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STEADY STATES OF A STRONGLY COUPLED PREY–PREDATOR MODEL

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Abstract. We study an elliptic system arising from a prey-predator model, where cross diffusions are included to reflect the influences of density gradients of prey and predator toward the fluxes of underlying populations. We establish existence and non–existence of non–constant positive solutions, or the possible pattern of population distribution.

1. **Introduction.** We consider positive solutions of an elliptic system

$$\begin{cases} \mathrm{div}(\mathbf{K}(\mathbf{u})\nabla\mathbf{u}) + \mathbf{G}(\mathbf{u}) = 0 & \text{in } \Omega, \\ \partial_\nu \mathbf{u} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbf{R}^N$ $(N \geq 1)$ with smooth boundary $\partial\Omega$, ∂_ν is the directional derivative normal to $\partial\Omega$, and

$$\mathbf{u} = \begin{pmatrix} u \\ v \end{pmatrix}, \quad \mathbf{K}(\mathbf{u}) = \begin{pmatrix} K_{11}(\mathbf{u}) & K_{12}(\mathbf{u}) \\ -K_{21}(\mathbf{u}) & K_{22}(\mathbf{u}) \end{pmatrix}, \quad \mathbf{G}(\mathbf{u}) = \begin{pmatrix} G_1(\mathbf{u}) \\ G_2(\mathbf{u}) \end{pmatrix}.$$

(1) is the system for steady state of the dynamical predator-prey model

$$\mathrm{diag}(\tau_1,\, \tau_2)\mathbf{u}_t = \mathrm{div}(\mathbf{K}(\mathbf{u})\nabla\mathbf{u}) + \mathbf{G}(\mathbf{u}).$$

From the biological consideration we assume that the growth rates of prey and predator take the following forms

$$G_1(\mathbf{u}) = u\varphi(u) - uv := u\, g_1(\mathbf{u}), \qquad G_2(\mathbf{u}) = uv - v\psi(v) := v\, g_2(\mathbf{u}),$$

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where φ and ψ are smooth functions on $[0, \infty)$ and satisfy, for a positive constant L,

$$\varphi(u) > 0 \text{ in } [0, L), \quad \varphi(u) < 0 \text{ in } (L, \infty), \quad \lim_{u\to\infty} u\,|\varphi(u)| = \infty, \tag{2}$$

$$\psi(v) > 0 \quad \text{in} \quad [0,\infty), \qquad \lim_{v\to\infty} v\,\psi(v) = \infty. \tag{3}$$

In addition, we assume the diffusion coefficients $K_{ij}(\mathbf{u})$ are smooth and satisfy

$$\begin{aligned} &K_{11}(\mathbf{u}),\, K_{22}(\mathbf{u}) \geq 1, \quad K_{12}(\mathbf{u}),\, K_{21}(\mathbf{u}) \geq 0, \\ &K_{12}(0,v) = K_{21}(u,0) = 0 \text{ in } \mathbf{R}^2_+. \end{aligned} \tag{4}$$

In this setting, $J_u := -K_{11}(\mathbf{u})\nabla u - K_{12}(\mathbf{u})\nabla v$ and $J_v := K_{21}(\mathbf{u})\nabla u - K_{22}(\mathbf{u})\nabla v$ are the population fluxes of prey and predator respectively. The terms $K_{11}(\mathbf{u})$ and $K_{22}(\mathbf{u})$ represent the "self–diffusions". The terms $K_{12}(\mathbf{u})$ and $K_{21}(\mathbf{u})$ are called "cross–diffusions" coefficients. The condition $K_{12}(\mathbf{u}) \geq 0$ implies that the part $-K_{12}\nabla v$ of flux of u is directed toward decreasing population density of v, i.e. the prey avoids the predator, whereas $K_{21}(\mathbf{u}) \geq 0$ implies that the part $-K_{21}\nabla u$ of flux of v is directed toward increasing population density of u, i.e. the predator chases the prey. See Okubo [6, Chpt.10] for a more detailed discussion on biological models.

For mathematical analysis, we assume that there exists a positive constant M_0 such that $\forall\,\mathbf{u} \in \mathbf{R}^2_+$,

$$\|\mathbf{K}^{-1}(\mathbf{u})\| \leq M_0 \qquad \text{i.e.,} \quad \frac{\max\{K_{11},\, K_{12},\, K_{21},\, K_{22}\}}{K_{11}K_{22} + K_{12}K_{21}} \leq M_0. \tag{5}$$

In literature, almost all theoretical results assumed that $K_{12}(\mathbf{u}) = K_{21}(\mathbf{u}) \equiv 0$. The only literature we know of that includes cross diffusion is Lou, Martinez and Ni [1, 2, 3].

