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GLOBAL EXACT CONTROLLABILITY OF SEMILINEAR WAVE EQUATIONS BY A DOUBLE COMPACTNESS/UNIQUENESS ARGUMENT

Irena Lasiecka and Roberto Triggiani

Department of Mathematics
University of Virginia
Charlottesville, VA 22904 USA

Abstract. We prove exact controllability in the energy space of semilinear wave equations with L_2-Neumann boundary controls. The present proof integrates a double compactness/uniqueness PDE-based argument in establishing the uniform continuous observability inequality of the linearized, dual, uncontrolled problem with the abstract operator-theoretic approach proposed in [11], [21]. The latter approach analyzes suitable families of collectively compact operators [1] and ultimately culminates with the application of a global inversion theorem (homeomorphism) [4], [17] to the original controlled semilinear problem.

1. **Controlled Dynamics: Wave Equation with Neumann Control.** Let Ω be an open bounded domain of $\mathbb{R}^n$, $n \geq 2$, with sufficiently smooth boundary $\Gamma = \overline{\Gamma_0 \cup \Gamma_1}$, $\Gamma_0 \cap \Gamma_1 = \emptyset$, $\Gamma_0 \neq \emptyset$. In Ω we consider the following semilinear problem for the wave equation in the solution $w(t,x)$, with Neumann-boundary control u based on Σ_1:

$$\begin{cases} w_{tt} = \Delta w + f(w) & \text{in } (0,T] \times \Omega \equiv Q; & (1.1a) \\ w(0,\cdot) = w_0, w_t(0,\cdot) = w_1 & \text{in } \Omega; & (1.1b) \\ w|_{\Sigma_0} \equiv 0; \ \left.\dfrac{\partial w}{\partial \nu}\right|_{\Sigma_1} = u & \text{in } (0,T] \times \Gamma_i \equiv \Sigma_i, \ i = 0,1. & (1.1c) \end{cases}$$

Here $\nu(x)$ is the unit outward normal on $x \in \Gamma$. The standing *assumption* on the non-linearity f is a global Lipschitz condition as follows:

$$\begin{gathered} f : \mathbb{R} \to \mathbb{R} \text{ is an absolutely continuous function with first derivative} \\ \text{a.e. which is uniformly bounded on } \mathbb{R}: \\ |f'(r)| \leq \text{const.} \quad \text{a.e. } r \in \mathbb{R}. \end{gathered} \tag{1.2}$$

Remark 1.1. For sake of definiteness (and lack of space), this note is written with reference to a specific problem: the exact controllability on the energy state space $H \equiv H^1_{\Gamma_0}(\Omega) \times L_2(\Omega)$ of the semilinear wave equation (1.1a–c) within the class of $L_2(0,T; L_2(\Gamma_1))$-Neumann boundary controls u. However, the ideas and approach pursued here apply also to other models: for instance, semilinear wave equations in different state spaces with different classes of Dirichlet boundary controls as in [11]. See Final Remarks.

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Exact controllability. Under some geometrical conditions and for all $T >$ some universal $T_0 > 0$, the linear problem (1.1) with $f \equiv 0$ is exactly controllable over $[0,T]$, on the state space $H^1_{\Gamma_0}(\Omega) \times L_2(\Omega)$, within the class of controls $u \in L_2(0,T;L_2(\Gamma_1))$ [3], [Lag.1–2], [13], [9], [15], [Ta.1–2], etc. One could take $T_0 = 2\sup_{x\in\Omega} \operatorname{dist}(\Gamma_1, x)$. The result below extends the same (global) exact controllability property to the semilinear problem (1.1) subject to (1.2), over the same time interval.

Theorem 1.1. *Let $T > 0$ be a time for which exact controllability of the linear problem with $f \equiv 0$ holds true in the state space $H^1_{\Gamma_0}(\Omega) \times L_2(\Omega)$ within the class of $L_2(0,T;L_2(\Gamma_1))$-controls u. Let f satisfy assumption (1.2). Then, a similar exact controllability result holds true for the original problem (1.1) for the same time T: for any pair $\{w_0, w_1\} \in H^1_{\Gamma_0}(\Omega) \times L_2(\Omega)$ of initial conditions, and any pair $\{\tilde{w}_0, \tilde{w}_1\} \in H^1_{\Gamma_0}(\Omega) \times L_2(\Omega)$ of target states, there exists a suitable control function $u \in L_2(0,T;L_2(\Gamma_1))$ such that the corresponding solution $\{w, w_t\}$ of problem (1.1) satisfies $\{w(T,\cdot), w_t(T,\cdot)\} = \{\tilde{w}_0, \tilde{w}_1\}$.*

Theorem 1.1 is proved in [21], to which we refer for an overview of available relevant literature. Paper [21] follows, at least in its Part I, the path of the abstract approach proposed in [11] [revisited, however, as in the present case the input $\to$ solution operator $\mathcal{L}_T$ of the correspondent linear problem ($f \equiv 0$) is *unbounded* [10] from the space $L_2(\Sigma_1)$ of controls to the state space $H \equiv H^1_{\Gamma_0}(\Omega) \times L_2(\Omega)$, *unlike the treatment in* [11]]. To this end, [21] verifies that three key structural properties and two critical control-theoretic properties are satisfied in the present setting of problem (1.1). The proof of the exact controllability Theorem 1.1 given in [21, Part I] is based on the global inversion approach (homeomorphism) [4], [17] applied to the *original, controlled* problem (1.1), as proposed in [11]. This way the control u which performs the transfer over the interval $[0,T]$ from the initial condition in H to the target state in H along the semilinear dynamics (1.1) is explicitly linked to the control v which performs the very same transfer, this time, however, along the corresponding linear problem ($f \equiv 0$).

