PROPERTIES OF KERNELS AND EIGENVALUES FOR THREE POINT BOUNDARY VALUE PROBLEMS*

K. Q. Lan
Department of Mathematics
Ryerson University
Toronto, Ontario, Canada M5B 2K3

Abstract. We investigate the properties of a kernel arising from a three point boundary value problem. We seek a lower bound for the kernel and evaluate the optimal values for the integrals related to the kernel. The smallest positive characteristic value for a linear second ordinary differential equation with a three point boundary condition is estimated by using our lower bound. These optimal values and the estimates for characteristic values are useful in studying the existence of nonzero positive solutions for the boundary value problem.

1. Introduction. A second order ordinary differential equation of the form

\[ z''(t) + g(t)f(t, z(t)) = 0, \quad \text{a.e on } [0, 1], \]

with boundary condition

\[ z'(0) = 0, \quad \alpha z(\eta) = z(1), \quad 0 < \alpha, \eta < 1. \]

(2)

can be changed into a Hammerstein integral equation of the form

\[ z(t) = \int_0^1 k(t, s)g(s)f(s, z(s)) \, ds \equiv Az(t), \]

(3)

where \( k \) is the Green’s function to \( -z'' = 0 \) subject to (2). The existence of one or several positive solutions for (1)-(2) can be derived from well-known results on the existence of one or several positive solutions for (3) (see for example [24]).

In order to employ the results from (3), one need prove that the kernel \( k \) has an upper bound \( \Phi(s) \) for \( t, s \in [0, 1] \) and a lower bound \( c(a, b)\Phi(s) \) for \( t \in [a, b] \), evaluate the two values \( m \) and \( M(a, b) \) which are related to the kernel \( k \) (the precise definitions for these symbols will be given later) and seek the the smallest positive characteristic value \( \mu_1 \) with positive eigenfunctions of the second order differential equation

\[ z''(t) + \mu g(t)z(t) = 0, \quad \text{a.e. on } [0, 1]. \]

(4)

with the boundary condition (2).

The function \( \Phi \) and the value \( c(a, b) \) were obtained by Webb in [22] and improve corresponding results obtained in [21]. The minimums of \( M(a, b) \) and \( M(a, b)c(a, b) \) on the whole \( [0, 1] \times [0, 1] \) were discussed in [22]. It is known that the conditions under which the positive solutions exist depend on the product \( M(a, b)c(a, b) \) and the

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interval \([c(a, b), \rho, \rho]\). The best choice for \(a, b\) would be the case when \(M(a, b)c(a, b)\) is its minimum value and \(c(a, b)\) is the maximum value of the function \(c\). However, it is hard to find such \(a, b\) which satisfy the requirement.

In this paper, we improve the above results obtained in [22]. We will seek a function \(c(t)\) and show that \(c(t)\Phi(t)\) is the lower bound of \(k\) for \(t, s \in [0, 1]\). We also apply the result to study the properties of positive solutions for (1)-(2) and functions satisfying (2). We look for the minimum values of \(M(a, b)\) and \(M(a, b)c(a, b)\) in several cases. We shall show that \(\mu_1 \in (m, M(a, b))\) and provide a new estimate for \(\mu_1\) by using \(\Phi\) and \(c(t)\).

2. Results on the existence of positive solutions for Hammerstein integral equations. In this section, we mention results obtained in [24] on the existence of positive solutions of a Hammerstein equation of the form

\[
    u(t) = Au(t) := \int_0^1 k(t, s)g(s)f(s, u(s))\,ds, \quad t \in [0, 1].
\]

To obtain positive solutions of (5), some weaker assumptions on \(g, k,\) and \(f\) were imposed in [24]. Here, we assume that \(k\) is continuous because many kernels arising from boundary value problems are continuous. Throughout this section we make the following hypotheses on \(g, k,\) and \(f:\)

\(C_1\) \(k : [0, 1] \times [0, 1] \rightarrow \mathbb{R}_+\) is continuous.

\(C_2\) There exists a continuous function \(\Phi : [0, 1] \rightarrow \mathbb{R}_+\) and for \(a, b \in [0, 1]\) with \(a < b\), there exists a constant \(c(a, b) \in (0, 1]\) such that

\[
    k(t, s) \leq \Phi(s) \quad \text{for} \quad t, s \in [0, 1]
\]

and

\[
    k(t, s) \geq c(a, b)\Phi(s) \quad \text{for} \quad t \in [a, b] \quad \text{and} \quad s \in [0, 1].
\]

\(C_3\) \(g : [0, 1] \rightarrow \mathbb{R}_+\) is measurable such that \(g\Phi \in L^1[0, 1]\) and \(\int_a^b \Phi(s)g(s)\,ds > 0\).