We are interested in the existence of non-degenrate non-constant positive solutions. For this purpose, we assume that there is a unique positive constant solution $\mathbf{u}^*$. More precisely, with $\mathbf{u}^* = (1,\, 1)^T$ for definiteness, there holds

$$\begin{cases} \varphi(1) = \psi(1) = 1, \quad \varphi'(1)\psi'(1) < 1, \\ g_1(\mathbf{u}) = g_2(\mathbf{u}) = 0, \quad \mathbf{u} \geq 0 \iff \mathbf{u} = \mathbf{u}^*. \end{cases} \tag{6}$$

The lack of structure makes conventional approaches such as variational analysis, (1) hard to use in analyzing (1). We consider in this paper the 1D case, i.e., the system

$$\begin{cases} -[K_{11}(\mathbf{u})u_x + K_{12}(\mathbf{u})v_x]_x = u\,g_1(\mathbf{u}), & 0 < x < \ell, \\ [K_{21}(\mathbf{u})u_x - (K_{22}(\mathbf{u})v_x]_x = v\,g_2(\mathbf{u}), & 0 < x < \ell, \\ u_x = v_x = 0, & x = 0,\ \ell. \end{cases} \tag{7}$$

The approach we use is a bifurcation technique using Leray-Schauder degree theory (incorporated with hard a priori estimates). See also [1, 2].

The organization of this paper is as follows. In §2, we shall establish a priori upper and lower bounds for positive solutions of (7). In §3, we study the non–existence of non-constant positive solutions. In particular, we show that

1. (7) *has no positive non–constant solution if* $\ell \ll 1$ *or if both* K_{11} *and* K_{22} *are large.*

Finally, in §4, we show that (with $\mathbf{u}^* = (1,1)$)

2. *If $K_{11}(\mathbf{u}^*) < \varphi'(1)\ell^2/\pi^2$ and $K_{22}(\mathbf{u}^*)$ is large enough, then* (7) *admits a positive solution.*

2. **Upper and Lower Bounds for Positive Solutions.**

Theorem 1. *Assume* (2)–(5).

(i) *There exists a positive constant C_1, which depends only on φ, ψ, M_0 and ℓ, such that any positive $C^2([0,\,\ell])$ solution $\mathbf{u} = (u, v)^T$ to* (7) *satisfies*

$$\max\{u(x),\, v(x)\} < C_1 \qquad \forall\ 0 \le x \le \ell. \tag{8}$$

(ii) *Assume in addition that $\psi(0) \neq L$. Then there exists a positive constant C_2, which depends only on ℓ, φ, ψ, M_0 and the norm $\|\mathbf{K}\|_{C^2([0,C_1]^2)}$, such that any positive $C^2([0,\,\ell])$ solution $\mathbf{u} = (u, v)^T$ to* (7) *satisfies*

$$\min\{u(x),\, v(x)\} > C_2 \qquad \forall\ 0 \le x \le \ell. \tag{9}$$

We first state without proof a simple lemma which will be needed to prove (9).

Lemma 1. *Assume that $\mathbf{K}_m(\mathbf{u})$, $m = 1, 2, 3, \cdots$, satisfy* (4)*, and the corresponding positive solution $\mathbf{u}_m = (u_m,\, v_m)^T$ of* (7) *satisfies $\mathbf{u}_m \to \hat{\mathbf{u}} = (\hat{u},\, \hat{v})^T$ as $m \to \infty$. If $\hat{\mathbf{u}}$ is constant vector, then $g_1(\hat{\mathbf{u}}) = 0 = g_2(\hat{\mathbf{u}})$.*

Proof of Theorem 1. Let $\mathbf{u}$ be a positive solution. Integrating (7) over $[0,\,\ell]$ gives

$$\int_0^\ell u\varphi(u)\,dx = \int_0^\ell uv\,dx = \int_0^\ell v\,\psi(v)\,dx. \tag{10}$$

This implies

$$\int_{\{u\ge L\}} u|\varphi(u)|\,dx + \int_0^\ell uv\,dx = \int_{\{u<L\}} u\varphi(u)\,dx \le \ell C_0, \tag{11}$$

where $C_0 = \max_{0\le u\le L} u|\varphi(u)|$. Consequently, by (2) and (3), $\int_0^\ell v\psi(v)\,dx = \int_0^\ell uv \le \ell C_0$ and

$$\min_{[0,\,\ell]} u \le \tilde{C}, \quad \min_{[0,\,\ell]} v \le \tilde{C}. \tag{12}$$

