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GLOBAL STRUCTURE OF 2-D INCOMPRESSIBLE FLOWS

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Dedicated to Professor Chen Wenyuan on the Occasion of his 70th Birthday

Abstract. The main objective of this article is to classify the structure of divergence-free vector fields on general two-dimensional compact manifold with or without boundaries. First we prove a Limit Set Theorem, Theorem 2.1, a generalized version of the Poincaré-Bendixson to divergence-free vector fields on 2-manifolds of nonzero genus. Namely, the ω (or α) limit set of a regular point of a regular divergence-free vector field is either a saddle point, or a closed orbit, or a closed domain with boundaries consisting of saddle connections. We call the closed domain ergodic set. Then the ergodic set is fully characterized in Theorem 4.1 and Theorem 5.1. Finally, we obtain a global structural classification theorem (Theorem 3.1), which amounts to saying that the phase structure of a regular divergence-free vector field consists of finite union of circle cells, circle bands, ergodic sets and saddle connections.

Introduction. The main motivation of this article long with [1, 4, 5, 6, 7, 8] is to develop a geometric theory of 2-D incompressible fluid flows. Our main philosophy is to classify the topological structure and its transitions of the *instantaneous* velocity field (i.e. streamlines in the Eulerian coordinates), treating the time variable as a parameter. Based on this philosophy, our project contains studies in two directions: 1) the development of a global geometric theory of divergence-free vector fields on general two-dimensional compact manifolds with or without boundary, and 2) the connections between the solutions of the Navier-Stokes (or Euler) equations and the dynamics of the velocity fields in the physical space.

This article along with [4, 7, 8] is in the first direction. In [4, 7], we studied the case where M is a compact sub-manifold of S^2. In this case, the classical Poincaré-Bendixson theorem holds true. Hence in particular, we proved that a divergence-fee vector field is structurally stable with divergence-free vector fields perturbations if and only if (1) v is regular; (2) all interior saddle points of v are self-connected; and (3) each boundary saddle point is connected to boundary saddles on the same connected component of the boundary. These conditions are quite different from those in the classical M. Peixoto [11] structural stability theorem for general vector

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field with general vector fields perturbations on two-dimensional compact manifolds; they are (i) the field can have only a finite number of singularities and closed orbits (critical elements) which must be hyperbolic; (ii) there are no saddle connections; (iii) the non wandering set consists of singular points and closed orbits. A direct consequence of the above two sets of necessary and sufficient conditions is that no divergence-free vector field is structurally stable under general C^r vector fields perturbations. Such a drastic change in the stable configurations is explained by the fact that divergence-free fields preserve volume and so attractors and sources can never occur for these fields. In particular, this makes it natural the restriction that saddles in the boundary must be connected with saddles on the boundary on the same connected component, in the third condition.

Progress has also been made in the second direction recently in [1, 5]. In [5], we proved that when M is a compact sub-manifold of S^2 for any external forcing in an open and dense subset of $C^\alpha(TM)$ $(0 < \alpha < 1)$, all steady state solutions of the two-dimensional Navier-Stokes equations are structurally stable. In [1], we study the structural bifurcation of one-parameter families of divergence-free vector fields. In particular, we classified the flow trajectories and their bifurcation near degenerate singular points near boundary. Prototype of examples based on the bifurcation result and the classification of the flow structure match precisely numerical simulations for boundary separations of fluid flows. The nature of flow's separation from the boundary plays a fundamental role in many physical problems, we believe this study will lead to an essential understanding on boundary layer separation, a long standing problem in fluid dynamics.

The main objective of this article is to study the dynamics of divergence-free vector fields on general 2D compact manifolds with or without boundaries. As we know, the classical Poincaré-Bendixson theorem holds true only for sub-manifolds of S^2 (or P^2 or K^2). Hence to obtain a structural classification for divergence-free vector fields on general two-dimensional manifolds, we need a new version of Poincaré-Bendixson theorem to understand the ω and α limit sets. It is well known that flows on a torus may be non-trivially recurrent, and the ω and α limit sets can be very complicated sets. For instance, the ω and α limit sets of the Cherry flow can have the structure of Cantor sets; see [10].

