

## HIGHER ORDER BOUNDARY VALUE PROBLEMS WITH MULTIPLE SOLUTIONS: EXAMPLES AND TECHNIQUES

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**Abstract.** Conditions are derived on the nonlinear function  $f(y)$  so that two boundary value problems associated with the fifth order differential equation  $y^{(5)} = f(y)$  has one or more positive solutions. These examples are not included in any previous results.

We seek conditions on the nonlinear function  $f(y)$  for which certain boundary value problems associated with the ordinary differential equation  $y^{(n)} = f(y)$  has one or more positive solutions. Typical of the kinds of boundary value problems which interest us are these two examples:

$$y^{(5)} = f(y), \quad 0 < x < 1, \quad (1)$$

with the five boundary conditions

$$y(0) = y'(0) = y'(1) = y'''(0) = y'''(1) = 0, \quad (2)$$

hereafter called Problem 1 and

$$y^{(5)} = -f(y), \quad 0 < x < 1, \quad (3)$$

with the five boundary conditions

$$y(0) = y'(0) = y''(0) = y^{(4)}(0) = y''(1) = 0, \quad (4)$$

hereafter called Problem 2. We shall always assume that  $f(y) \geq 0$ , so the sign change from (1) to (3) is significant.

Problems like this have attracted attention for a number of years, for example, Parter [13]), Brown, Ibrahim, and Shivaji [5], Leggett and Williams [12], Henderson and Thompson [9, 10], Graef, Qian, and Yang [7], and our previous work [1, 2, 3, 4]. (Drabek and Robinson [6] extended some of these results to semilinear elliptic partial differential equations.) Most of the work has dealt with even order problems with half the boundary conditions imposed at each end of the interval. In contrast, Henderson and Thompson [10] and Baxley and Houmand [4] studied problems (both even and odd order) where one boundary condition was placed at one end of the interval and the other  $n - 1$  were at the other end. However, neither paper included Problem 2 above; instead, they required boundary conditions of the form  $y(0) = y'(0) = y''(0) = y'''(0) = y^{(k)}(1) = 0$ , where  $0 \leq k \leq 3$ .

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For comparison with our results below, we include a statement of the basic theorem in [9].

**Theorem 1.** *Suppose that  $0 < a < b < c/2$  and  $f : \mathbb{R} \rightarrow [0, \infty)$  is continuous and satisfies*

1.  $f(y) < 8a$  for  $0 \leq y \leq a$ ,
2.  $f(y) \geq 16b$  for  $b \leq y \leq 2b$ ,
3.  $f(y) \leq 8c$  for  $0 \leq y \leq c$ .

*Then the boundary value problem (1), (2) has three symmetric positive solutions  $y_1, y_2, y_3$  with  $\max y_1(x) < a$ ,  $y_2(1/2) > a$ ,  $y_2(1/4) < b$ ,  $y_3(1/4) > b$ .*

A number of different methods have been used on these problems. Parter and his colleagues used some impressive hard analysis, including computer assistance. Motivated by this problem, which had three positive solutions, Leggett and Williams proved an abstract fixed point theorem implying the existence of three solutions and obtained Parter's result as a corollary. Henderson and Thompson used the Leggett-Williams theorem. Graef, Qian, and Yang and others (including the authors) have used the Krasnosel'skii fixed point theorem. We have also used shooting methods and Drabek and Robinson have used upper and lower solution techniques, even on partial differential equations. The method used here extends to a variety of other boundary value problems of different orders and other arrangements of boundary conditions. A full treatment will appear elsewhere.

Recall that a cone in a Banach space  $\mathbf{B}$  is a closed set  $\mathbf{K}$  containing the zero element of  $\mathbf{B}$  satisfying

1. For any  $u, v \in \mathbf{K}$ ,  $au + bv \in \mathbf{K}$  for any nonnegative real numbers  $a, b$ ;
2. For each  $0 \neq u \in \mathbf{K}$ , at least one of  $u, -u$  does not belong to  $\mathbf{K}$ .

Any cone  $\mathbf{K}$  in a Banach space can be used to impose a partial ordering as follows:  $u \leq v$  if  $v - u \in \mathbf{K}$ .

We will use the Krasnosel'skii theorem, in the following form [8, Theorem 2.3.3, p. 93].