In its Part II, [21] provides an alternative proof, which is based, instead, on establishing a uniform continuous observability inequality (UCOI), this time, however, for a corresponding linearized homogeneous *uncontrolled, dual* problem:

$$\begin{cases} \phi_{tt} = \Delta\phi + q(t,x)\phi & \text{in } (0,T]\times\Omega \equiv Q; & (1.3a)\\ \phi(T,\cdot) = \phi_0, \phi_t(T,\cdot) = \phi_1 & \text{in } \Omega; & (1.3b)\\ \phi|_{\Sigma_0} \equiv 0;\ \left.\dfrac{\partial\phi}{\partial\nu}\right|_{\Sigma_1} = 0 & \text{in } (0,T]\times\Gamma_i \equiv \Sigma_i. & (1.3c)\end{cases}$$

This alternative route is given by

Theorem 1.2. *Let the potential $q \in B(0,r)$, a ball in $L_\infty(Q)$ centered at the origin of fixed radius $r > 0$. Assume the Uniform COI on $[0,T]$: there exists a constant $C_{Tr} > 0$, independent of the I.C. and of the potential $q \in B(0,r)$, but dependent only on $T > 0$ and $r > 0$, such that the following inequality for problem (1.3) holds true,*

$$\|\{\phi_0,\phi_1\}\|^2_Y \le C_{Tr}\int_0^T\int_{\Gamma_1}\phi^2 d\Sigma_1, \quad Y = L_2(\Omega)\times[\mathcal{D}(\mathcal{A}^{\frac12})]'; \tag{1.4}$$

$$\mathcal{D}(\mathcal{A}^{\frac12}) \equiv H^1_{\Gamma_0}(\Omega),\ [\mathcal{D}(\mathcal{A}^{\frac12})]' \equiv [H^1_{\Gamma_0}(\Omega)]',\ \mathcal{A} = -\Delta \text{ with}$$

$$\mathcal{D}(\mathcal{A}) = \left\{ h \in H^2(\Omega) : \ h|_{\Gamma_0} = \frac{\partial h}{\partial \nu}|_{\Gamma_1} = 0 \right\}.$$

Then, the global inversion (homeomorphism) approach of [21], [11] *is applicable and thus problem (1.1) is exactly controllable on the state space* $H \equiv H^1_{\Gamma_0}(\Omega) \times L_2(\Omega)$, *within the* $L_2(\Sigma_1)$*-class of Neumann controls. The control* u *doing the transfer for the semilinear problem is explicitly linked to the control* v *doing the same transfer in the corresponding linear case* ($f \equiv 0$).

The proof of Theorem 1.2 as given in [21, Proposition 5.1] shows, in fact, that the Uniform COI (1.4) is *equivalent* to the uniform inversion of the global inversion approach of [11] provided that the operator $[I+\Lambda'_T(q)]$ in [21, Section 4], [11, Section 2] is *injective*, for each $q \in L_\infty(Q)$ [21, below Eqn. (5.9)]. The operator Λ'_T is the Frechet derivative of the operator Λ_T defined in [21, Eqn. (4.17)] in terms of the data of the problem. In our present case, as well as in the case of [11], this is true *a-fortiori*, since it turns out that $\Lambda'_T(q)$ is compact [21, Eqn. (4.19a)]. It is natural to relate the uniform COI (1.4) for the linearized dual, uncontrolled ϕ-problem (1.3) to the COI for the dual, uncontrolled ψ-problem below, which is the ϕ-problem with $q \equiv 0$. To this end, one introduces the following decomposition:

$$\phi = \psi + \chi, \tag{1.5}$$

where ψ and χ are solutions of the following problems

$$\begin{cases} \psi_{tt} = \Delta\psi \\ \psi(T,\cdot) = \phi_0;\ \psi_t(T,\cdot) = \phi_1 \\ \psi|_{\Sigma_0} \equiv 0;\ \left.\dfrac{\partial\psi}{\partial\nu}\right|_{\Sigma_1} \equiv 0 \end{cases} \qquad \begin{cases} \chi_{tt} = \Delta\chi + q(t,x)\phi & \text{in } Q; \quad (1.6a) \\ \chi(T,\cdot) = 0;\ \chi_t(T,\cdot) = 0 & \text{in } \Omega; \quad (1.6b) \\ \chi|_{\Sigma_0} \equiv 0;\ \left.\dfrac{\partial\chi}{\partial\nu}\right|_{\Sigma_1} \equiv 0 & \text{in } \Sigma_i, \quad (1.6c) \end{cases}$$

By assumption, the linear problem (1.1) with $f \equiv 0$ is exactly controllable on the energy state space $H \equiv H^1_{\Gamma_0}(\Omega) \times L_2(\Omega)$, within the class of $L_2(0,T;L_2(\Gamma_1))$-Neumann controls, and T sufficiently large (explicit estimates, in some cases optimal, are available). Equivalently, the above ψ-problem satisfies the corresponding COI (the first estimate below), so that: there exists a constant $c_T > 0$ independent on the I.C. such that

$$c_T \|\{\phi_0,\phi_1\}\|^2_{L_2(\Omega)\times[\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'} \leq \int_{\Sigma_1} \psi^2 d\Sigma_1 = \int_{\Sigma_1} (\phi-\chi)^2 d\Sigma_1 \tag{1.7}$$

$$\leq 2\int_{\Sigma_1} \phi^2 d\Sigma_1 + 2\int_{\Sigma_1} \chi^2 d\Sigma_1. \tag{1.8}$$