Thanks to the incompressibility conditions, the divergence-free vector fields have better properties. First, we prove in this article a Limit Set Theorem, Theorem 2.1. This theorem can be considered as a version of the Poincare-Bendixson theorem for divergence-free vector fields on general 2D compact manifolds with or without boundaries. The Limit Set Theorem says that the ω (resp. α) limit of a regular point of a regular divergence-free vector field is either a saddle, or a non-limiting closed orbit, or an ergodic set which is a closed domain with boundaries consisting of saddle connections of finite length. The structure of the Cherry flow shows that the same result is not true for general vector fields, and the divergence-free condition is crucial for the ergodic set being a closed domain.

Second, as a corollary of the Limit Set Theorem, we prove a global structural classification theorem, Theorem 3.1, for divergence-free vector fields on general two dimensional compact manifolds with or without boundaries. This theorem amounts to saying that for a regular divergence-free vector field on M, the manifold M can be decomposed into a finite union of invariant sets of v consisting of circle cells, circle bands, ergodic sets and saddle connections.

Third, in the Limit Set Theorem, we have shown that an ergodic set is a closed domain with boundaries consisting of saddle connections of finite length. The detailed structure of the ergodic sets is characterized by Theorems 4.1 and 5.1 in Sections 4 and 5.

This article is organized as follows. Some basic properties of divergence-free vector fields are given in Section 1. The Limit Set Theorem, Theorem 2.1, is stated and proved in Section 2. Section 3 establishes a global structural classification theorem for incompressible flows on general 2–D compact manifolds with or without boundaries. Detailed topological structure of ergodic sets are proved in Sections 4 and 5.

1. **Preliminaries.** Let M be a two dimensional differentiable manifold with boundary ∂M. For any integer $r \geq 0$, let $C^r_n(TM)$ be the space of all r-th differentiable vector fields v on M such that $v|_{\partial M} \in C^r(T\partial M)$, namely the restriction of any r-th differentiable vector field $v \in C^r(TM)$ on the boundary ∂M is a r-th differentiable vector field of the tangent bundle of ∂M.

Let $r \geq 1$. A point $p \in M$ is called a singular point of $v \in C^r_n(TM)$ if $v(p) = 0$; a singular point p of v is called non-degenerate if the Jacobian matrix $Dv(p)$ is invertible; v is called regular if all singular points of v are non-degenerate. For convenience, we set

$$D^r(TM) = \{v \in C^r_n(TM)| \text{ div } v = 0\},$$
$$D^r_0(TM) = \{v \in D^r(TM)| \ v \text{ is regular } \}.$$

Here we assume that M is a Riemannian manifold with the Riemannian metric g. The differential operator div is the divergence operator of a vector field, which can be defined in terms of the Levi-Civita connection.

Let $\Phi(x,t)$ be the orbit passing through $x \in M$ at $t = 0$ of the flow generated by v. The ω-limit set $\omega(x)$ and the α-limit set $\alpha(x)$ of the trajectory $\Phi(x,t)$ are defined by

$$\omega(x) = \{y \in M \mid \text{ there exist } t_n \to \infty \quad \text{ such that } \Phi(x,t_n) \to y\},$$
$$\alpha(x) = \{y \in M \mid \text{ there exist } t_n \to -\infty \quad \text{ such that } \Phi(x,t_n) \to y\}.$$

An orbit with its end points is called a saddle connection if its α and ω-limit sets are saddle points.

Thanks to the divergence-free conditions, the properties of divergence-free vector fields are quite different from those of general vector fields. In particular, it is easy to see that for any $v \in D^r(TM)(r \geq 1)$ an interior non-degenerate singular point of v can either be a center or a saddle, and a non-degenerate boundary singularity must be a saddle.

Furthermore, the divergence-free condition eliminates the existence of limiting cycles. Hence the following lemma:

Lemma 1.1. *Let M be $v \in D^r(TM)(r \geq 1)$, and $V \subset M$ be the set consisting of all closed orbits and centers of v. Then V is open.*

2. **Limit set theorem.**

2.1. **The Limit Set Theorem.** As we know, the Poincaré–Bendixson theorem is crucial for the understanding of the global structure of a vector field. In this section, we generalize this theorem to general two-dimensional compact manifolds. We proceed as follows:

Let $V \subset M$ the set of all closed orbits and centers of $v \in D^r(TM)$. By Lemma 1.1, V is an open set in M, therefore $K = \overline{M - \bar{V}}$ is a closed set. We show in next lemma that K is in fact a closed domain. Here a set Ω is called a closed domain if $\Omega = \mathrm{cl}\overset{\circ}{\Omega}$.