**Theorem 2.** (Krasnosel'skii) Suppose  $F$  is a completely continuous operator on a Banach space  $\mathbf{B}$ , and let  $\mathbf{K}$  be a cone in  $\mathbf{B}$ . If there exist bounded open sets  $\Omega_1, \Omega_2$  containing the zero element of  $\mathbf{B}$  such that  $\overline{\Omega}_1 \subset \Omega_2$ ,  $F : \mathbf{K} \cap (\overline{\Omega}_2 - \Omega_1) \rightarrow \mathbf{K}$ , and either:

1.  $u \not\leq Fu$  for  $u \in \mathbf{K} \cap \partial\Omega_1$  and  $Fu \not\leq u$  for  $u \in \mathbf{K} \cap \partial\Omega_2$ ; or
2.  $Fu \not\leq u$  for  $u \in \mathbf{K} \cap \partial\Omega_1$  and  $u \not\leq Fu$  for  $u \in \mathbf{K} \cap \partial\Omega_2$ ,

then there is an element  $u_0 \in \mathbf{K} \cap (\Omega_2 - \overline{\Omega}_1)$  satisfying  $Fu_0 = u_0$ .

Here we shall prove (compare to Theorem 1 above)

**Proposition 1.** *Let  $f(y) \geq 0$  be continuous for  $y \geq 0$ . Then there exists positive constants  $M_i, K_i, q_i$ ,  $i = 1, 2$ , so that if there exist  $a, b$  with  $0 < a < b < q_i b$  and*

$$f(y) < M_i a, \text{ for } 0 \leq y \leq a,$$

$$f(y) \geq K_i b, \text{ for } b \leq y \leq q_i b,$$

*then Problem  $i$  has a positive nondecreasing solution  $y_i$  satisfying*

$$a < y_i(1) < q_i b, \quad y_i(1/2) < b.$$

*Moreover, if there exists  $c$  with  $q_i b < c$  and*

$$f(y) < M_i c, \text{ for } 0 \leq y \leq c,$$

then Problem  $i$  has a second positive nondecreasing solution  $z_i$  satisfying

$$z_i(1/2) > b, \quad z_i(1) < c.$$

Requiring that  $f$  satisfy more inequalities of a similar nature leads to the existence of additional positive nondecreasing solutions.

To apply Theorem 2 successfully, we must design a Banach space, describe a cooperative cone and two bounded open sets, and then verify some inequalities. The challenge is that everything must fit properly. The Banach space, the cone, and the open sets have to be sufficiently compatible that the needed inequalities can be verified without unreasonable difficulty.

For our Banach space, we take  $C[0, 1]$ . Then we have to transform each of our problems into a fixed point problem for some operator. We prepare for this task by letting  $\ell$  denote the differential operator defined on the set  $C_0^1[0, 1]$  of continuously differentiable functions  $y$  defined on  $[0, 1]$  satisfying  $y(0) = 0$  by  $\ell y = y'$  and we let  $\tau$  denote the differential operator defined on the set  $C_{(0,1)}^2[0, 1]$  of twice continuously differentiable functions  $y$  defined on  $[0, 1]$  satisfying  $y(0) = y(1) = 0$  by  $\tau y = -y''$ . Then the problem (1), (2) takes the form

$$(\tau \circ \tau \circ \ell)y = f(y) \tag{5}$$

and the problem (3), (4) takes the form

$$(\ell \circ \tau \circ \ell \circ \ell)y = f(y). \tag{6}$$

Note that the negative sign has now disappeared in the second problem. It is easy to see that  $\ell$  is inverted by the integral operator

$$(Ly)(x) = \int_0^x y(t) dt,$$

and  $\tau$  is inverted by the integral operator

$$(Ty)(x) = \int_0^1 G(x, t)y(t) dt,$$

where

$$G(x, t) = \left\{ \begin{array}{l} t(1-x), 0 \leq t \leq x, \\ x(1-t), x < t \leq 1 \end{array} \right\}$$

is the Green's function for  $\tau$ . Note that for  $y$  in the domain of  $\ell$ , we have  $L(y') = y$  and for  $y$  in the domain of  $\tau$ , we have  $T(y'') = -y$ . Now our two problems can be re-written in the form

$$F_1(y) \equiv (L \circ T \circ T \circ f)y = y \tag{7}$$

and

$$F_2(y) \equiv (L \circ L \circ T \circ L \circ f)y = y, \tag{8}$$

each a fixed point problem. So we need to find fixed points of each of the operators  $F_i$  on  $C[0, 1]$ . It is straightforward to verify that each of these operators is completely continuous on  $C[0, 1]$ .

We now need to choose a cone for each problem. For the operator  $F_i$ ,  $i = 1, 2$ , we choose  $\mathbf{K}_i$  to be all  $u \in C[0, 1]$  for which

$$u(0) = 0, u(x) \text{ nondecreasing, } u(1) \leq q_i u(1/2),$$

and  $q_1 = 4$ ,  $q_2 = 8$ . We begin by explaining why these are the appropriate values for  $q_1, q_2$ .

