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# WEIGHTED HARDY-LITTLEWOOD-SOBOLEV INEQUALITIES AND SYSTEMS OF INTEGRAL EQUATIONS

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**Abstract.** In this paper, we consider systems of integral equations related to the weighted Hardy-Littlewood-Sobolev inequality. We present the symmetry, monotonity, and regularity of the solutions. In particular, we obtain the optimal integrability of the solutions to a class of such systems. We also present a simple method for the study of regularity, which has been extensively used in various forms. The version we present here contains some new developments. It is much more general and very easy to use. We believe the method will be helpful to both experts and non-experts in the field.

1. **Introduction.** Let $0 < \lambda < n$ and let $1 < s, r < \infty$ such that $\frac{1}{r} + \frac{1}{s} + \frac{\lambda}{n} = 2$. Let $\|f\|_p$ be the $L^p(R^n)$ norm of the function $f$. The well-known classical Hardy-Littlewood-Sobolev inequality (HLS) states that:

$$\int_{R^n} \int_{R^n} \frac{f(x)g(y)}{|x-y|^\lambda} dxdy \leq C_{s,\lambda,n} \|f\|_r \|g\|_s \tag{1}$$

for any $f \in L^r(R^n)$ and $g \in L^s(R^n)$.

Hardy and Littlewood also introduced the double weighted inequality, which was generalized by Stein and Weiss in [23]. This inequality is called double weighted Hardy-Littlewood-Sobolev (WHLS) inequality:

$$\left| \int_{R^n} \int_{R^n} \frac{f(x)g(y)}{|x|^\alpha |x-y|^\lambda |y|^\beta} dxdy \right| \leq C_{\alpha,\beta,s,\lambda,n} \|f\|_r \|g\|_s \tag{2}$$

where $1 < r, s < \infty$, $0 < \lambda < n$, $\alpha + \beta \geq 0$ and the powers $\alpha, \beta$ of the weights satisfy

$$1 - \frac{1}{r} - \frac{\lambda}{n} < \frac{\alpha}{n} < 1 - \frac{1}{r}, \quad \text{and } \frac{1}{r} + \frac{1}{s} + \frac{\lambda + \alpha + \beta}{n} = 2. \tag{3}$$

To obtain the best constant in the weighted inequality (2), one can maximize the functional

$$J(f,g) = \int_{R^n} \int_{R^n} \frac{f(x)g(y)}{|x|^\alpha |x-y|^\lambda |y|^\beta} dxdy \tag{4}$$

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under the constraints $\|f\|_r = \|g\|_s = 1$. The corresponding Euler-Lagrange equations are the following system of integral equations:

$$\begin{cases} \lambda_1 r f(x)^{r-1} = \frac{1}{|x|^\alpha} \int_{R^n} \frac{g(y)}{|y|^\beta |x-y|^\lambda} dy \\ \lambda_2 s g(x)^{s-1} = \frac{1}{|x|^\beta} \int_{R^n} \frac{f(y)}{|y|^\alpha |x-y|^\lambda} dy \end{cases} \tag{5}$$

where $f, g \geq 0,\ x \in R^n$ and $\lambda_1 r = \lambda_2 s = J(f, g)$.

Letting $u = c_1 f^{r-1}$, $v = c_2 g^{s-1}$, $p = \frac{1}{r-1}$, $q = \frac{1}{s-1}$, when $pq \neq 1$, and by a proper choice of constants $c_1$ and $c_2$, system (5) becomes

$$\begin{cases} u(x) = \frac{1}{|x|^\alpha} \int_{R^n} \frac{v(y)^q}{|y|^\beta |x-y|^\lambda} dy \\ v(x) = \frac{1}{|x|^\beta} \int_{R^n} \frac{u(y)^p}{|y|^\alpha |x-y|^\lambda} dy \end{cases} \tag{6}$$

where $u, v \geq 0$, $0 < p, q < \infty$, $0 < \lambda < n$, $\frac{\alpha}{n} < \frac{1}{p+1} < \frac{\lambda+\alpha}{n}$, and $\frac{1}{p+1} + \frac{1}{q+1} = \frac{\lambda+\alpha+\beta}{n}$.

