DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Supplement Volume 2005

Website: http://AIMsciences.org

pp. **420–426**

STABILITY OF CELLULAR NEURAL NETWORK WITH SMALL DELAYS

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Abstract. We consider a system of cellular neural networks with delays. By using appropriate Lyapunov functions, we obtain sufficient conditions so that the system is globally stable when the delay is small enough.

1. **Introduction.** Cellular neural network is a novel structure for nonlinear analog signal processing. Nonlinear delayed cellular neural networks were first introduced by Chua and Roska [3] in 1992. They have been used in various types of motion-related applications (e.g., processing of moving images, and speed detection of moving objects) and pattern classification. Stability and convergence conditions are essential for the actual design of recurrent neural networks. Many results have been established regarding the convergence and stability for systems of cellular neural networks, see [1], [2], [4] and [6] for delay-independent stabilities and [7] for delay-dependent ones.

In this article we further investigate global stabilities of delayed cellular neural networks. In particular, by using appropriate Lyaponov functions, we obtain sufficient conditions so that when the delay is small enough, the system is globally asymptotically stable that means all trajectories converge to a unique and globally asymptotically stable equilibrium point.

2. **Global Stability Results.** The dynamics of the periodic cellular neural network can be described by the following system.

$$x'(t) = -x(t) + Af(x(t)) + Bf(x(t-\tau)) + U \tag{1}$$

where $x = (x_1, x_2, \cdots, x_n)^T \in R^n$ is the state vector, $A = (a^{ij})$ is the feedback matrix and $B = (b^{ij})$ is the delayed feedback matrix, τ is the transmission delay, U is the external constant input vector. Here $f(x(\cdot)) = [f(x_1(\cdot)), \cdots, f(x_n(\cdot))]^T$ is the output vector, f is a sufficiently smooth sigmoid amplification, normalized so that $f(0) = 0$ and $f'(0) = 1$. A commonly used amplification function is

2000 *Mathematics Subject Classification.* 34K20, 34K15, 92B20.
Key words and phrases. Stability, neural network, Lyapunov functions, delays.

$$f(x) = \tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

which satisfies the following monotonicity and concavity properties:
(H1). $f(0) = 0, f'(0) = 1$.
(H2). For all x, $f'(x) > 0$,
(H3). For all $x \neq 0$, $f''(x)x < 0$.
(H4). $\lim_{x\to\infty} f(x) = 1, \lim_{x\to-\infty} f(x) = -1$.

In this article, we assume that $U = 0$. When $U \neq 0$, our results can be established in the same way.

We rewrite system (1) as

$$x'(t) = -x(t) + (A + B)f(x(t)) + B(f(x(t - \tau)) - f(x(t)) \tag{2}$$

We say an $n \times n$ matrix $W > 0$ if for any vector $x \in R^n$, $x^T W x > 0$. Note that W is not necessarily symmetric. In fact, $W > 0$ if and only if $W + W^T$ is positive definite.

We also need to use the following lemma.

Lemma 1. *If x and y are both vectors in R^n, we have*

$$y^T y \leq \frac{x^T x + y^T y}{2}.$$

Using (H1)-(H4), we can also get the following lemma.

Lemma 2. *Suppose $x(t)$ is a smooth vector valued function in R^n and $f(x(\cdot)) = [f(x_1(\cdot)), \cdots, f(x_n(\cdot))]^T$, with f satisfies (H1)-(H4). Then*

$$x^T (D_x f)^T (D_x f)x \leq f^T(x)f(x),$$

$$(A + B)^T (D_x f)^T (D_x f)(A + B) \leq (A + B)^T (A + B),$$

and

$$B^T (D_x f)^T (D_x f)B \leq B^T B.$$

Our main theorem is as follows.

Theorem 1. *For system (2), if f satisfies (H1)–(H4) and*

$$I - A - B > 0$$

then when the delay τ is sufficiently small, system (2) has a unique equilibrium and it is globally asymptotically stable.

Proof. We will use the Lyapunov function method to prove the result. Let x be a solution to system (2). Define the following function

$$V(x) = V_1(x) + V_2(x) + V_3(x)$$

such that

$$
\begin{aligned}
V_1(x(t)) =& \frac{\epsilon}{2} x^T(t)x(t)\\
V_2(x(t)) =& \Sigma_{i=1}^n \int_0^{x_i(t)} f(s)ds\\
&+ a\int_{t-\tau}^{t} (f(x(s)) - f(x(t)))^T(f(x(s)) - f(x(t)))ds\\
V_3(x(t)) =& b\int_{t-\tau}^{t}\int_{s}^{t} f(x(\eta))^T f(x(\eta))d\eta ds.
\end{aligned}
$$

We will show next that when $I - A - B > 0$ and τ is sufficiently small, V is a Lyapunov function with the following property $V(x(t)) \geq 0$ and $\frac{d}{dt}V(x(t)) \leq 0$. Furthermore, $\frac{d}{dt}V(x(t)) = 0$ if and only if $x(t) = 0$. We know that $0 \in R^n$ is an equilibrium of system (2). The existence of Lyapunov function tells us that this equilibrium is unique and is globally asymptotically stable.