Now we can integrate (7) from x to ℓ and using (10) and (11) to obtain,

$$\begin{cases} K_{11}(\mathbf{u})u_x + K_{12}(\mathbf{u})v_x = \displaystyle\int_x^\ell ug_1(\mathbf{u})\,dx \in (-\ell C_0,\, \ell C_0), \\ K_{21}(\mathbf{u})u_x + K_{22}(\mathbf{u})v_x = \displaystyle\int_x^\ell vg_2(\mathbf{u})\,dx \in (-\ell C_0,\, \ell C_0). \end{cases} \tag{13}$$

In view of (5) we then conclude that

$$-\ell C_0 M_0 \le u_x(x),\ \ v_x(x) \le \ell C_0 M_0 \qquad \forall\ x \in [0,\,\ell].$$

Together with (12) this gives $\|u\|_\infty + \|v\|_\infty \le C_1$ for some positive constant C_1. (i) is proved.

To prove (ii), we use a contradiction argument. Suppose on the contrary that (9) does not hold. Then there exist a constant $M > 0$ and a sequence $\{\mathbf{K}_m, \mathbf{u}_m\}_{m=1}^\infty$ such that for each $m \ge 1$, $\mathbf{K}_m$ satisfies (4), (5) and $\|\mathbf{K}_m\|_{C^2([0,\,C_1]^2)} \le M$, and $\mathbf{u}_m$ is a positive solution to (7) with $\mathbf{K}$ replaced by $\mathbf{K}_m$, and $\min_{[0,\,\ell]}\{u_m,\, v_m\} \to 0$ as $m \to \infty$. From (i) and (13) we see that $\sup_m \|\mathbf{u}_m\|_{C^2([0,\,\ell]^2)}$ is bounded. Hence, by passing to a subsequence if necessary, we can find $\mathbf{u}$ and $\mathbf{K}$ such that $(\mathbf{K}_m,\, \mathbf{u}_m) \to (\mathbf{K},\, \mathbf{u})$ in

$C^1([0,\, C_1]^2 \to R^4) \times C^1([0,\, \ell] \to R^2)$ as $m \to \infty$. Then $\mathbf{u} = (u,\, v)$ is a non-negative solution of (7), $\mathbf{K_u}$ is Lipschitz continuous, and $\min\{\min_{[0,\,\ell]} u,\, \min_{[0,\,\ell]} v\} = 0$.

Suppose $\min_{[0,\,\ell]} u = 0$. Then there exists $x_0 \in [0,\, \ell]$ such that $u(x_0) = 0 = u'(x_0)$. Write the first equation of (7) as, for $0 < x < \ell$,

$$-(K_{11}(\mathbf{u})u_x)_x = u g_1(\mathbf{u}) + K_{12}(\mathbf{u})v_{xx} + \frac{\partial K_{12}(\mathbf{u})}{\partial v} v_x^2 + \frac{\partial K_{12}(\mathbf{u})}{\partial u} u_x v_x.$$

Since $K_{12}(0,\, v) = 0$ and $\frac{\partial K_{12}(0,\, v)}{\partial v} = 0$, from a uniqueness result of ODE (with initial value at x_0) we see that $u \equiv 0$. Consequently,

$$-(K_{22}(0,\, v)v_x)_x = -v\,\psi(v) \ \text{ in } \ (0,\, \ell), \quad v_x(0) = v_x(\ell) = 0.$$

Since $v \geq 0$ and $\psi(s) > 0$ for $s \geq 0$ (see (3)), it yields $v \equiv 0$. That is, $\mathbf{u}_m \to (0,\, 0)^T$ as $m \to \infty$. It contradicts Lemma 1 since $g_1(0, 0) > 0$.