\(C_4\) \(f : [0, 1] \times \mathbb{R}_+ \rightarrow \mathbb{R}_+\) satisfies Carathéodory conditions, that is, \(f(\cdot, u)\) is measurable for each fixed \(u \in \mathbb{R}_+\) and \(f(t, \cdot)\) is continuous for almost every \(t \in [0, 1]\), and for each \(r > 0\), there exists \(\phi_r \in L^\infty[0, 1]\) such that

\[
    0 \leq f(t, u) \leq \phi_r(t) \quad \text{for all} \quad u \in [0, r] \quad \text{and almost all} \quad t \in [0, 1].
\]

Let \(P = \{u \in C[0, 1] : u \geq 0\}\) denote the standard cone of nonnegative continuous functions defined on \([0, 1]\). We need smaller cones than \(P\). Let \(q : C[0, 1] \rightarrow \mathbb{R}\) denote the continuous function

\[
    q(u) = \min\{u(t) : t \in [a, b]\}.
\]

This, together with \(c = c(a, b)\) as in \((C_2)\), enables one to define a cone

\[
    K = \{u \in P, q(u) \geq c\|u\|\}.
\]

This type of cone has been used in, for example, [4] [5] [9], [13] [10].

The smallest positive characteristic value of the following integral operator

\[
    Lu(t) := \int_0^1 k(t, s)g(s)u(s)\,ds.
\]

has been widely used to study the existence of positive solutions, see for example [7] [17] [18] [23] [24]. It is known that \(L : C[0, 1] \rightarrow C[0, 1]\) is compact and maps \(P\) into \(K\). We recall that \(\lambda \in \mathbb{R}\) is an eigenvalue of \(L\) with corresponding eigenfunction \(\varphi \in C[0, 1]\) if \(\varphi \neq 0\) and \(\lambda \varphi = L\varphi\). The reciprocals of eigenvalues are called characteristic values of \(L\). The radius of the spectrum of \(L\), denoted \(r(L)\), is given
by the well-known spectral radius formula \( r(L) = \lim_{n \to \infty} \|L^n\|^{1/n} \). It is known that when \((C_1)-(C_3)\) hold, \( r(L) \) is an eigenvalue of \( L \) with eigenfunction \( \varphi_1 \) in \( K \) (see Theorem 2.6 in [24]). We write \( \mu_1 = 1/r(L) \). Let

\[
m = \left( \sup_{t \in [0,1]} \int_0^1 k(t, s) g(s) \, ds \right)^{-1}, \quad M(a, b) = \left( \inf_{t \in [a,b]} \int_a^b k(t, s) g(s) \, ds \right)^{-1}.
\] (9)

The following result obtained in [24] gives the relation among \( m, M(a, b) \) and \( \mu_1 \).

**Theorem 1.** \( m \leq \mu_1 \leq M(a, b) \).

Recall that \( L \) satisfies \((UPE)\) if \( r(L) \) is the only positive eigenvalue of \( L \) with an eigenfunction in the cone \( P \). It is known that \( L \) satisfies \((UPE)\) if one of the following conditions holds: (i) \( L \) maps \( P \backslash \{0\} \) into the interior of \( P \), that is, \( L \) is strongly positive (see Theorem 3.2 in [3]), (ii) \( k(t, s) > 0 \) on \([0,1] \times [0,1]\), and \( g(s) > 0 \) for almost all \( s \in [0,1] \) and (iii) \( k \) is symmetric, that is, \( k(t, s) = k(s, t) \) for \( t, s \in [0,1] \).

**Notation** Let \( f \) satisfy \((C_4)\) and let \( E \) be a fixed subset of \([0,1]\) of measure zero. We make the following definitions.

\[
\overline{f}(u) := \sup_{t \in [0,1]\backslash E} f(t, u), \quad \underline{f}(u) := \inf_{t \in [0,1]\backslash E} f(t, u);
\]

\[
f^0 = \lim_{u \to 0^+} \overline{f}(u)/u, \quad f_0 = \liminf_{u \to 0^+} f(u)/u;
\]

\[
f^\infty = \lim_{u \to \infty} \overline{f}(u)/u, \quad f_\infty = \liminf_{u \to \infty} f(u)/u.
\]

Now, we are in a position to mention results on the existence of at least one or two nonzero positive solutions for \((5)\). These results were proved in [24] by using the fixed point index for compact maps defined in Banach spaces (see [3,9]).