In the special case, where $\alpha = 0$ and $\beta = 0$, system (6) reduces to

$$\begin{cases} u(x) = \int_{R^n} \frac{v^q(y)}{|x-y|^\lambda} dy \\ v(x) = \int_{R^n} \frac{u^p(y)}{|x-y|^\lambda} dy \end{cases} \tag{7}$$

with

$$\frac{1}{q+1} + \frac{1}{p+1} = \frac{\lambda}{n}. \tag{8}$$

The integral system is closely related to the system of partial differential equations

$$\begin{cases} (-\Delta)^{\gamma/2} u = v^q, \ \ u > 0, \text{ in } R^n, \\ (-\Delta)^{\gamma/2} v = u^p, \ \ v > 0, \text{ in } R^n, \end{cases} \tag{9}$$

where $\gamma = n - \lambda$

In the special case where $p = q = \frac{n+\gamma}{n-\gamma}$, and $u(x) = v(x)$, the system (7) becomes:

$$u(x) = \int_{R^n} \frac{u(y)^{\frac{n+\gamma}{n-\gamma}}}{|x-y|^{n-\gamma}} dy, \ \ u > 0 \text{ in } R^n. \tag{10}$$

The corresponding PDE is the well-known family of semi-linear equations

$$(-\Delta)^{\gamma/2} u = u^{(n+\gamma)/(n-\gamma)}, \ \ u > 0, \quad \text{in } R^n \tag{11}$$

In particular, when $n \geq 3$, and $\gamma = 2$, (11) becomes

$$-\Delta u = u^{(n+2)/(n-2)}, \ \ u > 0, \text{ in } R^n. \tag{12}$$

The classification of the solutions of (12) has provided an important ingredient in the study of the well-known Yamabe problem and the prescribing scalar curvature problem. It is also essential in deriving a priori estimates in many related nonlinear elliptic equations.

Solutions to (12) were studied by Gidas, Ni, and Nirenberg [14]. They proved that all the positive solutions of (12) with reasonable behavior at infinity, namely

$$u(x) = O(\frac{1}{|x|^{n-2}}) \tag{13}$$

are radially symmetric about some point and therefore assume the form of

$$c(\frac{t}{t^2 + |x - x_o|^2})^{(n-2)/2} \tag{14}$$

with some positive constants c and t.

Later, in [5], Caffarelli, Gidas, and Spruck removed the growth condition (13) and obtained the same result. Then Chen and Li [7], and Li [17] simplified their proof. Recently, Wei and Xu [25] generalized this result to the solutions of more general equation (11) with $\alpha$ being any even numbers between 0 and n.

Apparently, for other real values of $\alpha$ between 0 and n, equation (11) is also of practical interest and importance. For instance, it arises as the Euler-Lagrange equation of the functional

$$I(u) = \int_{R^n} |(-\Delta)^{\frac{\gamma}{4}} u|^2 dx / (\int_{R^n} |u|^{\frac{2n}{n-\gamma}} dx)^{\frac{n-\gamma}{n}}.$$

The classification of the solutions would provide the best constant in the inequality of the critical Sobolev imbedding from $H^{\frac{\gamma}{2}}(R^n)$ to $L^{\frac{2n}{n-\gamma}}(R^n)$:

$$(\int_{R^n} |u|^{\frac{2n}{n-\gamma}} dx)^{\frac{n-\gamma}{n}} \leq C \int_{R^n} |(-\Delta)^{\frac{\gamma}{4}} u|^2 dx.$$

In [18], Lieb classified all the maximizers of the functional (1) under the constraints $\|f\|_r = 1 = \|g\|_s$ in the special case where $p = q = \frac{n+\gamma}{n-\gamma}$, and thus obtained the best constant in the HLS inequalities in that case. He then posed the classification of all the critical points of the functional – the solutions of the integral equation (10) as an open problem.

Chen, Li, and Ou solved this open problem in [9]. They proved:

**Proposition 1.** *All solutions of partial differential equation (11) satisfy the integral equation (10), and vise versa. Every positive solution* $u(x) \in L_{loc}^{\frac{2n}{n-\gamma}}(R^n)$, $0 < \gamma < n$, *of (10) or (11) is radially symmetric and decreasing about some point* $x_o$ *and therefore assumes the form of (14).*

This proposition unifies and extends all the previous results on the family of partial differential equations (11).