If $\min_{[0,\,\ell]} v = 0$, then similarly we have $v \equiv 0$. Hence u satisfies

$$-(K_{11}(u,\, 0)u_x)_x = u\varphi(u) \ \text{ in } \ (0,\, \ell), \quad u_x(0) = u_x(\ell) = 0.$$

In view of (2), one can show by a maximum principle that $u \equiv 0$ or $u \equiv L$, i.e. $\mathbf{u}_m \to (0,\, 0)$ or $(L,\, 0)$ as $m \to \infty$. This again contradicts Lemma 1 since $g_2(L, 0) = L - \psi(0) \neq 0$. The proof is complete. □

3. **Non-existence of Non-constant Positive Solutions.**

Theorem 2. *Assume* (2)–(5). *There exists $\ell_0 > 0$ such that if $\ell \in (0,\, \ell_0)$, then* (1) *does not have any non-constant positive solution.*

Proof. Let $\mathbf{u}$ be an arbitrary positive solution. From $\int_0^\ell u\, g_1(\mathbf{u})\, dx = \int_0^\ell v\, g_2(\mathbf{u})\, dx = 0$ we see that there exist $x_1,\, x_2 \in [0,\, \ell]$ such that $g_1(\mathbf{u}(x_1)) = 0 = g_2(\mathbf{u}(x_2))$. It then follows from (13) that

$$\begin{aligned} \|\mathbf{u}_x\| &\leq C \int_0^\ell [\|g_1(\mathbf{u}) - g_1(\mathbf{u}(x_1))\| + \|g_2(\mathbf{u}) - g_2(\mathbf{u}(x_2))\|]\, dx \\ &\leq \ell C[\|\mathbf{u} - \mathbf{u}(x_1)\|_\infty + \|\mathbf{u} - \mathbf{u}(x_2)\|_\infty] \leq \ell^2 C \|\mathbf{u}_x\|_\infty. \end{aligned}$$

Hence, $\mathbf{u}_x \equiv 0$ if ℓ is small enough. □

Next, we will prove that if $K_{11}(\mathbf{u})$ and $K_{22}(\mathbf{u})$ are large then (7) has no positive non–constant solution. To this end, we set

$$K_{11}(\mathbf{u}) = d_1 + \tilde{K}_{11}(\mathbf{u}), \qquad K_{22}(\mathbf{u}) = d_2 + \tilde{K}_{22}(\mathbf{u})$$

with large positive constants d_1, d_2 and fixed positive functions $\tilde{K}_{11}(\mathbf{u})$, $\tilde{K}_{22}(\mathbf{u})$. If $d_1,\, d_2 \geq 1$, then the constant M_0 in (5) does not depend on d_1 and d_2. Therefore, the positive constant C_1 given by Theorem 1 does not depend on d_1 and d_2.

Theorem 3. *There exists $d_0 \geq 1$ such that if $d_1,\, d_2 \geq d_0$, then* (7) *does not have any non-constant positive solution.*

Proof. Let $\mathbf{u}$ be an arbitrary positive solution to (7), and $\bar{u}$ and $\bar{v}$ be the average of u and v over $(0,\, \ell)$. Multiplying (7) by $(u - \bar{u},\, v - \bar{v})$ and integrating the result over $(0, \ell)$ gives

$$\begin{aligned}
&\int_0^\ell [K_{11}(\mathbf{u})u_x^2 + K_{22}(\mathbf{u})v_x^2 + (K_{12}(\mathbf{u}) - K_{21}(\mathbf{u}))u_x v_x]dx \\
&= \int_0^\ell [(u-\bar{u})G_1(\mathbf{u}) + (v-\bar{v})G_2(\mathbf{u})]\,dx \\
&= \int_0^\ell \{(u-\bar{u})[G_1(\mathbf{u}) - G_1(\bar{\mathbf{u}})] + (v-\bar{v})[G_2(\mathbf{u}) - G_2(\bar{\mathbf{u}})]\}\,dx \\
&\le C\int_0^\ell [(u-\bar{u})^2 + (v-\bar{v})^2]\,dx
\end{aligned}$$

by (8) and a mean value theorem. Note that

$$K_{11}(\mathbf{u}) = d_1 + \tilde{K}_{11}(\mathbf{u}) > d_1,\ K_{22}(\mathbf{u}) = d_2 + \tilde{K}_{22}(\mathbf{u}) > d_2,$$

and (8) implies that

$$|K_{12}(\mathbf{u}) - K_{21}(\mathbf{u})| \le \|K_{12}(\mathbf{u}) - K_{21}(\mathbf{u})\|_{C([0,\,\ell]^2)} \equiv M.$$

Thus

$$K_{11}(\mathbf{u})u_x^2 + [K_{12}(\mathbf{u}) - K_{21}(\mathbf{u})]u_x v_x + K_{22}(\mathbf{u})v_x^2 \ge d_1\,u_x^2 + d_2 v_x^2 - \frac{M}{2}(u_x^2 + v_x^2).$$

Hence, $u \equiv \bar{u}$, $v \equiv \bar{v}$ if d_1 and d_2 are large enough. The proof is completed. □

We remark that in Theorem 3, d_1 and d_2 can be of different large sizes.