**Theorem 2.** Assume that one of the following conditions holds.

\( (H_1) \) \( L \) satisfies \((UPE)\), \( 0 \leq f^0 < \mu_1 \) and \( \mu_1 < f_\infty \leq \infty \).

\( (H_2) \) \( \mu_1 < f_0 \leq \infty \) and \( 0 \leq f^\infty < \mu_1 \).

Then \((5)\) has a solution \( u \in K \) with \( \|u\| > 0 \).

We will use the following conditions:

\( (H_{1\rho}) \): \( f(s, u) \leq mp \) for all \( u \in [c(a, b)\rho, \rho] \) and almost all \( s \in [0,1] \).

\( (H_{2\rho}) \): \( f(s, u) \geq M(a, b)c(a, b)\rho \) for all \( u \in [c(a, b)\rho, \rho] \) and almost all \( s \in [0,1] \).

Let \( K_\rho = \{ u \in K : \|u\| < \rho \} \) and \( \Omega_\rho = \{ x \in K : q(x) < \rho c \} \). We mention the results on the existence of at least two positive solutions of \((5)\).

**Theorem 3.** Assume that one of the following conditions holds.

\( (S_1) \) \( 0 \leq f^0 < \mu_1 \), \( (H_{1\rho}) \) and \( x \neq Ax \) for \( x \in \partial \Omega_\rho \), and \( 0 \leq f^\infty < \mu_1 \).

\( (S_2) \) \( L \) satisfies \((UPE)\), \( \mu_1 < f_0 \leq \infty \), \( (H_{2\rho}) \), \( x \neq Ax \) for \( x \in \partial K_\rho \) and \( \mu_1 < f_\infty \leq \infty \).

Then \((5)\) has two nonzero solutions in \( K \).

In order to apply the results mentioned above to boundary value problems, one needs to prove the kernels arising from the boundary value problems satisfy condition \((C_2)\), calculate the values \( m \) and \( M(a, b) \) and find the characteristic value \( \mu_1 \) and show that \( \mu_1 \) is unique. For some boundary value problems, it is not easy to find the exact value \( \mu_1 \), so it is of interest to look for a suitable upper bound and a lower bound for \( \mu_1 \). It is known that \( \mu_1 \in [m, M(a, b)] \), so it is best to choose \( a, b \) such that \( M(a, b) \) is smallest because we can replace \( \mu_1 \) in the above theorems by
m and M(a,b). For example, the conditions 0 \leq f^0 < \mu_1 \text{ and } \mu_1 < f_\infty \leq \infty \text{ can be replaced by } 0 \leq f^0 < m \text{ and } M(a,b) < f_\infty \leq \infty.

In \((H^p_\infty)\), we wish \(c(a,b)\) to be the maximum value of \(c\) while in \((H^p_\infty)\), one has to find suitable \(a, b\) such that the product \(M(a,b)c(a,b)\) is its minimum value (Note that for such \(a, b\), \(c(a,b)\) may not be the maximum value of \(c\) or find \(a, b\) such that \(c(a,b)\) is the maximum value of \(c\) (Note that for such \(a, b\), \(M(a,b)c(a,b)\) may not be its minimum value).

3. Properties of the kernel \(k\) for the boundary value problem (1)-(2). Let \(k\) be the Green’s function to \(-z'' = 0\) subject to (2). It is well-known (see for example, [21] [22]) that \(k : [0,1] \times [0,1] \to \mathbb{R}_+\) is defined by

\[
k(t,s) = \frac{1}{1-\alpha} (1-s) - \begin{cases} 
\frac{\alpha}{1-\alpha}(\eta - s) & s \leq \eta \\
0 & s > \eta 
\end{cases} \quad \begin{cases} 
t - s & s \leq t \\
0 & s > t 
\end{cases}
\]  


To study the properties of \(k\), it is convenient to rewrite \(k\) in the following form:

\[
k(t,s) = \frac{1}{1-\alpha} \begin{cases} 
1 - \alpha \eta - (1-\alpha)s & \text{if } t \leq s \leq \eta, \\
1 - \alpha \eta - (1-\alpha)t & \text{if } s \leq t \text{ and } s \leq \eta, \\
1 - s & \text{if } t \leq s \text{ and } \eta < s, \\
1 - \alpha s - (1-\alpha)t & \text{if } \eta < s \leq t.
\end{cases}
\]  