Thus, in order to establish the desired Uniform COI (1.4), it suffices—and indeed necessary as well: see [21, Remark 8.1]—to show that: there exists a constant $\text{const}_{Tr} > 0$ depending on T and r, but independent of the I.C. or of $q \in B(0,r)$, such that

$$\int_{\Sigma_1} \chi^2 d\Sigma_1 \leq \text{const}_{Tr} \int_{\Sigma_1} \phi^2 d\Sigma_1, \tag{1.9}$$

uniformly on $q \in B(0,r) \subset L_\infty(Q)$.

Goal of the present note. A *direct* operator-theoretic proof of inequality (1.9) was given in [21, Part III, Section 8 or Part IV, Section 9], ultimately relying—in line with the basic technical approach of [21] and [11]—on the analysis of suitable families of *collectively compact* operators [1]. This way compactness/uniqueness

arguments are entirely dispensed with in [11], [21]. All proofs in [21] are operator-theoretic, with a PDE-interpretation. By contrast, the goal of the present note is to give a PDE-based proof of the uniform comparison trace inequality (1.9), which relies on a double compactness/uniqueness argument. This approach is inspired by [5], where a double compactness/uniqueness argument was used for the purpose of obtaining a relevant stabilization inequality of a boundary feedback Kirchhoff plate problem, *independent of (uniform with respect to)* the thickness h of the plate, for all $0 < h \leq$ some h_0. In the present paper, instead, we seek to prove inequality (1.9), *uniformly with respect to the potential* $q \in B(0,r) \subset L_\infty(Q)$.

Once the uniform inequality (1.9) is proved—as through the double compactness/uniqueness proof of Sections 2 and 3—then (1.9) substituted in (1.8) yields the Uniform COI (1.4). Accordingly, via Theorem 1.2, this proves Theorem 1.1.

2. Proof of Theorem 1.1 via the UCOI (1.4) by Establishing Inequality (1.9). Case: $q \in L_\infty(Q)$ fixed.

The proof of the uniform inequality (1.9) proceeds in two phases: first, we prove, in Proposition 2.1 below, inequality (1.9) for a *fixed* $q \in L_\infty(Q)$, hence with a constant C_{Tq} depending only on $T > 0$ and q; next, we prove inequality (1.9) in its full strength, that is, uniformly in $q \in B(0,r) \subset L_\infty(Q)$, in Proposition 3.1 below.

Throughout this Section 2, $q \in L_\infty(Q)$ fixed.

Proposition 2.1. *Let $T > T_0$ (defined above Theorem 1.1). With reference to the ϕ-problem in (1.6) with $q \in L_\infty(Q)$ fixed and with I.C. $\{\phi_0, \phi_1\}$ at $t = 0$ and the χ-problem on the RHS of (1.6), the following estimate holds true: there exists a constant $C_{T,q}$, depending on T and $q \in L_\infty(Q)$, but not on the I.C., such that*

$$\int_{\Sigma_1} \chi^2 d\Sigma_1 \leq C_{T,q} \int_{\Sigma_1} \phi^2 d\Sigma_1. \tag{2.1}$$

Proof. (By compactness/uniqueness, as usual.) By contradiction, let there be a sequence $\{\phi_n\}$, $\{\chi_n\}$, with

$$\phi_n = \psi_n + \chi_n, \tag{2.2}$$

solving the following problems (with a fixed q):

$$\begin{cases} (\phi_n)_{tt} = \Delta\phi_n + q(t,x)\phi_n; \\ \phi_n(0,\,\cdot\,) = \phi_{0n},\ (\phi_n)_t(0,\,\cdot\,) = \phi_{n1}; \\ \phi_n|_{\Sigma_0} \equiv 0;\ \left.\dfrac{\partial\phi_n}{\partial\nu}\right|_{\Sigma_1} \equiv 0; \end{cases} \qquad \begin{cases} (\psi_n)_{tt} = \Delta\psi_n & \text{in } Q; \\ \psi_n(0,\,\cdot\,) = \phi_{0n},\ (\phi_n)_t(0,\,\cdot\,) = \phi_{n1} & \text{in } \Omega; \\ \psi_n|_{\Sigma_0} \equiv 0;\ \left.\dfrac{\partial\psi_n}{\partial\nu}\right|_{\Sigma_1} \equiv 0 & \text{in } \Sigma_i; \end{cases} \tag{2.3}$$

$$\{\phi_{0n}, \phi_{1n}\} \in Y \equiv L_2(\Omega) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]';$$

$$\begin{cases} (\chi_n)_{tt} = \Delta\chi_n + q(t,x)\chi_n + q(t,x)\psi_n & \text{in } Q; \\ \chi_n(0,\,\cdot\,) = 0;\ (\chi_n)_t(0,\,\cdot\,) = 0 & \text{in } \Omega; \\ \chi_n|_{\Sigma_0} \equiv 0;\ \left.\dfrac{\partial\chi_n}{\partial\nu}\right|_{\Sigma_1} \equiv 0 & \text{in } \Sigma_i \end{cases} \tag{2.4}$$