4. **Existence of Non–constant Positive Solutions.** In this section we shall establish the existence of non–constant positive solutions to (7), under the assumption (2)–(6).

Denote $a = \varphi'(1)$, $b = \psi'(1)$, and $k_{ij} = K_{ij}(1,1)$. Then $ab < 1$ by (6), and

$$\mathbf{G_u}(\mathbf{u}^*) = \begin{pmatrix} a & -1 \\ 1 & -b \end{pmatrix}, \quad \mathbf{K}(\mathbf{u}^*) = \begin{pmatrix} k_{11} & k_{12} \\ -k_{21} & k_{22} \end{pmatrix}.$$

We choose a working space $\mathbf{X} = \{\mathbf{u}\,|\,\mathbf{u} \in [C^1([0,\,\ell])]^2,\ \mathbf{u}_x(0) = \mathbf{u}_x(\ell) = 0\,\}$. Let $\{(\mu_i,\,\phi_i)\}_{i=0}^\infty = \{((i\pi/\ell)^2,\,\cos(i\pi/\ell))\}_{i=0}^\infty$ be a complete set of eigen-pairs of the eigenvalue problem

$$-\,\Delta\phi = -\phi'' = \mu\,\phi \quad \text{in}\ \ (0,\,\ell), \qquad \phi'(0) = \phi'(\ell) = 0.$$

Decompose $\mathbf{X} = \oplus_{i=0}^\infty \mathbf{X}_i$, where $\mathbf{X}_i := \{\mathbf{c}\phi_i \mid \mathbf{c} \in \mathbf{R}^2\,\}$. Define

$$H(\mu,\,\vec{k}\,) = (k_{11}k_{22} + k_{12}k_{21})\mu^2 + (bk_{11} + k_{12} + k_{21} - ak_{22})\mu + 1 - ab.$$

and $\vec{k} = (k_{11},\,k_{22},\,k_{12},\,k_{21})$.

Theorem 4. *If $H(\mu_m,\,\vec{k}\,) = H((m\pi/\ell)^2,\,\vec{k}\,) < 0$ for some $m \ge 1$, then (7) has at least one non-constant positive solution.*

To prove Theorem 4, we first prove a Lemma.

Lemma 2. *If $H((\pi/\ell)^2,\,\vec{k}\,) < 0 < H((m\pi/\ell)^2,\,\vec{k}\,)$ for all $m \ge 2$, then (7) has at least one non-constant positive solution.*

Proof. Let d be a large positive constant and for $t \in [0,\, 1]$, define

$$\hat{\mathbf{K}}(\mathbf{u};\, t) = \begin{pmatrix} t\, K_{11}(\mathbf{u}) + (1-t)d & t\, K_{12}(\mathbf{u}) \\ -t\, K_{21}(\mathbf{u}) & t\, K_{22}(\mathbf{u}) + (1-t)d \end{pmatrix}.$$

Consider

$$-\,\hat{\mathbf{K}}(\mathbf{u};\, t)\mathbf{u}_{xx} = \mathbf{G}(\mathbf{u}) + \mathbf{u}_x^T \hat{\mathbf{K}}_{\mathbf{u}}(\mathbf{u};\, t)\mathbf{u}_x, \quad \text{in} \quad (0,\, \ell), \tag{14}$$

with $\mathbf{u}_x(0) = \mathbf{u}_x(\ell) = 0$. Then $\mathbf{u}$ is a solution of (7) if and only if it is a solution of (14) for $t = 1$. Also for any $0 \le t \le 1$, $\mathbf{u}$ is a solution of (14) if and only if it solves

$$h(\mathbf{u};\, t) := \mathbf{u} - (\mathbf{I} - \Delta)^{-1}\{\hat{\mathbf{K}}^{-1}(\mathbf{u},\, t)[\mathbf{G}(\mathbf{u}) + \mathbf{u}_x^T \hat{\mathbf{K}}_{\mathbf{u}}(\mathbf{u};\, t)\mathbf{u}_x] + \mathbf{u}\} = 0, \tag{15}$$

where $(\mathbf{I} - \Delta)^{-1}$ is the inverse of $\mathbf{I} - \Delta$ in $\mathbf{X}$. By (6), $\mathbf{u} \equiv \mathbf{u}^*$ is the unique positive constant solution of (15). Direct computation gives

$$D_{\mathbf{u}} h(\mathbf{u}^*;\, t) = \mathbf{I} - (\mathbf{I} - \Delta)^{-1}\{\hat{\mathbf{K}}^{-1}(\mathbf{u}^*,\, t)\mathbf{G}_{\mathbf{u}}(\mathbf{u}^*) + \mathbf{I}\}.$$