Using (2) and Lemma 1, we get

$$
\begin{aligned}
&\frac{d}{dt}V_1(x(t)) = \epsilon x^T(t)\frac{d}{dt}x(t)\\
=& -\epsilon x^T(t)x(t) + \epsilon x^T(t)(A+B)f(x(t)) + \epsilon x^T(t)B(f(x(t-\tau)) - f(x(t)))\\
\leq& -\epsilon x^T(t)x(t) + \frac{p\epsilon}{2}x^T(t)x(t) + \frac{\epsilon}{2p}f^T(x(t))(A+B)^T(A+B)f(x(t))\\
&+ \frac{q\epsilon}{2}x^T(t)x(t) + \frac{\epsilon}{2q}[f(x(t-\tau)) - f(x(t))]^T B^T B[f(x(t-\tau)) - f(x(t))]\\
=&(-\epsilon + \frac{p\epsilon}{2} + \frac{q\epsilon}{2})x^T(t)x(t) + f^T(x(t))[\frac{\epsilon}{2p}(A+B)^T(A+B)]f(x(t))\\
&+\frac{\epsilon}{2q}(f(x(t-\tau)) - f(x(t)))^T B^T B(f(x(t-\tau)) - f(x(t)))
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{d}{dt}V_2(x(t))\\
=& f(x(t))^T\frac{d}{dt}x(t) - a(f(x(t-\tau)) - f(x(t)))^T(f(x(t-\tau)) - f(x(t)))\\
&- a\int_{t-\tau}^{t} 2(f(x(s)) - f(x(t)))^T\frac{df(x(t))}{dt}ds\\
=& -f^T(x(t))x(t) + f^T(x(t))(A+B)f(x(t)) + f^T(x(t))B(f(x(t-\tau)) - f(x(t)))\\
&- a(f(x(t-\tau)) - f(x(t)))^T(f(x(t-\tau)) - f(x(t)))\\
&- 2a\int_{t-\tau}^{t}(f(x(s)) - f(x(t)))^T\frac{df(x(t))}{dt}ds
\end{aligned}
$$

Using Lemma 1, we get

$$\begin{aligned}
&\frac{d}{dt}V_2(x(t))\\
\leq &- f^T(x(t))f(x(t)) + f^T(x(t))(A+B)f(x(t)) + \frac{1}{2}cf^T(x(t))f(x(t))\\
&+ \frac{1}{2c}[f(x(t-\tau)) - f(x(t))]^T B^T B[f(x(t-\tau)) - f(x(t))]\\
&- a(f(x(t-\tau)) - f(x(t)))^T(f(x(t-\tau)) - f(x(t)))\\
&+ a\int_{t-\tau}^{t} f^T(x(s))f(x(s))ds + a\int_{t-\tau}^{t} \frac{d}{dt}f^T(x(t))\frac{d}{dt}f(x(t))ds\\
&+ a\int_{t-\tau}^{t} f^T(x(t))f(x(t))ds + a\int_{t-\tau}^{t} \frac{d}{dt}f^T(x(t))\frac{d}{dt}f(x(t))ds\\
=&f^T(x(t))[-I + (A+B) + \frac{1}{2}cI + a\tau]f(x(t)) + (f(x(t-\tau))\\
&- f(x(t)))^T[-aI + \frac{1}{2c}I](f(x(t-\tau)) - f(x(t)))\\
&+ a\int_{t-\tau}^{t} f^T(x(s))f(x(s))ds + 2a\tau\frac{d}{dt}f^T(x(t))\frac{d}{dt}f(x(t)),
\end{aligned}$$

meanwhile,

$$\begin{aligned}
&\frac{d}{dt}V_3(x(t))\\
=&- b\int_{t-\tau}^{t} f(x(\eta))^T f(x(\eta))d\eta + b\int_{t-\tau}^{t} f(x(t))^T f(x(t))ds\\
=&- b\int_{t-\tau}^{t} f(x(\eta))^T f(x(\eta))d\eta + b\tau f(x(t))^T f(x(t))\\
=&- b\int_{t-\tau}^{t} f(x(s))^T f(x(s))ds + b\tau f(x(t))^T f(x(t))
\end{aligned}$$

We know that

$$\frac{d}{dt}f^T(x(t))\frac{d}{dt}f(x(t)) = \frac{d}{dt}x^T(t)(D_x f)^T(D_x f)\frac{d}{dt}x(t) \tag{3}$$

where $D_x f$ is a diagonal matrix whose diagonal terms are $\frac{d}{dx}f|_{x_i(t)}$.
Using the fact that

$$(u+v+w)^T(u+v+w) \leq 3u^T u + 3v^T v + 3w^T w.$$

we get

$$\begin{aligned}
&\frac{d}{dt}f^T(x(t))\frac{d}{dt}f(x(t))\\
=&\frac{d}{dt}x^T(t)(D_xf)^T(D_xf)\frac{d}{dt}x(t)\\
=&[-x(t)+(A+B)f(x(t))+B(f(x(t-r))-f(x(t))]^T\\
&\cdot(D_xf)^T(D_xf)[-x(t)+(A+B)f(x(t))+B(f(x(t-r))-f(x(t)))]\\
\leq&3x^T(t)(D_xf)^T(D_xf)x(t)+3f^T(x(t))(A+B)^T(D_xf)^T(D_xf)(A+B)f(x(t))\\
+&3[f(x(t-r))-f(x(t))]^TB^T(D_xf)^T(D_xf)B[f(x(t-r))-f(x(t))]
\end{aligned}$$

