DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Supplement Volume 2005

Website: http://AIMsciences.org

pp. **164–172**

WEIGHTED HARDY-LITTLEWOOD-SOBOLEV INEQUALITIES AND SYSTEMS OF INTEGRAL EQUATIONS

Wenxiong Chen[†], Chao Jin[‡], Congming Li[‡] and Jisun Lim[‡]

[†]Department of Mathematics
Yeshiva University

[‡]Department of Applied Mathematics
University of Colorado at Boulder

Abstract. In this paper, we consider systems of integral equations related to the weighted Hardy-Littlewood-Sobolev inequality. We present the symmetry, monotonity, and regularity of the solutions. In particular, we obtain the optimal integrability of the solutions to a class of such systems. We also present a simple method for the study of regularity, which has been extensively used in various forms. The version we present here contains some new developments. It is much more general and very easy to use. We believe the method will be helpful to both experts and non-experts in the field.

1. **Introduction.** Let $0 < \lambda < n$ and let $1 < s, r < \infty$ such that $\frac{1}{r} + \frac{1}{s} + \frac{\lambda}{n} = 2$. Let $\|f\|_p$ be the $L^p(R^n)$ norm of the function f. The well-known classical Hardy-Littlewood-Sobolev inequality (HLS) states that:

$$\int_{R^n}\int_{R^n} \frac{f(x)g(y)}{|x-y|^\lambda} dxdy \leq C_{s,\lambda,n}\|f\|_r\|g\|_s \tag{1}$$

for any $f \in L^r(R^n)$ and $g \in L^s(R^n)$.

Hardy and Littlewood also introduced the double weighted inequality, which was generalized by Stein and Weiss in [23]. This inequality is called double weighted Hardy-Littlewood-Sobolev (WHLS) inequality:

$$\left|\int_{R^n}\int_{R^n} \frac{f(x)g(y)}{|x|^\alpha|x-y|^\lambda|y|^\beta} dxdy\right| \leq C_{\alpha,\beta,s,\lambda,n}\|f\|_r\|g\|_s \tag{2}$$

where $1 < r, s < \infty$, $0 < \lambda < n$, $\alpha + \beta \geq 0$ and the powers α, β of the weights satisfy

$$1 - \frac{1}{r} - \frac{\lambda}{n} < \frac{\alpha}{n} < 1 - \frac{1}{r}, \quad \text{and } \frac{1}{r} + \frac{1}{s} + \frac{\lambda+\alpha+\beta}{n} = 2. \tag{3}$$

To obtain the best constant in the weighted inequality (2), one can maximize the functional

$$J(f,g) = \int_{R^n}\int_{R^n} \frac{f(x)g(y)}{|x|^\alpha|x-y|^\lambda|y|^\beta} dxdy \tag{4}$$

2000 *Mathematics Subject Classification.* 35J99, 45E10, 45G05.

Key words and phrases. Weighted Hardy-Littlewood-Sobolev inequalities, integral equations and systems, radial symmetry, monotonicity, moving planes in integral forms.

The first author is partially supported by NSF Grant DMS-0072328.

The third author is partially supported by NSF Grant DMS-0401174.