Lemma 2.1. *Let M be a two-dimensional compact manifold with or without boundary, and $v \in D_0^r(TM)$ $(r \geq 1)$ be a regular vector field.*

1). *If $K = \emptyset$, then $\overline{V} = M$ and $M - V$ consists of saddle connections;*
2). *If $K \neq \emptyset$, then K is a closed domain, i.e. cl $\overset{\circ}{K} = K$.*

The main theorem in this section is

Theorem 2.1. (Limit Set Theorem) *Let M be a two-dimensional compact manifold with or without boundary and $v \in D_0^r(TM)(r \geq 1)$ be a regular vector field. Let $x_0 \in M$ be a regular point of v. Then the α and ω — limit sets $\alpha(x_0)$ and $\omega(x_0)$ must be one of the following types:*

1). *a closed orbit, but not a limit cycle,*
2). *a saddle point, or*
3). *a connected closed domain $\Omega \subset K = \overline{M - \bar{V}}$, with $\partial\Omega$ consisting of saddle connections having finite length.*

Remark 2.1. Results similar to the Limit Set Theorem is not true for general vector fields on torus. The Cherry flow, for instance, provides a nice counterexample; see [10]. Detailed structure of the closed domain Ω, also called ergodic sets is given in Sections 4 and 5. When $M \subset S^2$, the classical Poincare-Bendixon asserts that only first two could happen.

2.2. **The proof of the Limit Set Theorem.** We start with the following lemma:

Lemma 2.2. *Let $v \in D_0^r(TM)$ $(r \geq 1)$ be a regular divergence-free vector field and $\Gamma \subset M$ be a closed curve consisting of saddle connections of v. Assume that there exists an open set $D \subset M$ such that Γ is a connected component of ∂D, and for any saddle point $p \in \Gamma$, p has no connecting orbits in $\overset{\circ}{D}$. Then for any $x \in \overset{\circ}{D}$ sufficiently close to Γ the orbit $\Phi(x,t)$ is closed.*

Proof. Let $p_1, \cdots, p_k \in \Gamma$ be the saddle points of v on Γ. These saddle points divide Γ into m orbit curves $\gamma_1, \cdots, \gamma_m$. Notice that there exists an open set $D \subset M$ such that Γ is a connected component of ∂D, and for any p_i $(1 \leq i \leq k)$, p_i has no connecting orbits in $\overset{\circ}{D}$. Hence the limit sets of the orbits $\gamma_j (1 \leq j \leq m)$ satisfies $\omega(\gamma_j) = \alpha(\gamma_{j+1}), \quad j \in \mathbb{Z}_m$, where $\mathbb{Z}_m = \mathbb{Z}/m+1$ is the mod m group.

Let $\Sigma \subset D$ be a closed arc transversal to v containing $x_0 \in \Gamma$; see Figure 1. Since each saddle point p_i $(1 \leq i \leq k)$ has no connecting orbits in $\overset{\circ}{D}$, for any $x \in \Sigma$ $(x \neq x_0)$ sufficiently close to x_0, the orbit $\Phi(x,t)$ will return to intersect with Σ at x_1; see Figure 1.

If $x_1 \neq x_0$, let C be the closed curve defined by

$$C = \{\Phi(x,t) \mid 0 \leq t \leq t_1\} \cup [x, x_1],$$

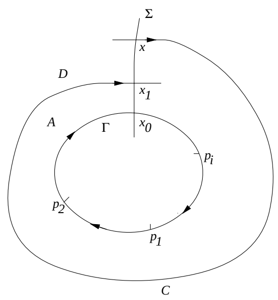

FIGURE 1

where $[x, x_1] \subset \Sigma$, $\Phi(x, 0) = x$, $\Phi(x, t_1) = x_1$. Let A be the domain enclosed by C and γ. Since Σ is transversal to v which points inwards into A, the positive orbit of x_1 is contained in A. Hence $\Phi(C, t) \subset A$ $\forall t > 0$. This yields $|\Phi(A, t)| < |A|$, $\forall t > 0$, which contradicts to $v \in D^r(TM)$ being divergence-free. Hence $\Phi(x, t)$ is a closed orbit. □

Proof of the Limit Set Theorem. We divide the proof into a few steps as follows:

Step 1. Thanks to the divergence-free condition, a non-degenerate interior singular point of $v \in D_0^r(TM)$ must be either a center or a saddle, and a non-degenerate boundary singular point must be a saddle. Hence if $\alpha(x_0)$ (resp. $\omega(x_0)$) is a singular point, then it must be a saddle.