(10)

We define a continuous function \(\Phi : [0,1] \to [0,1]\) by

\[\Phi(s) = k(s,s) = \frac{1}{1-\alpha} \begin{cases} 
1 - \alpha \eta - (1-\alpha)s & \text{if } s \leq \eta \\
1 - s & \text{if } s > \eta.
\end{cases}\]

The above function \(\Phi\) was given by Webb [22]. It is clear that \(k\) has following properties: (i) \(k(t,s) \leq \Phi(s)\) for \(t, s \in [0,1]\), (ii) \(k(t,s) = k(s,s)\) for \(t \leq s\) and (iii) \(k(t,s) = k(s,t)\) for \(t, s \in [0,0,\eta]\).

The following new result gives a lower bound for \(k\).

**Proposition 1.** The kernel \(k\) defined in (10) has the following property.

\(P\) \(k(t,s) \geq c(t)\Phi(s)\) for \(t, s \in [0,1]\),

where \(c(t) = \frac{1 - \alpha \eta - (1-\alpha)t}{1 - \alpha \eta}\) for \(t \in [0,1]\).

**Proof.** By calculation, we have for \(t, s \in [0,1]\),

\[
k(t,s) - c(t)\Phi(s) = \frac{1}{1-\alpha} \begin{cases} 
[1 - \alpha \eta - (1-\alpha)s]t & \text{if } t \leq s \leq \eta, \\
(1 - \alpha \eta - (1-\alpha)t)s & \text{if } s \leq t \text{ and } s \leq \eta, \\
(1-s)t & \text{if } t \leq s \text{ and } \eta < s, \\
(1 - \alpha \eta)(s-t) + (1-s)t & \text{if } \eta < s \leq t.
\end{cases}
\]

This implies \(k(t,s) - c(t)\Phi(s) \geq 0\) for \(t, s \in [0,1]\). \qed

For \(a, b \in [0,1]\), we choose \(c(a,b) = c(b)\) for \(a, b \in [0,1]\), so \(k\) satisfies \((C_2)\). Note that \(c(a,b)\) is independent of \(a, c(.)\) is decreasing on \([0,1]\) and \(c(b) \geq c(1) = \frac{\alpha(1-\eta)}{1-\alpha \eta} > 0\) for \(b \in [0,1]\).

(1)-(2) can be transformed into (5) with the kernel \(k\) defined in (10) and if \(z \in P\) is a solution of (5), then \(z\) is also a solution of (1)-(2). Therefore, Theorems 2 and 3 hold for (1)-(2).

By Proposition 1, we obtain a new property of positive solutions of (1)-(2). The new property will be used to prove Theorem 10 in section 4.
Theorem 4. Assume that \( z \in P \) is a solution of (1)-(2). Then \( z(t) \geq c(t)\|z\| \) for each \( t \in [0,1] \).

Proof. Assume that \( z \in P \) is a solution of (1)-(2). Then \( z = Az \). Since \( k(t,s) \leq \Phi(s) \) for \( t, s \in [0,1] \), we have

\[
\|Az\| \leq \int_0^1 \Phi(s)g(s)f(s,z(s))\,ds
\]

and by Proposition 3 we have

\[
Az(t) \geq c(t)\int_0^1 \Phi(s)g(s)f(s,z(s))\,ds \quad \text{for} \quad t \in [0,1].
\]

This implies \( Az(t) \geq c(t)\|Az\| \) for \( t \in [0,1] \) and \( z(t) \geq c(t)\|z\| \) for \( t \in [0,1] \).

By Proposition 1 we can obtain the following inequality on a function which satisfies (2).

Theorem 5. Assume that \( y \in C^1[0,1] \) satisfies (2) and \( y'' : [0,1] \rightarrow (-\infty,0] \) is measurable such that \( \Phi y'' \) is integrable. Then \( y(t) \geq c(t)\|y\| \) for \( t \in [0,1] \).

Proof. Let \( z(t) = -y''(t) \) for \( t \in [0,1] \). Then \( y(t) = \int_0^1 k(t,s)z(s)\,ds \) for \( t \in [0,1] \). Hence, we have \( \|y\| \leq \int_0^1 \Phi(s)z(s)\,ds \) and by Proposition 1 we have

\[
y(t) \geq c(t)\int_0^1 \Phi(s)z(s)\,ds \quad \text{for} \quad t \in [0,1].
\]

This implies \( y(t) \geq c(t)\|y\| \) for \( t \in [0,1] \).