Then in [10], Chen, Li, and Ou considered more general system (7) and obtained the symmetry and monotonicity of the solutions.

**Proposition 2.** *Let* $(u, v)$ *be a pair of solutions of (7) and* $p, q \geq 1$. *Assume that* $u \in L^{p+1}(R^n)$ *and* $v \in L^{q+1}(R^n)$. *Then* $u$ *and* $v$ *are radially symmetric and decreasing about some point* $x_o$.

In [8], Chen and Li also obtained the regularity of the solutions.

**Proposition 3.** *Assume that* $(u(x), v(x))$ *is a pair of positive solutions of (7), and* $u \in L^{p+1}(R^n)$ *and* $v \in L^{q+1}(R^n)$. *Then* $u(x)$ *and* $v(x)$ *are uniformly bounded in* $R^n$.

It follows from Proposition 3, the standard integration theory, and regularity argument that $u(x)$ and $v(x)$ are continuous and hence smooth everywhere.

To establish the symmetry of the solution, Chen, Li, and Ou [9] [10] introduced a new idea, an integral form of the method of moving planes. It is entirely different from the traditional method used for partial differential equations. Instead of relying on maximum principles, certain integral norms were estimated. We believe that this new idea will become a powerful tool in studying qualitative properties of other integral equations and systems.

Following Chen, Li, and Ou's work, Jin and Li [15] studied the symmetry of the solutions to the more general system (6).

**Proposition 4.** *Let the pair (u, v) be a positive solution of system (6) with* $u \in L^{p+1}, v \in L^{q+1}$ *and* $p,q \geq 1, pq \neq 1$, *and* $\alpha, \beta \geq 0$. *Then* $u$ *and* $v$ *are radially symmetric and decreasing about some point* $x_o$.

Jin and Li [16] also thoroughly discussed the regularity of the solutions to (6).

In this paper, we first present a simple method to study regularity of solutions. It has been used extensively in various forms in the authors previous works. The essence of the approach is well-known in the analysis community. However, the version we present here contains some new developments. It is much more general and is very easy to use. We believe that our method will provide convenient ways, for both experts and non-experts in the field, in obtaining regularities. Essentially, it is based on the following "regularity lifting" theorem.

Let $Z$ be a given vector space. Let $\|\cdot\|_X$ and $\|\cdot\|_Y$ be two norms on $Z$. Define a new norm $\|\cdot\|_Z$ by

$$\|\cdot\|_Z = \sqrt[p]{\|\cdot\|_X^p + \|\cdot\|_Y^p}$$

For simplicity, we assume that $Z$ is complete with respect to the norm $\|\cdot\|_Z$. Let $X$ and $Y$ be the completion of $Z$ under $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. Here, one can choose $p$, $1 \leq p \leq \infty$, according to what one needs. It's easy to see that $Z = X \cap Y$.

**Theorem 1.** *Let* $T$ *be a contraction map from* $X$ *into itself and from* $Y$ *into itself. Assume that* $f \in X$, *and that there exits a function* $g \in Z$ *such that* $f = Tf + g$. *Then* $f$ *also belongs to* $Z$.

As an immediate application, we show how this "Regularity Lifting Theorem" can be applied to obtain the regularity for the solutions of the weighted integral system. In [16], a more thorough discussion about the regularity can be found.