By the Poincaré–Bendixson theorem for divergence-free vector fields and Lemma 2.4, it suffices then to prove Assertion 3) in the Limit Set Theorem. Without loss of generality we assume that the closed domain $K = \overline{M - \bar{V}}$ is connected.

Step 2. Since the number of saddle connections is finite, we choose $x_0 \in K$ such that $\omega(x_0)$ is not a saddle point. It suffices then to prove that $\omega(x_0) = \text{cl} \overset{\circ}{\omega} (x_0)$, where $\text{cl} \overset{\circ}{\omega} (x_0)$ is the closure of the interior of $\omega(x_0)$.

Assume otherwise, i.e., $\omega(x_0) \neq \text{cl} \overset{\circ}{\omega} (x_0)$, then $\text{cl} \overset{\circ}{\omega} (x_0) \subset \omega(x_0)$. We need to derive a contradiction. To this end, we define a set $N = \overset{\circ}{K} - \omega(x_0)$. Obviously N is open and non-empty, otherwise $K = \omega(x_0)$, a contradiction to the assumption that $\omega(x_0)$ is not a closed domain.

Moreover, N is an invariant set, since both $\overset{\circ}{K} = M - \bar{V}$ and $\omega(x_0)$ are invariant. Hence ∂N is also an invariant set consisting of orbits and saddle points of v.

Step 3. Claim: each connected component of ∂N is of finite length. To see this, let $\Sigma \subset \partial N$ be a connected component of ∂N having infinite length.

Since there are only finite number of saddle points of v, we choose $\widetilde{x} \in \Sigma \subset \partial N$ such that the orbit $\Phi(\widetilde{x}, t) \subset \Sigma$ $(0 \leq t < \infty)$ is of infinite length. Let $x_1 = \Phi(\widetilde{x}, t_1)$ $(0 \leq t_1 < \infty)$ and $\Gamma \subset M$ be an arc transversal to v starting from x_1 and entering N. Since M is compact and the orbit $\Phi(\widetilde{x}, \cdot)$ is of infinite length, there is a time t_2 $(t_1 < t_2)$ such that $x_2 = \Phi(\widetilde{x}, t_2) \in \Gamma$ and $\Phi(\widetilde{x}, t) \notin \Gamma$ for any $t_1 < t < t_2$. For simplicity, let $(x_1, x_2) \subset \Gamma$ be the open segment from x_1 to x_2. Then for any $x \in (x_1, x_2) \cap N$, there is a time $\tau > 0$ such that either $\Phi(x, \tau) \in \Gamma$, $\Phi(x, \tau) \notin (x_1, x_2)$ and $\Phi(x, t) \notin \Gamma$ for any $0 < t < \tau$, or $\Phi(x, \tau) \in (x_1, x_2)$ and $\Phi(x, t) \notin \Gamma$ for any $0 < t < \tau$. It implies that there is a point $y \in (x_1, x_2)$ such that $y \in \partial N$, $(x_1, y) \subset N$ and $(x_1, y) \cap \partial N = \emptyset$. Among all those arcs Γ, there is an arc Γ_1 such that the

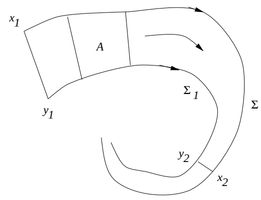

FIGURE 2

length of the open segment $(x_1, y_1) \subset \Gamma_1$ is the smallest. We call the length of the line segment (x_1, y_1) the width of N at x_1, and y_1 the dual point of x_1.

Since the area of M is finite, the width $h(t)$ of N at the point $\Phi(\widetilde{x}, t) \subset \Sigma \subset \partial N$ converges to zero as $t \to \infty$; see Figure 2.

Let

$$\Sigma_1 = \{y_t \in \partial N \,|\, y_t \text{ is the dual point of } \Phi(\widetilde{x}, t), 0 \le t < \infty\}.$$

Obviously, Σ_1 is of infinite length. Let $\widetilde{N} \subset N$ be the sub-domain enclosed by the line segment $[x_1, y_1]$, $\widetilde{\Sigma} = \{x = \Phi(\widetilde{x}, t) | t_1 \le t < \infty\}$ and the dual points of $\widetilde{\Sigma}$. Without loss of generality, we assume that there is no any singular point of v in $\widetilde{N}$.

