. 84–115.

[13] L. de Teresa, *Insensitizing controls for a semilinear heat equation*, Com. P.D.E., **25** (2000), no. 1–2, 39–72.

[14] L. de Teresa and E. Zuazua, *Controllability of the linear system of thermoelastic plates*, Adv. Differential Equations, **1** (1996), no. 3, 369–402.

[15] I. Ekeland and R. Temam, Analyse convexe et problèmes variationnels, Dunod, 1974.

[16] C. Fabre, J.-P. Puel, and E. Zuazua, *Approximate controllability of the semilinear heat equation*, Proc. Roy. Soc. Edinburgh Sect. A, **125** (1995), no. 1, 31–61.

[17] E. Fernández-Cara, *Null controllability of the semilinear heat equation*, ESAIM Control Optim. Calc. Var., **2** (1997), 87–103.

[18] E. Fernández-Cara and E. Zuazua, *Null and approximate controllability for weakly blowing up semilinear heat equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire, **17** (2000), no. 5, 583–616.

[19] A. V. Fursikov and O. Yu. Imanuvilov, *Controllability of evolution equations*, Seoul National University Research Institute of Mathematics Global Analysis Research Center, Seoul, 1996.

[20] P. Gervasio and M.G. Naso, *Numerical approximation of controllability of trajectories for Euler-Bernoulli thermoelastic plates*, Math. Models Methods Appl. Sci., **14** (2004), no. 5, 701–733.

[21] O.Y. Imanuvilov and M. Yamamoto, *Carleman inequalities for parabolic equations in Sobolev spaces of negative order and exact controllability for semilinear parabolic equations*, Publ. Res. Inst. Math. Sci., **39** (2003), no. 2, 227–274.

[22] O. A. Ladyženskaja, V. A. Solonnikov, and N. N. Ural′ceva, *Linear and quasilinear equations of parabolic type*, Translated from the Russian by S. Smith. Translations of Mathematical Monographs, Vol. 23, American Mathematical Society, Providence, R.I., 1967.

[23] I. Lasiecka and R. Triggiani, *Exact null controllability of structurally damped and thermoelastic parabolic models*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl., **9** (1998), no. 1, 43–69.

[24] I. Lasiecka and R. Triggiani, *Two direct proofs on the analyticity of the s.c. semigroup arising in abstract thermo-elastic equations*, Adv. Differential Equations, **3** (1998), no. 3, 387–416.

[25] R. T. Rockafellar, *Extension of Fenchel's duality theorem for convex functions*, Duke Math. J., **33** (1966), 81–89.

[26] R. Triggiani, *Optimal estimates of norms of fast controls in exact null controllability of two non-classical abstract parabolic systems*, Adv. Differential Equations, **8** (2003), no. 2, 189–229.

Received September, 2004; revised March 2005.

E-mail address: naso@ing.unibs.it