**Theorem 2.** *Let* $(u,v) \in L^{p+1}(R^n) \times L^{q+1}(R^n)$ *be a pair of solutions of (6). Assume that* $p,q > 1$, $\alpha, \beta \geq 0$, $\frac{1}{p+1} + \frac{1}{q+1} = \frac{\lambda+\alpha+\beta}{n}$ *and*

$$\frac{\alpha}{n-\lambda} \leq \frac{1}{p+1} \leq \frac{\lambda+\alpha}{n+\lambda}, \quad \frac{\beta}{n-\lambda} \leq \frac{1}{q+1} \leq \frac{\lambda+\beta}{n+\lambda}. \tag{15}$$

*Then*

$$u \in L^r \quad \text{for } \frac{1}{r} \in I_r = (\frac{\alpha}{n}, \frac{\lambda+\alpha}{n}) \text{ and } v \in L^s \quad \text{for } \frac{1}{s} \in I_s = (\frac{\beta}{n}, \frac{\lambda+\beta}{n}). \tag{16}$$

**Remark 1.** If we have $|K_1(x,y)|,\ |K_2(x,y)| \leq C$ , then Theorem 2 also holds for the following system:

$$\begin{cases} u(x) = \frac{1}{|x|^\alpha} \int_{R^n} \frac{K_1(x,y)v(y)^q}{|y|^\beta |x-y|^\lambda} dy \\ v(x) = \frac{1}{|x|^\beta} \int_{R^n} \frac{K_2(x,y)u(y)^p}{|y|^\alpha |x-y|^\lambda} dy \end{cases} \tag{17}$$

where $\lambda, \alpha, \beta, p, q$ satisfy the same condition as in the previous theorem.

The following theorem shows that the integrability intervals (16) are optimal.

**Theorem 3.** $(u,v) \in L^{p+1}(R^n) \times L^{q+1}(R^n)$ *be a pair of positive solutions of (6). Assume that* $\alpha, \beta \geq 0$, $\frac{1}{p+1} + \frac{1}{q+1} = \frac{\lambda+\alpha+\beta}{n}$, *then*

$$u \notin L^r(R^n) \quad \forall \quad \frac{1}{r} \notin (\frac{\alpha}{n}, \frac{\lambda+\alpha}{n}) \text{ and } v \notin L^s(R^n) \quad \forall \quad \frac{1}{s} \notin (\frac{\beta}{n}, \frac{\lambda+\beta}{n})$$

**Remark 2.** If we have $K_1(x,y), K_2(x,y) \geq c > 0$, then Theorem 3 also holds for system (17).

In section 2, we prove Theorem 1, 2 and 3.

## 2. **Regularity Lifting and Its Applications.**

Here, we prove Theorem 1.

*Proof. Step 1.* First show that $T : Z \longrightarrow Z$ is a contraction. Since $T$ is a contraction on $X$, there exists a constant $\theta_1,\ 0 < \theta_1 < 1$ such that

$$\|Th_1 - Th_2\|_X \leq \theta_1 \|h_1 - h_2\|_X.$$

Similarly, we can find a constant $\theta_2,\ 0 < \theta_2 < 1$ such that

$$\|Th_1 - Th_2\|_Y \leq \theta_2 \|h_1 - h_2\|_Y.$$

Let $\theta = \max\{\theta_1, \theta_2\}$. Then, for any $h_1,\ h_2 \in Z$,

$$\begin{aligned} \|Th_1 - Th_2\|_Z &= \sqrt[p]{\|Th_1 - Th_2\|_X^p + \|Th_1 - Th_2\|_Y^p} \\ &\leq \sqrt[p]{\theta_1^p \|h_1 - h_2\|_X^p + \theta_2^p \|h_1 - h_2\|_Y^p} \\ &\leq \theta \|h_1 - h_2\|_Z. \end{aligned}$$

*Step 2.* Since $T : Z \longrightarrow Z$ is a contraction, given $g \in Z$, we can find a solution $h \in Z$ such that $h = Th + g$. We see that $T : X \longrightarrow X$ is also a contraction and $g \in Z \subset X$. The equation $x = Tx + g$ has a unique solution in $X$. Thus, $f = h \in Z$ since both $h$ and $f$ are solutions of $x = Tx + g$ in $X$. □

Now, an application is given of this technique. Let $X = L^p(R^n)$, $Y = L^q(R^n)$ and $T$ be a contraction. Assume that $f$ satisfies the equation

$$f = Tf + g,$$

with $g \in L^p(R^n) \cap L^q(R^n)$.

If $f \in L^p(R^n)$, then we have $f \in L^p(R^n) \cap L^q(R^n)$ by the above theorem.