Some similar inequalities related to other boundary value problems can be found in, for example, [1, 2, 6, 12, 15, 25].

Now, we consider \( m \) and \( M(a,b) \) when \( g(t) \equiv 1 \). By calculation, we obtain for each \( t \in [a,b] \),

\[
\int_a^b k(t,s)\,ds = \begin{cases} 
\frac{-1}{2}t^2 + at + \frac{(1-\alpha)(b-a)}{1-\alpha} - \frac{b^2}{2} & \text{if } a < b \leq \eta \\
\frac{-1}{2}t^2 + at + \frac{-\alpha^2 - b^2 + 2\alpha b - 2a}{2(1-\alpha)} & \text{if } a \leq \eta \leq b \\
\frac{-1}{2}t^2 + at + \frac{-b^2 + \alpha b^2 + 2b - 2a}{2(1-\alpha)} & \text{if } \eta \leq a < b
\end{cases}
\]

When \( a = 0 \) and \( b = 1 \), \( m = \frac{2(1-\alpha)}{1-\alpha^2} \), which was given in [22].

Let \( M(a,b) = (\min_{t \in [a,b]} \int_a^b k(t,s)\,ds)^{-1} \). Noting that \( \int_a^b k(t,s)\,ds \) is decreasing on \([a,b]\), we have \( M(a,b) = (\int_a^b k(b,s)\,ds)^{-1} \).

As we mention in the above section, we wish to find \( a, b \) such that \( M(a,b) \) and \( M(a,b)c(a,b) \) are their minimum values. We consider three possible choices for \( a, b \): (i) \( 0 \leq a < b \leq \eta \); (ii) \( 0 \leq a \leq \eta \leq b \leq 1 \) and (iii) \( \eta \leq a < b \leq 1 \).

Theorem 6. Let \( 0 \leq a < b \leq \eta \). Then (i)

\[
M(a,b) \geq \begin{cases} 
M(0,\eta) = \frac{1-\alpha}{\eta(1-\eta)} & \text{if } \eta \leq \frac{1}{2-\alpha} \\
M(0,\frac{1-\alpha}{2(1-\alpha)}) = \frac{4(1-\alpha)^2}{(1-\alpha^2)^2} & \text{if } \eta \geq \frac{1}{2-\alpha}
\end{cases}
\]
(ii) \( M(a, b)c(b) \geq M(0, \eta)c(\eta) = \frac{1 - \alpha}{\eta(1 - \alpha \eta)} \) for \( 0 \leq a < b \leq \eta \).

Proof. (i) Let \( h(a, b) = \int_a^b k(b, s) ds = -b^2 + ab + \frac{(1 - \alpha \eta)(b - a)}{1 - \alpha} \) for \( 0 \leq a < b \leq \eta \). Then \( h(a, b) \leq h(0, b) \) for \( 0 \leq a < b \leq \eta \). Let \( g(b) = -b^2 + \frac{1 - \alpha \eta}{1 - \alpha} b \) for \( b \in [0, 1] \).

Then \( g(b) = h(0, b) \) for \( b \in (0, \eta] \) and \( g'(b) = -2(b - \frac{1}{2(1 - \alpha)}) \) for \( b \in [0, 1] \). Note that \( \frac{1 - \alpha \eta}{2(1 - \alpha)} \leq \eta \) if and only if \( \frac{1}{2 - \alpha} \leq \eta \). Hence,

\[
g(b) \leq \begin{cases} 
  g(\eta) = h(0, \eta) = \frac{\eta(1 - \eta)}{1 - \alpha} & \text{if } \eta \leq \frac{1}{2 - \alpha} \\
  g\left(\frac{1 - \alpha \eta}{2(1 - \alpha)}\right) = h(0, \frac{1 - \alpha \eta}{2(1 - \alpha)}) = \frac{(1 - \alpha \eta)^2}{4(1 - \alpha)^2} & \text{if } \eta \geq \frac{1}{2 - \alpha}
\end{cases}
\]

This implies (6) holds.