It is easy to see that for any $\varepsilon > 0$ sufficiently small, there is a time $T > 0$ such that for any $t > T$ the area $|\widetilde{N}_t| < \varepsilon$. Here $\widetilde{N}_t \subset N$ is the sub-domain enclosed by the line segment $[x_t, y_t]$, $\widetilde{\Sigma}_t = \{x = \Phi(\widetilde{x}, \tau) | t \le \tau < \infty\}$ and the dual points of $\widetilde{\Sigma}_t$, where $x_t = \Phi(\widetilde{x}, t)$ and y_t is the dual point of x_t.

Consider a domain $A \subset \widetilde{N}$ with $|A| > \epsilon_0 > 0$. Then there is a time $T_0 > 0$ such that for any $t > T_0$, $|\widetilde{N}_t| < \varepsilon_0$. Since there is no singular point of v in $\widetilde{N}$, there exists $T_1 > 0$ sufficiently large such that when $t > T_1$, $\Phi(A, t) \subset \widetilde{N}_{T_0}$. Hence

$$|\Phi(A, t)| \le |\widetilde{N}_{T_0}| < \varepsilon_0,$$

a contradiction to the flow Φ being area-preserving.

Step 4. By Step 3, it is easy to see that each connected component of ∂N consists of saddle connections. Therefore, both N and ∂N contains only finite number of connected components. Hence,

$$\partial N = \{x \in M \quad | \quad \exists x_n \in \overset{\circ}{N}, x_n \to x\},$$

thanks to N being an open set with finite number of connected components.

Step 5. Let

$$L_1 = \omega(x_0) \cap \partial K, \qquad L_2 = \omega(x_0) - \overset{\circ}{\omega}(x_0) - L_1.$$

We claim that $L_2 \subset \partial N$.

For any $x \in L_2$, we have $x \in \overset{\circ}{K}$. For any sufficiently small neighborhood $\mathcal{O}(x)$ of x, we have $\mathcal{O}(x) \subset \overset{\circ}{K}$. We now prove that

$$\mathcal{O}(x) \cap N \ne \emptyset.$$

Otherwise, noticing that $N = \overset{\circ}{K} - L_2 - \overset{\circ}{\omega}(x_0)$, we have

$$\mathcal{O}(x) \subset L_2 \cup \overset{\circ}{\omega}(x_0), \qquad \mathcal{O}(x) \cap \overset{\circ}{\omega}(x_0) \ne \emptyset.$$

There are two possibilities. First, $\mathcal{O}(x) \subset \overset{\circ}{\omega}(x_0)$, which contradicts with $x \notin \overset{\circ}{\omega}(x_0)$. Second,

$$\mathcal{O}(x) - \operatorname{cl}[\mathcal{O}(x) \cap \overset{\circ}{\omega}(x_0)]$$

is a nonempty open subset of L_2, a contradiction to $\overset{\circ}{L_2} = \emptyset$.

Therefore by Step 4 we showed that $x \in \bar{N}$ and $x \notin N$. Namely, $L_2 \subset \partial N$.

Step 6. By Steps 2–5, we conclude that

$$\omega(x_0) - \overset{\circ}{\omega}(x_0) \neq \partial \overset{\circ}{\omega}(x_0), \qquad \omega(x_0) = \overset{\circ}{\omega}(x_0) \cup L,$$

with L consisting of saddle connections. This is a contradiction to $\Phi(x_0, \cdot)$ being non-trivially recurrent. The proof is complete. □

3. **Structural Classification.** We prove in this section a global structural classification theorem of divergence-free vector fields. Consider a regular divergence-free vector field $v \in D_0^r(TM)$ $(r \geq 1)$.

1) Near a regular point $p \in M$ of v, the flow is clearly described by the tubular flow box theorem.
2) Let $p \in M$ be a center, then there is an open neighborhood C of p, such that for any $x \in C(x \neq p)$, the orbit $\Phi(x, t)$ is closed. We call the largest neighborhood C of p of this type a **circle cell** of v.
3) Let γ be a closed orbit of v. Then there is a neighborhood of γ containing closed orbits. Let $B \subset M$ be the largest open neighborhood of γ containing closed orbits. If any connected component Σ of ∂B is not a single point, then we call B a **circle band** of v.
4) Let $F \subset M$ be a closed domain such that for any $x \in F$, $\omega(x)$ is either a singular point of v or the whole domain F. We call F an ergodic set. Obviously, ergodic sets corresponds to the closed domain obtained in the Limit Set Theorem, and when $M \subset S^2$, the Poincare-Bendixon theorem asserts the non-existence of ergodic sets.

With the above considerations, we obtain immediately the following global structural classification theorem.