Next, as an immediate application, we prove Theorem 2. The proof consists of two main steps to extend the integrability of the solutions $(u,v)$ from $L^{p+1} \times L^{q+1}$ to $L^r \times L^s$ for the optimal integrability intervals of $r, s$.

In the first step, we use the weighted HLS inequality followed by the Holder inequality to set up a contraction mapping, so that we can use theorem 1 to lift the integrability. To apply the weighted HLS inequality, we need the $r, s$ to satisfy (18). To apply the Holder inequality, we need our $r$ and $s$ to satisfy (20). Based on the two requirements (18) and (20), we then divide the proof into two cases. In the second step, we obtain the optimal integrability by applying the weighted HLS inequality back to the original system (6) again.

*Proof.* Define the operators $T_1$ and $T_2$ by

$$(T_1 g)(x) = \int_{R^n} \frac{v^{q-1} g}{|x|^\alpha |x-y|^\lambda |y|^\beta} dy, \quad (T_2 f)(x) = \int_{R^n} \frac{u^{p-1} f}{|x|^\beta |x-y|^\lambda |y|^\alpha} dy$$

For

$$\frac{1}{r} \in (\frac{\alpha}{n}, \frac{\lambda+\alpha}{n}), \quad \frac{1}{s} \in (\frac{\beta}{n}, \frac{\lambda+\beta}{n}), \tag{18}$$

we can apply weighted HLS inequality to get

$$\|T_1 g\|_r \leq C \left\|v^{q-1} g\right\|_{\frac{nr}{n+(n-\alpha-\beta-\lambda)r}}, \quad \|T_2 f\|_s \leq C \left\|u^{p-1} f\right\|_{\frac{ns}{n+(n-\alpha-\beta-\lambda)s}} \tag{19}$$

Now for $(r,s)$ satisfying (18) and

$$\frac{1}{s}-\frac{1}{r}=\frac{1}{q+1}-\frac{1}{p+1}. \tag{20}$$

One can verify that

$$s>\frac{nr}{n+(n-(\alpha+\lambda+\beta))r} \tag{21}$$

This guarantees that there exists an $l>1$, such that

$$\frac{l-1}{l}s=\frac{nr}{n+(n-(\alpha+\lambda+\beta))r} \tag{22}$$

By (20), (22) and the condition $\frac{1}{p+1}+\frac{1}{q+1}=\frac{\lambda+\alpha+\beta}{n}$, we obtain that

$$q+1=l\frac{nr}{n+(n-\lambda-\alpha-\beta)r}(q-1) \tag{23}$$

then with (23), we can apply Holder inequality to the right hand sides of the inequalities in (19) to obtain:

$$\|T_1g\|_r\leq C\,\|v\|_{q+1}^{q-1}\,\|g\|_s,\ \ \|T_2f\|_s\leq C\,\|u\|_{p+1}^{p-1}\,\|f\|_r \tag{24}$$

We divide the proof of our theorem into the following two cases. Let $d_1=\frac{1}{p+1}-\frac{\alpha}{n},\quad d_2=\frac{\lambda+\alpha}{n}-\frac{1}{p+1}$ and $l_1=\frac{1}{q+1}-\frac{\beta}{n},\quad l_2=\frac{\lambda+\beta}{n}-\frac{1}{q+1}$ then $d_1,d_2,l_1,l_2>0$ and $d_1+d_2=l_1+l_2=\frac{\lambda}{n}$.

Case 1. $d_1\geq l_1$, i. e. $d_2\leq l_2$.

In this case, based on (20) and (18), we have

$$\frac{1}{r}\in(\frac{1}{p+1}-l_1,\frac{\alpha+\lambda}{n}),\ \ \frac{1}{s}\in(\frac{\beta}{n},\frac{1}{q+1}+d_2). \tag{25}$$

One can verify that, for any $r$ satisfying (25), we can find an $s$ such that (24) holds. This is also true for $s$.

Now we define $T:L^r(R^n)\times L^s(R^n)\longrightarrow L^r(R^n)\times L^s(R^n)$ by

$$T(f,g)=(T_1g,T_2f)$$

with the norm $\|(f,g)\|_{r\times s}=\|f\|_r+\|g\|_s$.