(refer to (1.6)), such that

$$\int_{\Sigma_1} \chi_n^2 d\Sigma_1 \equiv 1, \text{ while } \int_{\Sigma_1} \phi_n^2 d\Sigma_1 \to 0, \text{ as } n \to \infty. \tag{2.5}$$

Then, under the assumed hypothesis that the corresponding linear problem (1.1) with $f \equiv 0$ is exactly controllable on the state space $H \equiv H^1_{\Gamma_0}(\Omega) \times L_2(\Omega)$, within the class of $L_2(0,T;L_2(\Gamma_1))$-Neumann controls with the triple $\{\Omega, \Gamma_0, \Gamma_1\}$, $\Gamma_0 \neq \emptyset$, estimate (1.8) holds true for $\{\phi_n\}$, $\{\psi_n\}$, $\{\chi_n\}$: that is, via also (2.5), we have:

$$C_T \|\{\phi_{0n}, \phi_{1n}\}\|^2_{L_2(\Omega)\times[\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'} \leq \int_{\Sigma_1} \psi_n^2 d\Sigma_1 \leq 2\int_{\Sigma_1} \phi_n^2 d\Sigma_1 + 2\int_{\Sigma_1} \chi_n^2 d\Sigma_1 \leq \text{const.}, \tag{2.6}$$

uniformly in n. Thus, (2.6) implies that

$$\phi_{0n} \to \text{some } \phi_0, \text{ weakly in } L_2(\Omega); \tag{2.7a}$$

$$\phi_{1n} \to \text{some } \phi_1, \text{ weakly in } [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'; \tag{2.7b}$$

$$\phi_n|_{\Sigma_1} \to \text{some } \tilde{\phi} \equiv 0, \text{ strongly in } L_2(\Sigma_1) \text{ via (2.5).} \tag{2.7c}$$

Let now ϕ and ψ be solutions of the following problems corresponding to the pair $\{\phi_0, \phi_1\}$ obtained in (2.7), taken as Initial Conditions:

$$\begin{cases} \phi_{tt} = \Delta\phi + q(t,x)\phi & \text{in } Q; \\ \phi(0,\cdot) = \phi_0, \ \phi_t(0,\cdot) = \phi_1 & \text{in } \Omega; \\ \phi|_{\Sigma_0} = 0; \ \left.\dfrac{\partial\phi}{\partial\nu}\right|_{\Sigma_1} = 0 & \text{in } \Sigma_i; \end{cases} \quad \begin{cases} \psi_{tt} = \Delta\psi & \text{in } Q; \\ \psi(0,\cdot) = \phi_0, \ \psi_t(0,\cdot) = \phi_1 & \text{in } \Omega; \\ \psi|_{\Sigma_0} = 0, \ \left.\dfrac{\partial\psi}{\partial\nu}\right|_{\Sigma_1} \equiv 0 & \text{in } \Sigma_i. \end{cases} \tag{2.8}$$

Well-posedness and regularity estimates such as in [21, Eqns. (C.16) and (C.4) of Appendix C] apply to the ϕ-, respectively, the ψ-problems in (2.8). With reference to (2.8), let now

$$\chi = \phi - \psi, \text{ so that } \chi \text{ solves} \begin{cases} \chi_{tt} = \Delta\chi + q(t,x)\chi + q(t,x)\psi & \text{in } Q; \\ \chi(0,\cdot) = 0, \ \chi_t(0,\cdot) = 0 & \text{in } \Omega; \\ \chi|_{\Sigma_0} = 0, \ \left.\dfrac{\partial\chi}{\partial\nu}\right|_{\Sigma_1} \equiv 0 & \text{in } \Sigma_i. \end{cases} \tag{2.9}$$

Then, recalling the ψ_n-problem in (2.3), in the notation of [21, (2.3) of Section 2], we have the convergence

$$\begin{cases} \psi_n(t) = C(t)\phi_{0n} + S(t)\phi_{1n} \\ (\psi_n)_t(t) = C(t)\phi_{1n} - \mathcal{A}S(t)\phi_{0n} \end{cases} \longrightarrow \begin{cases} \psi(t) = C(t)\phi_0 + S(t)\phi_1 \\ \psi_t(t) = C(t)\phi_1 - \mathcal{A}S(t)\phi_0 \end{cases} \tag{2.10}$$

weakly (or weak-star) in $L_\infty(0,T;L_2(\Omega)\times[\mathcal{D}(\mathcal{A}^{\frac{1}{2}}]'$, by (2.7a–b), by standard cosine operator (or semigroup) theory. Hence, recalling the χ_n-problem in (2.4) and the

evolution operator $U(t,s)$ in [21, Appendix B], we obtain the convergence

$$\begin{aligned}\begin{bmatrix} \chi_n(t) \\ (\chi_n)_t(t) \end{bmatrix} &= \int_0^t U(t,s) \begin{bmatrix} 0 \\ q(s,\,\cdot\,)\psi_n(s) \end{bmatrix} ds \to \begin{bmatrix} \chi(t) \\ \chi_t(t) \end{bmatrix} \\ &= \int_0^t U(t,s) \begin{bmatrix} 0 \\ q(s,\,\cdot\,)\psi(s) \end{bmatrix} ds, \end{aligned} \tag{2.11}$$

weakly in $L_2(0,T;\mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega))$ by the top convergence in (2.10); hence, by Aubin's compactness theorem [2], recalling $\mathcal{D}(\mathcal{A}^{\frac{1}{2}}) = H^1_{\Gamma_0}(\Omega)$, it follows from (2.11) that

$$\chi_n \to \chi, \text{ strongly in } L_2(0,T;H^{\frac{1}{2}+\epsilon}_{\Gamma_0}(\Omega)),\ \epsilon < \frac{1}{2}, \tag{2.12}$$

and further, by trace theory, that

$$\chi_n|_{\Sigma_1} \to \chi|_{\Sigma_1} \text{ strongly in } L_2(\Sigma_1). \tag{2.13}$$