Theorem 3.1. (Structural Classification Theorem) *Let M be a two-dimensional compact manifold with or without boundary, and $v \in D^r(TM)(r \geq 1)$ be regular. Then M can be decomposed into finite union of the following invariant sets of v: 1). circle cells, 2). circle bands, 3). saddle connections, and 4). ergodic sets.*

Remark 3.1. When $M \subset S^2$, no ergodic sets can occur. Therefore, in this case, the flow structure consists of finite union of circle cells, circle bands and saddle connections.

4. **Structure of Ergodic Sets.**

4.1. **The main theorem.** The topological structure of ergodic sets is more complex than that of circle cells and circle bands. On the non-orientable compact manifold with genus $k \geq 4$, there exists non-orientable ergodic sets of vector fields, see [2]. In this subsection, we study topological properties of ergodic sets.

First we recall some topological properties of two–dimensional compact manifolds with boundary. Let M be a two–dimensional compact manifold with boundary, and

∂M have r connected components ($r \geq 1$). The genus $g = g(M)$ of M is defined as follows:

$$g = g(M) = \begin{cases} 1 - \frac{1}{2}(\chi(M) + r) & \text{if } M \text{ orientable,} \\ 2 - \chi(M) - r & \text{if } M \text{ non-orientable,} \end{cases} \tag{1}$$

where $\chi(M)$ is the Euler characteristics of M. Then it is easy to see that

Lemma 4.1. *Let M_1 and M_2 be two two-dimensional compact manifolds with the same orientation, the same genus and the same Euler characteristic. Then M_1 is homeomorphic to M_2.*

To characterize a two–dimensional compact manifold with boundary, we introduce a concept of standard manifolds.

Definition 4.1. *Let N be a compact manifold without boundary and with genus $k \geq 0$. A sub-manifold $M \subset N$ is called a standard manifold of genus k with boundary if each connected component of ∂M is retractable to a point in N.*

By Lemma 4.1, we have

Lemma 4.2. *Any compact manifold of genus k with boundary must be homeomorphic to a standard manifold of genus k with boundary. Hence if $M_1 \subset M$ is a sub-manifold, then $g(M_1) \leq g(M)$.*

Lemmas 4.1 and 4.2 are essentially known; we introduce them to make clear the topological structure of ergodic sets. We need a more general concept of a topological set with genus.

Definition 4.2. *Let N be a compact manifold without boundary and with genus $k \geq 0$. A closed domain $\Omega \subset N$ is called a pseudo–manifold with genus g if*

1). *Ω is connected and $\partial\Omega$ is homeomorphic to a union of finite number of circles S^1, each of which has finite number of common points with the other, and*
2). *there exists a sub-manifold $M \subset N$, such that $\Omega \subset M$ and M is retractable to Ω in N.*

The genus g of Ω is defined to be the genus of M, and M is called an extended manifold of Ω.

Remark 4.1. The genus g of Ω is independent of the extended manifold. In fact, if $M_1, M_2 \subset N$ are two extended manifolds of Ω, then as M_1, M_2 are retractable to Ω in N, we have

$$H_1(M_1, \partial M_1) = H_1(M_2, \partial M_2). \tag{2}$$

Let g_i be the genus of M_i, and b_i be the Betti numbers of $H_1(M_i, \partial M_i)$ ($i = 1, 2$). Then (2) implies $b_1 = b_2$. Hence $g_1 = g_2$.

Remark 4.2. By definition, the genus g of a pseudo–manifold Ω is given by

$$g = \begin{cases} 1 - \frac{1}{2}(\chi(\Omega) + r), & \text{if } \Omega \text{ orientable,} \\ 2 - \chi(\Omega) - r, & \text{if } \Omega \text{ nonorientable,} \end{cases} \tag{3}$$

where $r \geq 0$ is the number of connected components of the boundary of the extended manifold M of Ω.

Remark 4.3. The topological structure of a pseudo–manifold Ω is essentially the same as that of its extended manifold. Therefore, by Lemma 4.1, the topological properties of Ω can be characterized by a standard manifold with the same genus and the same Euler characteristic as Ω.