Let

$$v_A(x)=\begin{cases} v(x) & \text{if } v(x)\geq A \text{ or } |x|\geq A\\ 0 & \text{otherwise}\end{cases} \tag{26}$$

and similarly for $u_A(x)$.

Define

$$T_1^Ag=\int_{R^n}\frac{v_A^{q-1}g(y)}{|x|^\alpha|x-y|^\lambda|y|^\beta}dy,\ \ T_2^Af=\int_{R^n}\frac{u_A^{p-1}f(y)}{|x|^\beta|x-y|^\lambda|y|^\alpha}dy$$

and

$$T_A(f,g)=(T_1^Ag,T_2^Af).$$

It follows from (24) that

$$\|T_1^Ag\|_r\leq C\|v_A\|_{q+1}^{q-1}\|g\|_s \text{ and } \|T_2^Af\|_s\leq C\|u_A\|_{p+1}^{p-1}\|f\|_r \tag{27}$$

Since $u\in L^{p+1}(R^n)$ and $v\in L^{q+1}(R^n)$, one can choose A sufficiently large such that

$$\|T_A(f,g)\|_{r\times s}\leq\frac{1}{2}\|g\|_s+\frac{1}{2}\|f\|_r=\frac{1}{2}\|(f,g)\|_{r\times s}. \tag{28}$$

Therefore, $T_A$ is a contract mapping from $L^r(R^n)\times L^s(R^n)$ to itself, where $r,s$ are the same as in (25).

Let $(u,v) \in L^{p+1}(R^n) \times L^{q+1}(R^n)$ be a pair of solutions to the integral system (6) , it's easy to verify that

$$(u,v) = T_A(u,v) + (F,G) \tag{29}$$

where $F = \int_{R^n} \frac{(v-v_A)^q}{|x|^\alpha |x-y|^\lambda |y|^\beta} dy$ and $G = \int_{R^n} \frac{(u-u_A)^p}{|x|^\beta |x-y|^\lambda |y|^\alpha} dy$.

Obviously, $v - v_A$ is bounded on a bounded set in $R^n$. By weighted Hardy-Littlewood-Sobolev inequality, we know that $(F(x),G(x)) \in L^r(R^n) \times L^s(R^n)$ for $r,s$ as in (18).

Let

$$(f,g) = T_A(f,g) + (F,G). \tag{30}$$

we have known that $(u,v)$ is a pair of solutions to (30) in $L^{p+1} \times L^{q+1}$. By Theorem 1, $(u,v)$ is also a pair solution to (30) in $(L^{p+1} \times L^{q+1}) \cap (L^r \times L^s)$ for $r,s$ satisfying (20) and (18).

For any $r$ satisfying (25), we can find a $s$ such that $r,s$ satisfy (20) and (18). Thus, $u \in L^r$, $\forall\ \frac{1}{r} \in (\frac{1}{p+1} - l_1, \frac{\lambda+\alpha}{n})$. Similarly, we derive $v \in L^s$, $\forall\ \frac{1}{s} \in (\frac{\beta}{n}, \frac{\lambda+\beta}{n} + d_2)$.

For $\frac{1}{r} \in (max\{\frac{q\beta+\bar{\lambda}-n}{n}, \frac{\alpha}{n}\}, \frac{1}{p+1})$, where it's easy to verify that $\frac{q\beta+\bar{\lambda}-n}{n} < \frac{1}{p+1} - l_1$, we have $\frac{n+(n-\bar{\lambda})r}{qnr} \in (\frac{\beta}{n}, \frac{\lambda+\beta}{n} + d_2)$. So, we can apply weighted Hardy-Littlewood-Sobolev inequality to $u = \int \frac{v^q}{|x|^\alpha |x-y|^\lambda |y|^\beta} dy$,

$$\|u\|_r \leq C \|v\|^{\frac{1}{q}}_{\frac{qnr}{n+(n-\bar{\lambda})r}} < \infty \tag{31}$$

Thus, $u \in L^r$ when $\frac{1}{r} \in (max\{\frac{q\beta+\bar{\lambda}-n}{n}, \frac{\alpha}{n}\}, \frac{\lambda+\alpha}{n})$.