Then, (2.13) and $\int_{\Sigma_1} \chi_n^2 d\Sigma_1 \equiv 1$ by (2.5) yield

$$\int_{\Sigma_1} \chi^2 d\Sigma_1 = 1. \tag{2.14}$$

On the other hand, we also obtain the convergence

$$\begin{bmatrix} \phi_n(t) \\ (\phi_n)_t(t) \end{bmatrix} = U(t,0) \begin{bmatrix} \phi_{0n} \\ \phi_{1n} \end{bmatrix} \to \begin{bmatrix} \phi(t) \\ \phi_t(t) \end{bmatrix} = U(t,0) \begin{bmatrix} \phi_0 \\ \phi_0 \end{bmatrix}, \tag{2.15}$$

weakly in $L_2(0,T;L_2(\Omega) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]')$, by (2.7a–b). It follows that the limit $\tilde{\phi} \equiv 0$ in (2.7c) must then be $\tilde{\phi} = \phi|_{\Sigma_1} \equiv 0$. Thus, the ϕ-problem on the LHS of (2.8) has actually the over-determined boundary conditions:

$$\phi|_{\Sigma} \equiv 0 \text{ and } \left.\frac{\partial \phi}{\partial \nu}\right|_{\Sigma_1} \equiv 0, \text{ where } T > T_0. \tag{2.16}$$

As in the proof (from [11, Theorem 3.6, p. 133–4], [21, Theorem 3.2], we can use the over-determined B.C. (2.16) to boost the regularity of the solution ϕ to $\{\phi, \phi_t\} \in C([0,T];\mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega))$. It then follows from available uniqueness theorems for $T > T_0$ (under assumption $q \in L_\infty(Q)$), [12], [16], [23], [22], that the solution ϕ of (2.8) (left) and (2.16) satisfies in fact: $\phi \equiv 0$ in $Q = (0,T] \times \Omega$; in particular, that $\phi_0 = 0$, $\phi_1 = 0$. It then follows from the RHS of (2.8) that, in fact, $\psi \equiv 0$ in Q, and hence by (2.9), that $\chi = \phi - \psi \equiv 0$ in Q. Therefore, we obtain $\int_{\Sigma_1} \chi^2 d\Sigma_1 = 0$, and this contradicts (2.14). This contradiction shows that our assumptions in (2.5) are false; hence that inequality (2.1) holds true. □

Remark 2.1. We could use the variational formulation $((\phi_n)_{tt}, \rho) + (\nabla\phi_n, \nabla\rho) = (q\phi_n, \rho)$, $\rho =$ test function.

3. **Proof of Theorem 1.1 via the UCOI (1.4) by Establishing Inequality (1.9). Case: q Running in $B(0,r) \subset L_\infty(Q)$.** So far in our proof in Section 2, $q \in L_\infty(Q)$ was fixed. The constant $C_{T,q}$ in the COI depends plainly on q but not on the I.C. But how? In particular, does such $C_{T,q}$ blow up when q ranges on a fixed finite ball of $L_\infty(Q)$? The key is given by the following result which provides a positive answer.

Proposition 3.1. *Let $\{q^n\} \in B(0,r)$, where $B(0,r)$ is the ball of radius $r>0$ in $L_\infty(Q)$, centered at the origin, $n=1,2,\ldots$. Then, the constant in estimate (1.9) may be chosen as to depend only on the radius $r>0$, uniformly in n: $C_{T,r}>0$.*

Proof. Let $\{\phi_0,\phi_1\} \in L_2(\Omega)\times[\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'$. We consider the following problems:

$$\phi^n = \psi + \chi^n; \tag{3.1}$$

$$\begin{cases} \phi^n_{tt} = \Delta\phi^n + q^n(t,x)\phi^n & \text{in } Q;\\ \phi^n(0,\,\cdot\,) = \phi_0,\ \phi^n_t(0,\,\cdot\,) = \phi_1 & \text{in } \Omega;\\ \phi^n|_{\Sigma_0} = 0;\ \dfrac{\partial\phi^n}{\partial\nu}\Big|_{\Sigma_1} \equiv 0 & \text{in } \Sigma_i; \end{cases} \quad \begin{cases} \psi_{tt} = \Delta\psi & \text{in } Q;\\ \psi(0,\,\cdot\,) = \phi_0,\ \psi_t(0,\,\cdot\,) = \phi_1 & \text{in } \Omega;\\ \psi|_{\Sigma_0} \equiv 0,\ \dfrac{\partial\psi}{\partial\nu}\Big|_{\Sigma_1} \equiv 0 & \text{in } \Sigma_i; \end{cases} \tag{3.2}$$

$$\begin{cases} \chi^n_{tt} = \Delta\chi^n + q^n(t,x)\phi^n = \Delta\chi^n + q^n(t,x)\chi^n + q^n(t,x)\psi & \text{in } Q;\\ \chi^n(0,\,\cdot\,) \equiv 0,\ \chi^n_t(0,\,\cdot\,) \equiv 0 & \text{in } \Omega;\\ \chi^n|_{\Sigma_0} \equiv 0,\ \dfrac{\partial\chi^n}{\partial\nu}\Big|_{\Sigma_1} \equiv 0 & \text{in } \Sigma_i, \end{cases} \tag{3.3}$$