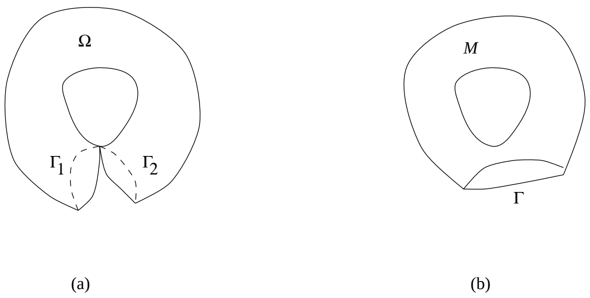

FIGURE 3. a). a pseudo–torus Ω with two circles Γ_1 and Γ_2 b). the extended manifold M of Ω with one boundary component Γ.

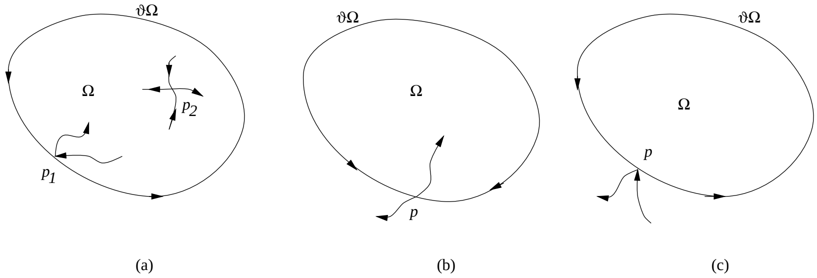

FIGURE 4. a). two Ω–interior saddles p_1 and p_2, b). an Ω–boundary saddle p, and c). an Ω–exterior saddle p.

Remark 4.4. For a pseudo–manifold Ω, $\partial\Omega$ is homeomorphic to a union of r_1 circles S^1. In general, r_1 is not equal to the number r of connected components of the boundary of an extended manifold M of Ω. Example, for the pseudo–torus Ω and its extended manifold M as shown in Figure 3, $r_1 = 2$ and $r = 1$.

Let $p \in \overset{\circ}{M}$ be an interior saddle point of v, then p is connected by four orbits, two of which are born from p, and the other two of which die in p. The orbits which are born from p are the unstable orbits, and the orbits which die in p are the stable orbits. If $p \in \partial M$ be a boundary saddle point, then p is connected by three orbits with two on the boundary and one in the interior of M.

Definition 4.3. *Let Ω be an ergodic set of v.*

1). *A saddle point $p \in \Omega$ of v is an Ω–interior saddle if there are four orbits connecting p in Ω;*
2). *A saddle point $p \in \Omega$ of v is an Ω–boundary saddle if there are only three orbits connecting p in Ω, and*
3). *A saddle point $p \in \Omega$ of v is an Ω–exterior saddle point if there are only two orbits connecting p in Ω.*

See Figure 4 below.

The following theorem characterizes the topological structure of ergodic sets.

Theorem 4.1. *Let M be a compact manifold with or without boundary and with genus $k \geq 1$. Let $v \in D_0^r(TM)$ ($r \geq 1$) be regular and $\Omega \subset M$ be an ergodic set of v. Then Ω is homeomorphic to a pseudo-manifold with or without boundary with the*

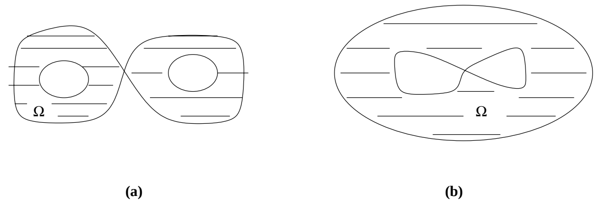

FIGURE 5

genus g of Ω *satisfying*

$$\begin{cases} 1 \leq g \leq k, & \text{if } \Omega \text{ orientable,} \\ 4 \leq g \leq k, & \text{if } \Omega \text{ nonorientable.} \end{cases} \tag{4}$$

Moreover, the Euler characteristic of Ω *is given by*

$$\chi(\Omega) = -s - \frac{b}{2}, \tag{5}$$

where s *is the number of* Ω*–interior saddles and* b *is the number of* Ω*–boundary saddles of* v *in* Ω*. Consequently, the genus of* Ω *is given by*

$$g = \begin{cases} 1 + \frac{1}{2}(s + \frac{1}{2}b - r), & \text{if } \Omega \text{ orientable,} \\ 2 + s + \frac{1}{2}b - r, & \text{if } \Omega \text{ nonorientable,} \end{cases} \tag{6}$$

where r *is the number of connected components of the boundary of the extended manifold* M_1*.*

4.2. **Proof of Theorem 4.1.** By the Limit Set Theorem in the previous subsection, Ω is a connected closed domain in $K = \overline{M - \overline{V}}$, where $V \subset M$ is the set of all closed orbits and centers of v.