Similarly, we prove $v \in L^s$ for $\frac{1}{s} \in (\frac{\beta}{n}, min\{\frac{p(\lambda+\alpha)+\bar{\lambda}-n}{n}, \frac{\lambda+\beta}{n}\})$.

As a summary of the case 1, we have obtained $u \in L^r, v \in L^s$ for any

$$\frac{1}{r} \in (max\{\frac{q\beta + \bar{\lambda} - n}{n}, \frac{\alpha}{n}\}, \frac{\lambda+\alpha}{n}); \quad \frac{1}{s} \in (\frac{\beta}{n}, min\{\frac{p(\lambda+\alpha) + \bar{\lambda} - n}{n}, \frac{\lambda+\beta}{n}\}) \tag{32}$$

Case 2. $d_1 \leq l_1$, i. e. $d_2 \geq l_2$.

Similarly as in case 1, we can show $u \in L^r, v \in L^s$ for any

$$\frac{1}{r} \in (\frac{\alpha}{n}, \min\{\frac{q(\lambda+\beta) + \bar{\lambda} - n}{n}, \frac{\lambda+\alpha}{n}\}); \quad \frac{1}{s} \in (max\{\frac{p\alpha + \bar{\lambda} - n}{n}, \frac{\beta}{n}\}, \frac{\lambda+\beta}{n}) \tag{33}$$

By (15), we have

$$\frac{q\beta + \bar{\lambda} - n}{n} \leq \frac{\alpha}{n}, \quad \frac{p(\lambda+\alpha) + \bar{\lambda} - n}{n} \geq \frac{\lambda+\beta}{n}$$

$$\frac{p\alpha + \bar{\lambda} - n}{n} \leq \frac{\beta}{n}, \quad \frac{q(\lambda+\beta) + \bar{\lambda} - n}{n} \geq \frac{\lambda+\alpha}{n}$$

Thus, both case 1 and case 2 lead to

$$u \in L^r, \ \forall\ \frac{1}{r} \in (\frac{\alpha}{n}, \frac{\lambda+\alpha}{n}); \quad v \in L^s, \ \forall\ \frac{1}{s} \in (\frac{\beta}{n}, \frac{\lambda+\beta}{n})$$

This completes the proof. □

Now, we prove Theorem 3.

*Proof.* Considering $u(x)$ for $|x| \leq 1$, we have

$$u(x) \geq \frac{1}{|x|^\alpha} \int_{B(0,1)} \frac{v^q}{|x-y|^\lambda |y|^\beta} dy \geq \frac{C}{|x|^\alpha} \int_{B(0,1)} v^q dy \geq \frac{C}{|x|^\alpha} \tag{34}$$

So, $\int_{R^n} u(x)^r dx = \infty$ for any $r \geq \frac{n}{\alpha}$.

Now consider $u(x)$ for $|x| > 2$, we have

$$u(x) \geq \frac{1}{|x|^\alpha} \int_{B(0,1)} \frac{v^q}{|x-y|^\lambda |y|^\beta} dy \geq \frac{C}{|x|^{\alpha+\lambda}} \int_{B(0,1)} v^q dy \geq \frac{C}{|x|^{\alpha+\lambda}} \tag{35}$$

This implies that $\int_{R^n} u(x)^r dx = \infty$ for any $r \leq \frac{n}{\alpha+\lambda}$.

Thus, we proved $u \notin L^r(R^n) \quad \forall \ \frac{1}{r} \notin (\frac{\alpha}{n}, \frac{\lambda+\alpha}{n})$. Similarly, $v \notin L^s(R^n) \quad \forall \ \frac{1}{s} \notin (\frac{\beta}{n}, \frac{\lambda+\beta}{n})$. Combining these with Theorem 2, we see that $u \in L^r(R^n)$ if and only if $\frac{1}{r} \in (\frac{\alpha}{n}, \frac{\lambda+\alpha}{n})$. Similarly, $v \in L^s(R^n)$ if and only if $\frac{1}{s} \in (\frac{\beta}{n}, \frac{\lambda+\beta}{n})$. □

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