where $q^n \in B(0,r)$, $r>0$, as assumed. Well-posedness and regularity estimates such as [21, (C.16), (C.4), (C.13), Appendix C] hold true for the ϕ^n-, ψ-, χ^n-problems. By Proposition 2.1, Eqn. (2.1), we have: there exists a constant C_{T,q^n}, independent on the I.C., $T>T_0$, such that

$$\int_{\Sigma_1} (\chi^n)^2 d\Sigma_1 \le C_{T,q^n}\int_{\Sigma_1}(\phi^n)^2 d\Sigma_1, \quad n=1,2\ldots \tag{3.4}$$

In order to establish Proposition 3.1, we need to prove that the constant C_{T,q^n} in (3.4) can be replaced by a constant $C_{T,r}$, depending only on the radius r of $B(0,r)$, uniformly in n. To this end, we shall use a second compactness/uniqueness argument. To begin with, there exists a subsequence, still denoted by q^n, such that

$$q^n \to \text{some } q \in B(0,r) \subset L_\infty(Q), \text{ weakly in } L_\infty(Q), \tag{3.5}$$

with corresponding solutions ϕ^n and χ^n in (3.2) and (3.3), respectively. Next, we return to the corresponding χ^n-problem in (3.3) and apply to it estimate [21, (C.13), Appendix C]. We obtain

$$\|\{\chi^n,\chi^n_t\}\|^2_{C([0,T];\mathcal{D}(\mathcal{A}^{\frac{1}{2}})\times L_2(\Omega))} \le C_T e^{M_{q^n}T}\|q^n\|_{L_\infty(Q)}\|\{\phi_0,\phi_1\}\|^2_{L_2(\Omega)\times[\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'}; \tag{3.6a}$$

$$M_{q^n} = \max\{c_P\|q^n\|_{L_\infty(Q)}, 2\|q^n\|_{L_\infty(Q)}\}, \quad c_P = \text{constant in Poincaré inequality}, \tag{3.6b}$$

see below [21, (C.8)]. By (3.5), (3.6), we then obtain

$$\|\{\chi^n,\chi^n_t\}\|^2_{C([0,T];\mathcal{D}(\mathcal{A}^{\frac{1}{2}})\times L_2(\Omega))} \le C_T r e^{M_r T}\|\{\phi_0,\phi_1\}\|^2_{L_2(\Omega)\times[\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'}; \tag{3.7}$$

$M_r = \max\{c_P r, 2r\}$, uniformly in n. Thus, it follows from (3.7) that we have the following convergence

$$\{\chi^n,\chi^n_t\} \to \text{some } \{\chi,\chi_t\} \text{ weakly in } C([0,T];\mathcal{D}(\mathcal{A}^{\frac{1}{2}})\times L_2(\Omega)). \tag{3.8}$$

By Aubin's compactness theorem as in [18], we conclude, as in (2.12), that

$$\chi^n \to \chi \text{ strongly in } C([0,T];H^{1-\epsilon}_{\Gamma_0}(\Omega)), \quad 0<\epsilon, \tag{3.9}$$

and hence by trace theory, as in (2.13), that

$$\chi^n|_{\Sigma_1} \to \chi|_{\Sigma_1} \text{ strongly in } L_2(\Sigma_1) \equiv L_2(0,T;L_2(\Gamma_1)). \tag{3.10}$$

Moreover, by (3.5) on q^n and (3.9) with $\epsilon = 1$ on χ^n, we also conclude that

$$q^n\chi^n - q\chi = q^n(\chi^n - \chi) + \chi(q^n - q) \to 0, \text{ weakly in } L_\infty(0,T;L_2(\Omega)). \tag{3.11}$$

On the other hand, $\psi \in C([0,T];L_2(\Omega))$ for $\{\phi_0,\phi_1\} \in Y = L_2(\Omega) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'$, and by (3.9) with $\epsilon = 1$ and (3.1), we conclude that

$$\phi^n \to \text{ some } \phi \text{ strongly in } C([0,T];L_2(\Omega)), \tag{3.12}$$

and, in fact,

$$\phi \equiv \psi + \chi \in C([0,T];L_2(\Omega)), \tag{3.13}$$

with ψ the fixed solution in (3.2) (RHS) with regularity $\psi \in C([0,T];L_2(\Omega))$, and χ the limit in (3.9). Furthermore, we have by (3.1), (3.13), (3.5), (3.11):

$$q^n\phi^n - q\phi = q^n(\chi^n + \psi) - q(\chi + \psi) = q^n\chi^n - q\chi + (q^n - q)\psi \to 0 \quad \text{weakly in } L_\infty(0,T;L_2(\Omega)). \tag{3.14}$$