(ii) Let \( \omega(b) = \frac{c(b)}{g(b)} \) for \( b \in (0, \eta] \). Then we have for \( b \in (0, \eta] \),

\[
\omega'(b) = -\frac{1 - \alpha}{(1 - \alpha \eta)(g(b))^2} (b - \frac{1 - \alpha \eta}{1 - \alpha})^2 < 0.
\]

This implies \( \omega(b) \geq \omega(\eta) \) for \( b \in (0, \eta] \) and \( M(a, b)c(b) \geq M(0, \eta)c(\eta) \) for \( a < b \leq \eta \). \( \square \)

Theorem 7. Let \( 0 \leq a \leq \eta \leq b \leq 1 \). Then

(i)

\[
M(a, b) \geq \begin{cases} 
  \frac{1}{2 - \alpha} & \text{if } \eta \leq \frac{1}{2 - \alpha} \\
  \frac{1 - \alpha}{\eta(1 - \eta)} & \text{if } \eta \geq \frac{1}{2 - \alpha}
\end{cases}
\]

(ii) \( M(a, b)c(b) \geq M(0, b_0)c(b_0) \) for \( a \leq \eta \leq b \), where \( b_0 = \frac{1 - \alpha \eta}{1 - \alpha} - \frac{1}{2 - \alpha} \sqrt{\frac{\alpha}{2 - \alpha}} \).

Proof. (i) Let \( h(a, b) = \int_a^b k(b, s) ds = -\frac{1}{2} b^2 + ab + \frac{-\alpha \eta^2 - 2(1 - \alpha)\eta a - b^2 + 2b}{2(1 - \alpha)} \) for \( 0 \leq a \leq \eta \leq b \leq 1 \). Then \( h(a, b) \leq h(0, b) \) for \( 0 \leq a \leq \eta \leq b \leq 1 \). Let \( g(b) = -\frac{1}{2} b^2 + \frac{-\alpha \eta^2 - b^2 + 2b}{2(1 - \alpha)} \) for \( b \in [0, 1] \). Then \( g(b) = h(0, b) \) for \( b \in [\eta, 1] \) and \( g'(b) = -\frac{1}{2} (1 - (2 - \alpha)b) \). This implies

\[
g(b) \leq \begin{cases} 
  g\left(\frac{1}{2 - \alpha}\right) = h(0, \frac{1}{2 - \alpha}) = \frac{1 - \alpha \eta^2(2 - \alpha)}{(2(1 - \alpha))(2 - \alpha)} & \text{if } \eta \leq \frac{1}{2 - \alpha} \\
  g(\eta) = h(0, \eta) = \frac{\eta(1 - \eta)}{1 - \alpha} & \text{if } \eta \geq \frac{1}{2 - \alpha}
\end{cases}
\]

This implies (7) holds.

(ii) Let \( \omega(b) = \frac{c(b)}{g(b)} \) for \( b \in (\eta, 1] \). Then \( \omega'(b) \) has the same sign as

\[
\omega_1(b) = -(1 - \alpha)[-(2 - \alpha)b^2 + 2b - \alpha \eta^2] - 2[1 - \alpha \eta - (1 - \alpha)b][1 - (2 - \alpha)b].
\]
Let $g$ have the form

\[
\omega(\phi) = -(1 - \alpha)(2 - \alpha)b^2 + 2(1 - \alpha \eta)(2 - \alpha)b + (1 - \alpha)\alpha \eta^2 - 2(1 - \alpha \eta)
\]

\[
= -(1 - \alpha)(2 - \alpha)[\frac{1 - \alpha \eta}{1 - \alpha} - b]^2 - \frac{\alpha(1 - \eta)^2}{(1 - \alpha)^2(2 - \alpha)}
\]

It is easy to verify that $b_0$ is the unique positive solution of the above equation and $b_0 \in (\eta, 1)$. Moreover, $\omega_1(b) \leq 0$ for $b \in (\eta, b_0)$ and $\omega_1(b) \geq 0$ for $b \in [b_0, 1)$. This implies $\omega(b) \geq \omega(b_0)$ for $b \in (\eta, 1)$. Hence, $M(a, b)c(b) \geq M(0, b_0)c(b_0)$ for $0 \leq a \leq \eta \leq b \leq 1$.

**Theorem 8.** Let $\eta \leq a < b \leq 1$. Then

(i) $M(a, b) \geq M(\eta, \frac{1}{2} + (1 - \alpha)\eta) = \frac{2(1 - \alpha)(2 - \alpha)}{1 - \eta}$.

(ii) $M(a, b)c(b) \geq M(\eta, b_0)c(b_0)$, where $b_0$ is the same as in Theorem 7.