In particular,

$$\begin{aligned} D_{\mathbf{u}} h(\mathbf{u}^*;\, 0) &= \mathbf{I} - (\mathbf{I} - \Delta)^{-1}\{\hat{\mathbf{K}}^{-1}(\mathbf{u}^*,\, 0)\mathbf{G}_{\mathbf{u}}(\mathbf{u}^*) + \mathbf{I}\} \\ &= \mathbf{I} - (\mathbf{I} - \Delta)^{-1}\left\{\mathbf{I} + \begin{pmatrix} d^{-1} & 0 \\ 0 & d^{-1} \end{pmatrix}\begin{pmatrix} a & -1 \\ 1 & -b \end{pmatrix}\right\}, \\ D_{\mathbf{u}} h(\mathbf{u}^*;\, 1) &= \mathbf{I} - (\mathbf{I} - \Delta)^{-1}\{\hat{\mathbf{K}}^{-1}(\mathbf{u}^*,\, 1)\mathbf{G}_{\mathbf{u}}(\mathbf{u}^*) + \mathbf{I}\} \\ &= \mathbf{I} - (\mathbf{I} - \Delta)^{-1}\{\mathbf{K}^{-1}(\mathbf{u}^*)\mathbf{G}_{\mathbf{u}}(\mathbf{u}^*) + \mathbf{I}\}. \end{aligned}$$

Let N_0 and N_1 be the numbers of negative eigenvalues (counting multiplicity) of $D_{\mathbf{u}} h(\mathbf{u}^*;\, 0)$ and $D_{\mathbf{u}} h(\mathbf{u}^*;\, 1)$ respectively. We first calculate N_1. Since $\mathbf{X} = \oplus_{i=0}^{\infty}\mathbf{X}_i$, and for each i, $\mathbf{X}_i$ is invariant for $D_{\mathbf{u}} h(\mathbf{u}^*;\, 1)$ and the number of negative eigenvalues of $D_{\mathbf{u}} h(\mathbf{u}^*;\, 1)$ on $\mathbf{X}_i$ is the same as that of the matrix $\mathbf{I} - (1+\mu_i)^{-1}[\hat{\mathbf{K}}^{-1}(\mathbf{u}^*;\, 1)\mathbf{G}_{\mathbf{u}}(\mathbf{u}^*) + \mathbf{I}]$ or the matrix $\mu_i \mathbf{I} - \hat{\mathbf{K}}^{-1}(\mathbf{u}^*;\, 1)\mathbf{G}_{\mathbf{u}}(\mathbf{u}^*) = \mu_i \mathbf{I} - \mathbf{K}^{-1}(\mathbf{u}^*)\mathbf{G}_{\mathbf{u}}(\mathbf{u}^*)$. Hence, the number of negative eigenvalues of $D_{\mathbf{u}} h(\mathbf{u}^*;\, 1)$ on $\mathbf{X}_i$ is the same as, denoting $\operatorname{sgn}\{z\}$ the sign of z,

$$\begin{aligned} \sigma_{1,i} &= \tfrac{1}{2}(1 - \operatorname{sgn}\{\det[\mu_i \mathbf{I} - \mathbf{K}^{-1}(\mathbf{u}^*)\mathbf{G}_{\mathbf{u}}(\mathbf{u}^*)]\}) \\ &= \tfrac{1}{2}(1 - \operatorname{sgn}\{\det[\mu_i \mathbf{K}(\mathbf{u}^*) - \mathbf{G}_{\mathbf{u}}(\mathbf{u}^*)]\}) = \tfrac{1}{2}(1 - \operatorname{sgn}\{H(\mu_i,\, \vec{k}\,)\}) \\ &= \begin{cases} 1 & \text{if } \ i = 1, \\ 0 & \text{if } \ i \ne 1 \end{cases} \end{aligned}$$

by our assumption. Similarly, denoting $\vec{d} = (d,\, d,\, 0,\, 0)$, the number of negative eigenvalues of $D_{\mathbf{u}} h(\mathbf{u}^*;\, 0)$ on $\mathbf{X}_i$ is the same as

$$\sigma_{0,i} = \frac{1}{2}(1 - \operatorname{sgn}\{H(\mu_i,\, \vec{d}\,)\}) = 0$$

for all i provided that $d \gg 1$. As $\mathbf{X} = \oplus_{i=0}^{\infty}\mathbf{X}_i$, $N_1 = \sum_{i=0}^{\infty}\sigma_{1,i} = 1$ and $N_0 = \sum_{i=0}^{\infty}\sigma_{0,i} = 0$.

