DISPERSE ESTIMATE FOR THE WAVE EQUATION
WITH THE INVERSE-SQUARE POTENTIAL

FABRICE PLANCHON
Laboratoire Analyse, Géométrie & Applications, UMR 7539, Institut Galilée
Université Paris 13, 99 avenue J.B. Clément, F-93430 Villetaneuse, FRANCE

JOHN G. STALKER
Department of Mathematics
Princeton University, Princeton N.J. 08544

A. SHADI TAHVILDAR-ZADEH
Department of Mathematics, Rutgers, The State University of New Jersey
110 Frelinghuysen Road, Piscataway NJ 08854

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Abstract. We prove that spherically symmetric solutions of the Cauchy problem
for the linear wave equation with the inverse-square potential satisfy a modified
dispersive inequality that bounds the $L^\infty$ norm of the solution in terms of certain
Besov norms of the data, with a factor that decays in $t$ for positive potentials. When
the potential is negative we show that the decay is split between $t$ and $r$, and the
estimate blows up at $r = 0$. We also provide a counterexample showing that the
use of Besov norms in dispersive inequalities for the wave equation are in general
unavoidable.

1. Introduction. Consider the following linear wave equation

$$\begin{cases}
\Box u + \frac{a}{|x|^2} u = 0 \\
u(0,x) = f(x) \\
\partial_t u(0,x) = g(x)
\end{cases} \tag{1.1}$$

where $\Box_n = \partial^2_t - \Delta_n$ is the D’Alembertian in $\mathbb{R}^{n+1}$ and $a$ is a real number. The
interest in this equation comes from the potential term being homogeneous of degree
-2 and therefore scaling the same way as the D’Alembertian term. This implies that
perturbation methods alone cannot be used in studying the effect of this potential.
In particular, the value of the constant $a$ is important.

In [8] we showed that in the radial case, i.e. when the data – and thus the
solution – are radially symmetric, the solution to (1.1) satisfies generalized space-
time Strichartz estimates as long as $a > -(n-2)^2/4$. In this paper we continue
the study of the radial case by proving a dispersive estimate, i.e. a decay-in-time
estimate for the $L^\infty$ norm of the solution. We note that for non-radial data, the
same estimate can be proven in the same way for each component in the spherical

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