**Proof.** Let $h(a, b) = \int_b^{\eta} k(b, s) \, ds = -\frac{1}{2}b^2 + ab + \frac{-b^2 + \alpha a^2 + 2b - 2a}{2(1 - \alpha)}$ for $\eta \leq a < b$. Then $h(a, b) \leq h(\eta, b)$ for $\eta \leq a < b \leq 1$. Let $g(b) = -\frac{1}{2}b^2 + \eta b + \frac{-b^2 + \alpha \eta^2 + 2b - 2\eta}{2(1 - \alpha)}$ for $b \in [0, 1]$. Then $g(b) = h(\eta, b)$ for $b \in (\eta, 1]$ and $g'(b) = \frac{2 - \alpha}{1 - \alpha} \frac{1 + (1 - \alpha)\eta}{2 - \alpha} - b$. Noting that $\eta < \frac{1}{2} + (1 - \alpha)\eta < 1$, we have for $b \in (\eta, 1]$,

\[
g(b) \leq g\left(\frac{1 + (1 - \alpha)\eta}{2 - \alpha}\right) = \frac{(1 - \eta)^2}{2(1 - \alpha)(2 - \alpha)}
\]

The result (i) follows.

(ii) Let $\omega(b) = \frac{c(b)}{g(b)}$ for $b \in (\eta, 1]$. Then $\omega'(b)$ has the same sign as

\[
\omega_2(b) = -(1 - \alpha)\left[-(2 - \alpha)b^2 + 2(1 + \eta - \alpha \eta)b - (2 - \alpha)(2 - \alpha)\eta - (1 - \alpha \eta) - (1 - \alpha)b\right] - [2(2 - \alpha)b + 2(1 + \eta - \alpha \eta)]
\]

Then $\omega_2(b) = \omega_1(b)$ and $M(a, b)c(b) \geq M(\eta, b_0)c(b_0)$ for $\eta \leq a < b \leq 1$.

By Theorems 3 and 7 we obtain the following inequality which was first obtained by Webb (see Theorem 3.1 in [22]): For $a, b \in [0, 1],

\[
M(a, b) \geq \begin{cases} 
M(0, \frac{1}{2 - \alpha}) & \text{if } \eta \leq \frac{1}{2 - \alpha} \\
M(0, \frac{1 - \alpha \eta}{2(1 - \alpha)}) & \text{if } \eta \geq \frac{1}{2 - \alpha}
\end{cases}
\]

4. Characteristic values of (1)-(2). In this section, we are interested in the smallest positive characteristic value $\mu_1$ with positive eigenfunctions of (1)-(2) with $g(t) \equiv 1$. It is known that $\mu_1 \in (0, (\pi/2)^2)$ (see [24]) and the positive eigenfunctions have the form $\varphi_1(t) = \cos(\sqrt{\mu_1}t)$. By Theorem 1 $m \leq \mu_1 \leq M(a, b)$. The following new result shows that these inequalities are strict.
Theorem 9. For $a, b \in [0,1]$, $m < \mu_1 < M(a, b)$.

Proof. Let $d = c(1) \int_0^1 \Phi(s)(1 - \varphi_1(s))\,ds$. Then $d > 0$. By Theorem 10, we have for $t \in [0,1]$,

$$\int_0^1 k(t, s)(1 - \varphi_1(s))\,ds \geq c(t) \int_0^1 \Phi(s)(1 - \varphi_1(s))\,ds \geq c(1) \int_0^1 \Phi(s)(1 - \varphi_1(s))\,ds = d > 0.$$ 

This implies $\int_0^1 k(t, s)\,ds \geq d + \int_0^1 k(t, s)\varphi_1(s)\,ds$. Therefore, we have

$$\varphi_1(t) = \mu_1 \int_0^1 k(t, s)\varphi_1(s)\,ds \leq \mu_1 (\int_0^1 k(t, s)\,ds - d).$$

Taking the supremum over $t \in [0,1]$ gives

$$1 \leq \mu_1 (\sup_{t \in [0,1]} \int_0^1 k(t, s)\,ds - d) = \mu_1/m - \mu_1 d < \mu_1/m,$$

so that $m < \mu_1$.