Now, we fix d large such that, by Theorem 3, $\mathbf{u} = \mathbf{u}^*$ is the only positive solution to (15) with $t = 0$. By Theorem 1, there exist positive constants C_1 and C_2 such that the positive solutions of (15), for all $t \in [0,\, 1]$, satisfy $C_2 < u,\, v < C_1$. Set

$$\Omega = \left\{(u,\, v) \in \mathbf{X} \;\middle|\; C_2 < u,\, v < C_1\right\}.$$

Then $h(\mathbf{u};\,t) \neq 0$ for all $\mathbf{u} \in \partial\Omega$ and $t \in [0,1]$. By the homotopy invariance of degree ([5]),

$$\deg(h(\cdot\,;\,0),\,0,\,\Omega) = \deg(h(\cdot\,;\,1),\,0,\,\Omega). \tag{16}$$

Since $\mathbf{u} = \mathbf{u}^*$ is the only positive solution to $h(\mathbf{u};\,0) = 0$ in $\mathbf{X}$,

$$\deg(h(\cdot\,;\,0),\,0,\,\Omega) = \mathrm{index}(h(\cdot\,;\,0);\mathbf{u}^*) = (-1)^{N_0} = 1. \tag{17}$$

Now, if (7) did not have non-constant positive solutions, then $\mathbf{u}^*$ would be the unique solution of $h(\mathbf{u};\,1) = 0$ in Ω. Hence

$$\deg(h(\cdot\,;\,1),\,0,\,\Omega) = \mathrm{index}(h(\cdot\,;\,1),\,\mathbf{u}^*) = (-1)^{N_1} = -1.$$

This contradicts (16) and (17), and then our lemma is proved. □

Proof of Theorem 4. . Without loss of generality we can assume that m is the largest integer such that $H(\mu_m,\,\vec{k}\,) < 0$. Since $H(\mu,\,\vec{k}\,)$ is quadratic, $H(\mu_n,\,\vec{k}\,) > 0$ for all $n \geq m+2$. If $m = 1$ and $H(\mu_2,\,\vec{k}\,) = H((2\pi/\ell)^2,\,\vec{k}\,) > 0$, then the assertion follows from Lemma 2. If $m > 1$, dividing the interval $[0,\,\ell]$ into m parts: $[0,\,\ell] = \cup_{i=1}^m[(i-1)\ell/m,\,i\ell/m]$. We consider the problem (7) with ℓ replaced by ℓ/m. Denote $\{\mu_i^{(m)}\}_{i=0}^\infty$ the set of eigenvalues of $-\Delta$ in $(0,\,\ell/m)$ with homogeneous Neumann boundary conditions. Then

$$\mu_1^{(m)} = [\pi/(\ell/m)]^2 = (m\pi/\ell)^2 = \mu_m, \qquad \mu_2^{(m)} = (2m\pi/\ell)^2 \geq \mu_{m+2}.$$

Thus, $H(\mu_1^{(m)},\,\vec{k}\,) < 0 < H(\mu_i^{(m)},\,\vec{k}\,)$ for all $i \geq 2$. Applying Lemma 2 we see that the problem (7), with ℓ replaced by ℓ/m, has at least one non-constant positive solution, say $\tilde{\mathbf{u}}$. Extending $\tilde{\mathbf{u}}$ first evenly over $[0,\,2\ell/m]$ and then periodically with period $2\ell/m$, we obtain a non-constant positive solution of (7) in $(0,\,\ell)$.