Writing the weak formulation (see Remark 2.1) of the ϕ^n-problem in (3.2) and of the χ^n-problem in (3.3) and taking weak limits, we obtain, after using $q^n\phi^n \to q\phi$ weakly in (3.14), that the limit functions ϕ and χ in (3.13) and (3.9) satisfy

$$\begin{cases} \phi_{tt} = \Delta\phi + q(t,x)\phi & \text{in } Q; \\ \phi(0,\cdot) = \phi_0,\ \phi_t(0,\cdot) = \phi_1 & \text{in } \Omega; \\ \phi|_{\Sigma_0} = 0;\ \left.\dfrac{\partial\phi}{\partial\nu}\right|_{\Sigma_1} = 0 & \text{in } \Sigma_i; \end{cases} \quad \begin{cases} \chi_{tt} = \Delta\chi + q(t,x)\phi & \text{in } Q; \\ \chi(0,\cdot) = 0,\ \chi_t(0,\cdot) = 0 & \text{in } \Omega; \\ \chi|_{\Sigma_0} \equiv 0,\ \left.\dfrac{\partial\chi}{\partial\nu}\right|_{\Sigma_1} \equiv 0 & \text{in } \Sigma_i. \end{cases} \tag{3.15}$$

The regularity $\{\phi,\phi_t\} \in C([0,T];L_2(\Omega)\times[\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]')$ and $\{\chi,\chi_t\} \in C([0,T];\mathcal{D}(\mathcal{A}^{\frac{1}{2}})\times L_2(\Omega))$ is then provided by [21, Theorem 3.1 - Theorem C.1, Eqn. (3.11) = (C.16) and (C.13)], respectively.

We can now run the contradiction argument on estimate (3.4). Since C_{T,q^n} is independent of the I.C. $\{\phi^n(0,\cdot),\phi^n_t(0,\cdot)\} = \{\phi_0,\phi_1\}$, we may think of C_{T,q^n} as being selected as the best such constant, so that (3.4) holds true as an equality: $\int_{\Sigma_1}(\chi^n)d\Sigma_1 = C_{T,q^n}\int_{\Sigma_1}(\phi^n)^2 d\Sigma_1$. With this choice, suppose by contradiction that

$$C_{T,q^n} \nearrow \infty, \text{ hence that } \int_{\Sigma_1}(\chi^n)^2 d\Sigma_1 \equiv 1 \text{ and } \int_{\Sigma_1}(\phi^n)d\Sigma_1 \to 0. \tag{3.16}$$

Then (3.10) would imply via (3.16) on χ^n that

$$\int_{\Sigma_1}\chi^2 d\Sigma_1 = 1. \tag{3.17}$$

On the other hand, (3.16) on ϕ^n and (3.12) would imply $\phi|_{\Sigma_1} = 0$ and thus the ϕ-problem in (3.15) (left) would be endowed with the over-determined B.C.: $\phi|_\Sigma \equiv 0$, $\frac{\partial\phi}{\partial\nu}|_{\Sigma_1} \equiv 0$, for $T > T_0$. We can now proceed as below (2.16). The over-determined B.C. boost the regularity of ϕ to $\{\phi,\phi_t\} \in C([0,T];\mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega))$. Thus, we can invoke available uniqueness theorems for $q \in L_\infty(Q)$ [12], [16], [23], [22], which would imply $\phi \equiv 0$ in $(0,T]\times\Omega$, hence $\phi_0 = 0$, $\phi_1 = 0$; then (3.2) on the ψ-problem would yield $\psi \equiv 0$ in Q, and thus $\chi = \phi - \psi \equiv 0$ in Q by (3.13). We would thus obtain $\int_{\Sigma_1}\chi^2 d\Sigma_1 = 0$, a contradiction with (3.17). This contradiction says that

$C_{T,q^n} \nearrow \infty$ is false, and hence $C_{T,q^n} \leq$ some $C_{T,r}$, with $\|q^n\|_{L_\infty(Q)} \leq r$, as desired. Proposition 3.1 is proved.□

Final Remarks. (1) Far more general second-order hyperbolic equations in place of (1.1a) can be handled in a similar way. To this end, one invokes [12] for the addition to (1.1a) of energy level terms and [22], for the Riemannian wave equation (hence variable coefficients also in the principal part). See also [21, Section 10] for the Riemannian wave equation with $L_2(0,T;L_2(\Gamma_1))$-Dirichlet control on the (optimal) state space $L_2(\Omega) \times H^{-1}(\Omega)$.

(2) In the case of *Dirichlet* boundary control: $w|_{\Sigma_0} \equiv 0$, $w|_{\Sigma_1} = u$, exact controllability of the semilinear wave equation (1.1a) under the same assumption (1.2) on the non-linearity was given in [11]: in the state space $H_0^1(\Omega) \times L_2(\Omega)$ within the class of $H_0^1(0,T;L_2(\Gamma_1))$-controls; and in the state space $L_2(\Omega) \times H^{-1}(\Omega)$ within the class of $L_2(0,T;L_2(\Gamma_1))$-controls. Hence, by interpolation, in the state space $H_0^\gamma(\Omega) \times H^{\gamma-1}(\Omega)$, $0 \leq \gamma \leq 1$, $\gamma \neq \frac{1}{2}$ (resp., in the state space $H_{00}^{\frac{1}{2}}(\Omega) \times [H_{00}^{\frac{1}{2}}(\Omega)]'$ for $\gamma = \frac{1}{2}$) within the class of $H_0^\gamma(0,T;L_2(\Gamma_1))$-controls (resp., within the class of $H_{00}^{\frac{1}{2}}(0,T;L_2(\Gamma_1))$-controls, for $\gamma = \frac{1}{2}$). These topological settings are optimal [8].