Let $d(a, b) = \min\{c(t) : t \in [a, b]\} \int_a^b \Phi(s)(\varphi_1(s) - q(\varphi_1))\,ds$. Then $d(a, b) > 0$. By Proposition 1, we have for $t \in [0,1]$,

$$\int_a^b k(t, s)(\varphi_1(s) - q(\varphi_1))\,ds \geq c(t) \int_a^b \Phi(s)(\varphi_1(s) - q(\varphi_1))\,ds \geq d(a, b) > 0.$$ 

This implies $\int_a^b k(t, s)\varphi_1(s)\,ds \geq d(a, b) + \int_a^b k(t, s)q(\varphi_1)\,ds$. Therefore, we have

$$\varphi_1(t) = \mu_1 \int_0^1 k(t, s)\varphi_1(s)\,ds \geq \mu_1 \int_a^b k(t, s)\varphi_1(s)\,ds \geq \mu_1 d(a, b) + \mu_1 q(\varphi_1) \frac{1}{M(a, b)}.$$ 

Taking minimum over $[a, b]$ gives

$$q(\varphi_1) \geq \mu_1 d(a, b) + \mu_1 q(\varphi_1) \frac{1}{M(a, b)} > \mu_1 q(\varphi_1) \frac{1}{M(a, b)}.$$ 

This implies $\mu_1 < M(a, b)$. \qed

By Theorem 4, we obtain the following new result which is independent of $a, b$.

Theorem 10. $\mu_1 < (\int_0^1 \Phi(s)c(s)\,ds)^{-1} = \frac{6(1 - \alpha)(1 - \alpha \eta)}{(1 + 2\alpha)\alpha \eta^3 - 3\alpha \eta^2 - 3\alpha \eta + 2 + \alpha}.$

Proof. Let $d_1 = \inf_{t \in [0,1]} \int_0^1 k(t, s) - c(t)\Phi(s)\,ds$. Then $d_1 > 0$. This implies

$$\int_0^1 k(t, s)\varphi_1(s)\,ds \geq c(t) \int_0^1 \Phi(s)\varphi_1(s)\,ds + d_1.$$ 

By Theorem 4, we have $\varphi_1(s) \geq c(s)$ for $s \in [0,1]$. Therefore, we have

$$\varphi_1(t) = \mu_1 \int_0^1 k(t, s)\varphi_1(s)\,ds \geq \mu_1 c(t) \int_0^1 \Phi(s)\varphi_1(s)\,ds + \mu_1 d_1 \geq \mu_1 c(t) \int_0^1 \Phi(s)c(s)\,ds + \mu_1 d_1.$$
Assume that

\[ D. \text{ Guo and V. Lakshmikantham, Nonlinear Problems in Abstract Cones, Academic Press,} \]

\[ L. \text{ Erbe, R. P. Agarwal and D. O’Regan,} \]

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\[ \text{L. H. Erbe, S. Hu and H. Wang,} \]

\[ \text{Chuanzhi Bai and Jinxuan Fang,} \]

\[ \text{R. P. Agarwal and D. O’Regan,} \]

This, together with \( 1 + \) By calculation, we obtain

\[ \text{Theorem 11. Assume that } \eta \leq \frac{1}{2 - \alpha}. \text{ Then} \]

\[ \mu_1 < (\int_0^1 \Phi(s)c(s) \, ds)^{-1} < M(0, \frac{1}{2 - \alpha}). \]

**Proof.** Let \( \eta \leq \frac{1}{2 - \alpha}. \) Then \( M(0, \frac{1}{2 - \alpha}) = \frac{2(1 - \alpha)(2 - \alpha)}{1 - \alpha \eta^2(2 - \alpha)} \) and the difference

\[ M(0, \frac{1}{2 - \alpha}) - (\int_0^1 \Phi(s)c(s) \, ds)^{-1} \]

has the same sign as the following value:

\[ h(\alpha, \eta) = (2 - \alpha)[(1 + 2\alpha)\alpha \eta^3 - 3\alpha \eta^2 - 3\alpha \eta + 2 + \alpha] - 3(1 - \alpha \eta^2(2 - \alpha)]. \]

By calculation, we obtain \( h(\alpha, \eta) = \alpha(1 - \alpha)(2 - \alpha)\eta^3 - 3\alpha(1 - \alpha)\eta + (1 - \alpha^2) \) and

\[ \frac{\partial h}{\partial \eta}(\alpha, \eta) = 3\alpha(1 - \alpha)(2 - \alpha)(\eta^2 - \frac{1}{2 - \alpha}). \]

This, together with \( 1 + \alpha > 2\alpha \) and \( \sqrt{2 - \alpha} > 1, \) implies

\[ h(\alpha, \eta) \geq h(\alpha, \frac{1}{\sqrt{2 - \alpha}}) = \frac{(1 - \alpha)(1 + \alpha)}{\sqrt{2 - \alpha}} > 0. \]

Hence, we obtain \( M(0, \frac{1}{2 - \alpha}) > (\int_0^1 \Phi(s)c(s) \, ds)^{-1}. \) \hfill \Box

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E-mail address: klan@ryerson.ca