It remains to consider the case $m = 1$ and $H(\mu_2,\,\vec{k}\,) = 0$, i.e., $H((\pi/\ell)^2,\,\vec{k}\,) < 0 = H((2\pi/\ell)^2,\,\vec{k}\,)$. Since $H(\mu,\,\vec{k}\,)$ is quadratic, $\frac{d}{d\mu}H(\mu,\,\vec{k}\,)|_{\mu=(2\pi/\ell)^2} \neq 0$. Moreover,

$$H((\pi/\tilde{\ell})^2,\,\vec{k}\,) < 0 < H((m\pi/\tilde{\ell})^2,\,\vec{k}\,) \quad \forall\, m \geq 2, \quad \tilde{\ell} \in (\ell/2,\,\ell).$$

Since with $\tilde{\ell} = \ell/2$, $H((\pi/\tilde{\ell})^2,\,\vec{k}\,) = 0 < H((m\pi/\tilde{\ell})^2,\,\vec{k}\,)$ for all $m = 0,\,2,\,\cdots$, $(\ell/2,\,\mathbf{u}^*)$ is a bifurcation point. By a local bifurcation theory, all solutions near $(\ell/2,\,\mathbf{u}^*)$ form a smooth curve $\mathbf{C}_1$. Similarly, $(\ell,\,\mathbf{u}^*)$ is also a bifurcation point, and the bifurcation curve $\mathbf{C}_2$ is obtained from $\mathbf{C}_1$ through the relation $(\tilde{\ell}/2,\,\mathbf{u}(x)) \to (\tilde{\ell},\,\mathbf{u}(x))$ (even extension of $\mathbf{u}$ is a default here).

We consider two cases: (i) $(\ell,\,\mathbf{u}^*) \notin \mathbf{C}_1$; (ii) $(\ell,\,\mathbf{u}^*) \in \mathbf{C}_1$.

In case (i), we conclude from a global bifurcation theory (since we have a priori bound for all positive solutions in $\mathbf{X}$) that $(\ell,\,\hat{\mathbf{u}}) \in \mathbf{C}_1$ for some $\hat{\mathbf{u}} \in \mathbf{X}$, $\hat{\mathbf{u}} \not\equiv \mathbf{u}^*$. Hence $\hat{\mathbf{u}}$ is a non-constant positive solution to (15) with $\tilde{\ell} = \ell$.

In case (ii), $\mathbf{C}_1$ connects $\mathbf{C}_2$. As $\mathbf{C}_2$ is smooth near $(\ell,\,\mathbf{u}^*)$ and is obtained from $\mathbf{C}_1$ via even extension, we see that there exists $\delta > 0$ such that (15) admits even solution for all $\tilde{\ell} \in (\ell-\delta,\,\ell)$. Thus, $\mathbf{C}_1 \cap \{(0,\,\ell/2) \times \Omega\} \neq \emptyset$. As $\mathbf{C}_1$ can only intersect $(\ell/2,\,\mathbf{u}^*)$ once, by a global bifurcation theory, there exists $\tilde{\mathbf{u}} \in \Omega \setminus \mathbf{u}^*$ such that $(\ell/2,\,\tilde{\mathbf{u}}) \in \mathbf{C}_1$. Evenly extending $\tilde{\mathbf{u}}$ we then obtain a non-constant positive solution to (15) with $\tilde{\ell} = \ell$. The proof is completed. □

We remark that through a lengthy argument it can be shown that indeed case (ii) cannot happen. We omit the rigorous proof here for simplicity.

Theorem 5. *Assume that* $a := \varphi'(1) > 0$ *and* $k_{11} := K_{11}(\mathbf{u}^*) < a\ell^2/\pi^2$. *Let* k_{12} *and* k_{21} *be fixed. Then there exists a positive constant* C*, which depends on* a, b *and* k_{ij}, $(i,\, j) \neq (2,\, 2)$*, such that* (7) *has at least one non–constant positive solution provided that* $k_{22} := K_{22}(\mathbf{u}^*) \geq C$.

Proof. Note that

$$\begin{aligned} H(\mu_1,\, \vec{k}\,) \quad &= \quad \mu_1^2\{k_{22}(k_{11} - a/\mu_1) + k_{12}k_{21} \\ &\quad +[bk_{11} + k_{12} + k_{21}]/\mu_1 + (1-ab)/\mu_1^2\}. \end{aligned} \tag{18}$$

Since $k_{11} < a/\mu_1$, there exists a positive constant C such that $H(\mu_1,\, \vec{k}\,) < 0$ provided that $k_{22} \geq C$. The assertion of the theorem thus follows from Theorem 4. □

Remark 1. For competition models, the cross diffusion coefficients satisfy $k_{12}k_{21} < 0$. We see from (18) that cross diffusion may help the formation of patterns (non constant solutions); see [1, 2, 3]. On the other hand, for the predator-prey model considered in this paper, cross-diffusions seem to perform a role preventing the formation of patterns.

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