Using Lemma 2, we get

$$x^T(D_x^+f)^T(D_x^+f)x \leq f^T(x)f(x),$$

$$(A+B)^T(D_x^+f)^T(D_x^+f)(A+B) \leq (A+B)^T(A+B),$$

and

$$B^T(D_x^+f)^T(D_x^+f)B \leq B^TB.$$

Therefore,

$$\begin{aligned}
\frac{d}{dt}f^T(x(t))\frac{d}{dt}f(x(t)) \leq& 3f^T(x(t))f(x(t))+3f^T(x(t))(A+B)^T(A+B)f(x(t))\\
&+3[f(x(t-r))-f(x(t))]^TB^TB[f(x(t-r))-f(x(t))]
\end{aligned}$$

We then get

$$\begin{aligned}
&\frac{d}{dt}V_2(x(t))\\
\leq&f^T(x(t))[-I+(A+B)+\frac{1}{2}cI+a\tau]f(x(t))\\
&+(f(x(t-\tau))-f(x(t)))^T[-aI++\frac{1}{2c}](f(x(t-\tau))-f(x(t)))\\
&+a\int_{t-\tau}^t f^T(x(s))f(x(s))ds+2a\tau[3f^T(x(t))f(x(t))\\
&+3f^T(x(t))(A+B)^T(A+B)f(x(t))\\
&+3[f(x(t-r))-f(x(t))]^TB^TB[f(x(t-r))-f(x(t))]\\
=&f^T(x(t))[-I+(A+B)+\frac{1}{2}cI+7a\tau I+6a\tau(A+B)^T(A+B)]f(x(t))\\
&+(f(x(t-\tau))-f(x(t)))^T[-aI+\frac{1}{2c}I\\
&+6a\tau B^TB](f(x(t-\tau))-f(x(t)))+a\int_{t-\tau}^t f^T(x(s))f(x(s))ds
\end{aligned}$$

As a result,

$$
\begin{aligned}
V'(x(t)) &= \frac{d}{dt}V_1(x(t)) + \frac{d}{dt}V_2(x(t)) + \frac{d}{dt}V_3(x(t)) \\
&\leq (-\epsilon + \frac{p\epsilon}{2} + \frac{q\epsilon}{2})x^T(t)x(t) + f^T(x(t))[-I + (A+B) + \frac{1}{2}cI + \frac{\epsilon}{2p}(A+B)^T(A+B) \\
&+ 7a\tau I + b\tau I + 6a\tau(A+B)^T(A+B)]f(x(t)) + (f(x(t-\tau)) - f(x(t)))^T[-aI \\
&+ \frac{\epsilon}{2q}B^TB + \frac{1}{2c}I + 6a\tau B^TB](f(x(t-\tau)) - f(x(t))) \\
&+ (-b+a)\int_{t-\tau}^{t} f^T(x(s))f(x(s))ds
\end{aligned}
$$

We need to choose the appropriate values of ϵ, p, q, a, b such that the above expression is negative. In particular, we let $a = b$. Then we need to have

$$
-I + (A+B) + 8a\tau I + \frac{1}{2}cI + (6a\tau + \frac{\epsilon}{2p})(A+B)^T(A+B) < 0 \tag{4}
$$

and

$$
-aI + \frac{1}{2c}I + (6a\tau + \frac{\epsilon}{2q})B^TB \leq 0. \tag{5}
$$

Since $I - A - B > 0$, from (4), we can pick the largest c such that

$$
cI < I - A - B
$$

and

$$
a\tau(8I + 6(A+B)^T(A+B)) \leq I - A - B - \frac{1}{2}cI.
$$

From (5), choose a so that

$$
aI > \frac{1}{2c}I + 6a\tau B^TB.
$$

Therefore, by choosing appropriate a and C, when τ is small enough, (4) and (5) can be satisfied. This proves the theorem. □

In our proof, with the Lyaponov function we defined, we are able to estimate the size of the delay τ so that the system is globally asymptotically stable. Furthermore, we can generalize our results to a broader class of systems, namely, when f is not restricted to be piecewise linear.

We also need to mention that when f is not smooth, for example, when

$$
f(x_i(t)) = \frac{1}{2}(|x_i(t)+1| - |x_i(t)-1|),
$$

our method can also be applied when we use the right derivative instead of derivatives and all the results will carry through.

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Received September, 2004; revised March, 2005.

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