Let  $u = F_1(y)$ . Among other assumptions, we will always assume that  $f(y) \geq 0$ . You see immediately that  $-u''' = (T \circ f)(y) \geq 0$ . Thus  $u''' \leq 0$  on  $(0, 1)$  and  $u''$  is non-increasing whence

$$u'(x) = \int_0^x u''(t) dt \geq xu''(x).$$

In particular, for  $x = 1/2$ , we get

$$u''(1/2) \leq 2u'(1/2).$$

We also obtain for  $1/2 \leq x \leq 1$

$$\begin{aligned} u'(x) - u'(1/2) &= \int_{1/2}^x u''(t) dt \\ &\leq u''(1/2)(x - 1/2) \\ &\leq 2u'(1/2)(x - 1/2), \end{aligned}$$

from which

$$u'(x) \leq 2xu'(1/2), 1/2 \leq x \leq 1.$$

Then

$$u(x) = \int_0^x u'(t) dt \geq \int_0^x tu''(t) dt.$$

Integrating by parts leads to

$$u(x) \geq xu'(x) - u(x),$$

or

$$2u(x) \geq xu'(x).$$

Thus

$$u'(1/2) \leq 4u(1/2).$$

Finally,

$$\begin{aligned} u(1) - u(1/2) &= \int_{1/2}^1 u'(t) dt \\ &\leq 2u'(1/2) \int_{1/2}^1 t dt \\ &= 3u'(1/2)/4 \leq 3u(1/2), \end{aligned}$$

so that

$$u(1) \leq 4u(1/2).$$

Also note that since  $u' = (T \circ T \circ f)(y) \geq 0$  on  $(0, 1)$ . Thus  $u$  is non-decreasing and, since  $u$  is in the range of  $L$ ,  $u(0) = 0$ . So any  $u = F(y)$  will belong to the cone  $\mathbf{K}_1$ . In other words, the number  $q_1 = 4$  is chosen so that  $F_1$  maps every member of  $C[0, 1]$  into the cone  $\mathbf{K}_1$ , so that the mapping hypothesis in Theorem 2 is satisfied.

For the operator  $F_2$ , we see that  $-u^{(4)} = (L \circ f)(y) \geq 0$  so  $u^{(4)} \leq 0$  and  $u''''$  is now non-increasing, whereas for  $F_1$ , it was the second derivative which was non-increasing. A similar argument, although longer, shows that in this case  $q_2 = 8$  has the property that  $F_2$  maps every member of  $C[0, 1]$  into the cone  $\mathbf{K}_2$ .

To get insight into how to set up the open sets, suppose that  $f$  satisfies  $0 \leq f(y) < Ma$  for  $0 \leq y \leq a$ . Then, focusing attention on  $F_1$  and letting  $w = (T \circ f)(y)$ , we have

$$\begin{aligned} 0 \leq w(x) &= \int_0^1 G(x,t)f(y(t)) dt \\ &< Ma \int_0^1 G(x,t) dt \\ &= Max(1-x)/2, \end{aligned}$$

and with  $v = T(w)$ , we have

$$\begin{aligned} 0 \leq v(x) &= \int_0^1 G(x,t)w(t) dt \\ &< \frac{Ma}{2} \int_0^1 t(1-t)G(x,t) dt. \end{aligned}$$

If you compute this elementary integral and then use this estimate in the integral for  $u = L(v)$  you will get

$$0 \leq u(1) \leq \int_0^1 v(t) dt < Ma/120.$$

Thus, if  $M = 120$ , we get  $u(1) < a$ . Thus if we let  $\Omega_1$  be all  $y \in C[0,1]$  so that  $\|y\| < a$ , then  $y \in \mathbf{K} \cap \partial\Omega_1$  implies that  $y(1) = a$ . Thus  $(Fy)(1) = u(1) < a = y(1)$  implies  $y \notin Fy$  for  $y \in \mathbf{K} \cap \partial\Omega_1$  and we have one of the required conditions of Theorem 2. So  $M_1 = 120$  is satisfactory.

For the operator  $F_2$ , we would let  $w = (L \circ f)(y)$  and then

$$w(x) = \int_0^x f(y(t)) dt < Max.$$

Then with  $v = T(w)$ , we have

$$\begin{aligned} 0 \leq v(x) &= \int_0^1 G(x,t)w(t) dt \\ &< Ma \int_0^1 tG(x,t) dt = \frac{Ma}{6}(x-x^3). \end{aligned}$$

Applying  $L$  in succession two times to this inequality, we conclude that  $u = F_2(y)$  satisfies

$$u(x) < \frac{Ma}{360}(10x^3 - 3x^5) \leq \frac{7Ma}{360}.$$

It follows that  $M_2 = 360/7$  is satisfactory.