Reference [11] was prompted by paper [24] which used a Schauder fixed point approach. The literature of semilinear controllability has been mostly dominated by fixed point approaches since about 1965 (H. Hermes) in the finite-dimensional case. However, reference [24] obtains exact controllability in the present Dirichlet-boundary control case only for the range $0 < \gamma < 1$. The limit case $\gamma = 0$ is excluded for lack of compactness in the Schauder fixed point argument. The limit case $\gamma = 1$ is excluded because of some difficulties arising in a passage to the limit argument. The approach of the present note, based on the double compactness/uniqueness argument, permits to overcome these difficulties and complete paper [24] by including also the case $\gamma = 1$. Moreover, as stated in the introduction, the present paper, combined with our global invertibility strategy [11], [21] (rather than use of Schauder fixed point) when applied to the Dirichlet boundary control case, will allow to include also the case $\gamma = 1$. Finally, the treatment of [23, Theorem 4.2, p. 831] could also be extended to include the cases $\gamma = 0$, $\gamma = 1$, by relying either on the present paper or on [11].

REFERENCES

[1] P. M. Anselone, Collectively Compact Operator Approximation Theory, Prentice Hall, Englewood Cliffs, NJ, 1971.

[2] J. P. Aubin, *Un theoreme de compacite*, C. R. Acad. Sci. Paris, **256** (1963), 5042–5044.

[3] C. Bardos, G. Lebeau, and J. Rauch, *Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary*, SIAM J. Control Optim., **30** (1992), 1024–1065.

[4] K. Deimling, Nonlinear Functional Analysis, Springer-Verlag, Berlin 1980, Section 15.2, p. 152.

[5] M. A. Horn and I. Lasiecka, *Asymptotic behavior with respect to thickness of boundary stabilizing feedback of the Kirchoff plate*, J. Diff. Eqns., **114** (Dec. 1994), 396–433.

[6] J. Lagnese, *Decay of solutions of wave equations in a bounded region with boundary dissipation*, J. Diff. Eqns., **50** (1983), 163–182.

[7] J. Lagnese, Boundary Stabilization of Thin Plates, SIAM, Philadelphia, 1989.

[8] I. Lasiecka, J. L. Lions, and R. Triggiani, *Non-homogeneous boundary value problems for second order hyperbolic operators*, J. Math. Pures et Appl., **65** (1986), 26–47.

[9] I. Lasiecka and R. Triggiani, *Exact controllability of the wave equation with Neumann boundary control*, Appl. Math. Optim., **19** (1989), 243–290.

[10] I. Lasiecka and R. Triggiani, *Sharp regularity for mixed second-order hyperbolic equations of Neumann type, Part I: The L_2-boundary case*, Annal. di Matemat. Pura e Appl. (IV), **157** (1990), 285–367.

[11] I. Lasiecka and R. Triggiani, *Exact controllability of the semilinear abstract system with application to waves and plates*, Appl. Math. Optim., **23** (1991), 109–154.

[12] I. Lasiecka, R. Triggiani, and X. Zhang, *Nonconservative wave equations with purely Neumann B.C.: Global uniqueness and observability in one shot*, Amer. Math. Soc., Cont. Math., **268** (2000), 227–326.

[13] J. L. Lions, Controllabilite Exacte, Stabilization et Perturbation des Systemes Distribues, Vol. 1, Masson, Paris, 1988.

[14] W. Littman, *Near-optimal-time boundary controllability for a class of hyperbolic equations*, Control Problems for Systems Described by Partial Differential Equations and Applications (Gainesville, FL, 1986), 307–312, Lecture Notes in Control and Inform. Sci. 97, Springer, Berlin-New York, 1987.

[15] W. Littman, *Boundary controllability for polyhedral domains*, m LNCIS, Springer Verlag, **178** (1992), 272–284.

[16] W. Littman, *Remarks on global uniqueness theorems for partial differential equations*, Contemporary Mathematics, AMS 268 (2000).

[17] J. T. Schwartz, Nonlinear Functional Analysis, Gordon & Breach, New York, 1969.

[18] J. Simon, *Compact sets on the space $L^p(0,T;B)$*, Ann. Mat. Pura & Appl. (4) CXLVI (1987), 65–96.

[19] D. Tataru, A-priori pseudoconvexity energy estimates in domains with boundary applications to exact boundary controllability for conservative PDEs, Ph.D. Thesis, University of Virginia, May 1992.

[20] D. Tataru, *Boundary controllability for conservative PDEs*, Appl. Math. & Optimiz., **31** (1995), 257–295.

[21] R. Triggiani, Global exact controllability on $H^1_{\Gamma_0}(\Omega \times L_2(\Omega)$ of semilinear wave equations with Neumann $L_2(0,T;L_2(\Gamma_1))$-boundary control, to appear in "Control Theory for Partial Differential Equations," Lecture Notes in Pure and Applied Mathematics, Chapter 19, CRC Press/Chapman and Hall, to appear May 2005.

[22] R. Triggiani and P. F. Yao, *Carleman estimates with no lower order terms for general Riemannian wave equations. Global uniqueness and observability in one shot*, Appl. Math. & Optim., **46** (Sept./Dec. 2002), 331–375.

[23] X. Zhang, *Explicit observability inequalities for the wave equation with lower order terms by means of Carleman inequalities*, SIAM J. Control & Optim., **3** (2000), 812–834.

[24] E. Zuazua, *Exact boundary controllability for the semilinear wave equation. Nonlinear PDEs and their applications*, Research Notes in Mathematics, Pitman **220** (1991),357–391, Longman, Chicago.

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E-mail address: `il2v@virginia.edu; rt7u@virginia.edu`