We now let  $\Omega_2$  be all  $y \in C[0,1]$  for which  $y(1/2) < b$  and  $\|y\| < q_i b$ , where  $b$  is any number greater than  $a$ . If  $y \in \mathbf{K} \cap \partial\Omega_2$ , then  $y(1/2) = b$ ,  $y(0) = 0$ ,  $y$  is nondecreasing and  $y(1) \leq q_i b$ , for if  $y(1/2) < b$ , then the cone condition forces  $y(1) \leq q_i y(1/2) < q_i b$  and  $y$  is then inside the open set  $\Omega_2$ . Then  $b \leq y(x) \leq 4b$  for all  $1/2 \leq x \leq 1$ . Consider first the case of  $F_1$ . If we assume that  $f(y) \geq Kb$  for

$b \leq y \leq 4b$ , we have for  $w = T(y)$ ,

$$\begin{aligned} w(x) &= \int_0^1 G(x,t)f(y(t)) dt \\ &> \int_{1/2}^1 G(x,t)f(y(t)) dt \\ &\geq Kb \int_{1/2}^1 G(x,t) dt. \end{aligned}$$

Computing this integral for  $0 \leq x \leq 1/2$  gives

$$w(x) > Kbx/8.$$

Computing for  $1/2 < x \leq 1$ , we break the integral into two parts (from  $1/2$  to  $x$  and from  $x$  to  $1$ ) and find

$$w(x) > Kb(1-x)(x-1/4)/2.$$

Next, we get for  $v = T(w)$

$$v(x) = \int_0^1 G(x,t)w(t) dt.$$

We only need a lower bound on this integral for  $0 \leq x \leq 1/2$  because we only need a lower bound on  $(F_1y)(1/2)$ . We break this integral into three parts, from  $0$  to  $x$ ,  $x$  to  $1/2$  and  $1/2$  to  $1$  and, omitting this tedious computation, we get

$$v(x) > \frac{Kb}{8}[x^3/3 + x/12 - x^3/2 + x/16].$$

Then for  $u = Lv$

$$u(1/2) = \int_0^{1/2} v(t) dt > \frac{Kb}{2^9}.$$

Thus  $u(1/2) > b$  if

$$K = 2^9 = 512.$$

Thus  $(Fy)(1/2) = u(1/2) > b = y(1/2)$  implies  $F(y) \not\leq y$  for  $y \in \mathbf{K} \cap \partial\Omega_2$  and we have the other required condition of Theorem 2 in the case of  $F_1$  if we assume that there exists  $b > a$  for which  $f(y) \geq K_1b = 512b$  for  $b \leq y \leq 4b$ .

Now consider the case of  $F_2$ . If we now assume that  $f(y) \geq Kb$  for  $b \leq y \leq 8b$ , we have for  $w = L(y)$

$$w(x) = \int_0^x f(y(t)) dt$$

and we can only conclude that

$$w(x) \geq 0, \text{ for } 0 \leq x < 1/2,$$

and

$$w(x) > \int_{1/2}^x Kb dt = Kb(x-1/2) \text{ for } 1/2 \leq x \leq 1.$$

Letting  $v = T(w)$ , we then get

$$v(x) = \int_0^1 G(x,t)w(t) dt > Kb \int_{1/2}^1 (t-1/2)G(x,t) dt.$$

Computing the integral gives

$$v(x) > \frac{Kbx}{48}, \text{ for } 0 \leq x \leq 1/2.$$

Applying  $L$  two more times, we find for  $u = F_2(y)$

$$u(x) > \frac{Kbx^3}{288}, \text{ for } 0 \leq x \leq 1/2.$$

Thus  $u(1/2) > b$  if  $K = 8(288) = 2304$ . Again, we obtain the second required condition for Theorem 2 in the case of  $F_2$  if we assume that there exists  $b > a$  for which  $f(y) \geq K_2b = 2304b$  for  $b \leq y \leq 8b$ .

It follows that there is a fixed point of  $F_i$  in  $\mathbf{K} \cap (\Omega_2 - \overline{\Omega}_1)$ , giving us a positive solution  $y_i$  of Problem i with these assumptions.

If  $f(y) \leq 120c$  for  $0 \leq y \leq c$ , where  $q_1b < c$ , we can show that there is a fixed point of  $F_1$  in  $\mathbf{K} \cap (\Omega_3 - \overline{\Omega}_2)$  where  $\Omega_3$  consists of all  $y \in C[0, 1]$  for which  $\|y\| < c$ , giving another positive solution of Problem 1. The verification of the needed hypothesis of Theorem 2 on the boundary of  $\Omega_3$  is a repeat of the verification on the boundary of  $\Omega_1$ . Similar comments apply for Problem 2.

From the work above, it is seen that the numbers asserted to exist in the statement of Proposition 1 are

$$M_1 = 120, K_1 = 512, q_1 = 4, M_2 = 360/7, K_2 = 2304, q_2 = 8.$$

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