We divide the proof into a few steps.

Step 1. We claim that Ω *is a pseudo-manifold.* By Theorem 2.1, $\overline{V} \subset M$ consists of circle cells, circle bands and saddle connections. Since each connected component of $\partial\overline{V}$ contains saddle connections, it must be homeomorphic to $S[x_i, y_i; i = 1, \cdots, n]$, where $S[x_i, y_i; i = 1, \cdots, n]$ is the quotient space

$$S[x_i, y_i; i = 1, \cdots, n] = S^1 / \{x_i = y_i \quad | \quad 1 \leq i \leq n\},$$

and x_i, $y_i \in S^1$ are $2n$ ($n \geq 0$) different points on the circle S^1.

On the other hand, by the Limit Set Theorem, $\partial\Omega$ consists of saddle connections, and $\partial\Omega = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1 \subset \partial\bar{V} \cup \partial M$ and Γ_2 is the common boundaries of two ergodic sets of v if $\Gamma_2 \cap \overset{\circ}{K} \neq \emptyset$. Hence $\partial\Omega$ has the boundary properties of a pseudo–manifold.

It suffices then to prove that there exists a manifold M_1 and a topological set $\Omega_1 \subset M_1$ such that M_1 is retractable to Ω_1 and Ω is homeomorphic to Ω_1.

Let Σ_i ($i = 1, \cdots, m$) be the connected components of $\partial\Omega$. For each Σ_i, there is a diffeomorphism between Σ_i and $S[x^i_j, y^i_j; j = 1, \cdots, n_i]$. It is easy to see that the self-intersection points corresponding to $x^i_j = y^i_j$ ($j = 1, \cdots, n_i$) are all Ω-interior saddles on Σ_i. The local topological structure of each Ω-interior saddle on $\partial\Omega$ is one of the two types illustrated in Figure 5.

Let A_i ($i = 1, \cdots, m$) be a closed domain with boundary $\partial A_i = \cup_{j=1}^{k_i} \sigma^j_i \cup \sigma^0_i$ such that

1). each σ^j_i ($j = 1, \cdots k_i$) is homeomorphic to S^1,

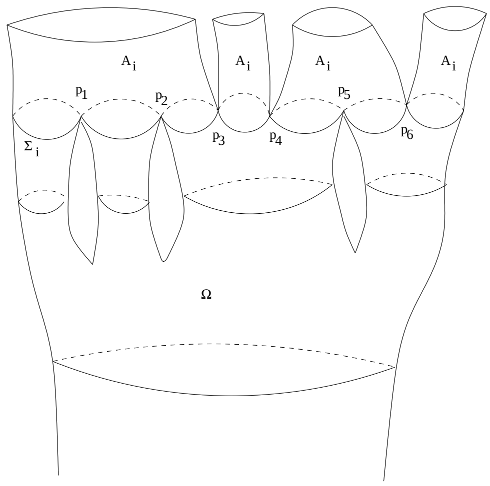

FIGURE 6

2). σ_i^0 is homeomorphic to Σ_i with the homeomorphism $h_i : \sigma_i^0 \to \Sigma_i$, and
3). A_i is retractable to σ_i^0 .

An example of A_i is given depicted in Figure 6, where p_1, p_2 and p_5 are of the type given in Figure 5(a), while p_3, p_4 and p_6 are of the type in Figure 5(b).

Then we define

$$M_1 = [\sum_{i=1}^{m} A_i + \Omega]/[h_1, ..., h_m],$$

and it is easy to see that M_1 is a two dimensional compact manifold retractable to Ω.

Step 2. It is easy to see that

$$g = g(\Omega) = g(M_1) \leq g(M) = k. \tag{7}$$

By the Poincaré–Bendixson Theorem, on an orientable manifold with genus $k = 0$ there is no ergodic set. In [3, 9] it has been proven that there is no nonorientable ergodic set on nonorientable manifolds with genus $k \leq 3$. Therefore

$$g \geq \begin{cases} 1, & \text{for } \Omega \text{ orientable,} \\ 4, & \text{for } \Omega \text{ nonorientable.} \end{cases} \tag{8}$$

Then (4) follows from (7) and (8).

Step 3. To prove (5), we extend the vector field v on Ω to its extended manifold. To this end, let $\omega = v|_\Omega$ be the restriction of v on Ω. Then ω is a vector field on Ω which is tangent to $\partial\Omega$. Let M_1 be an extended manifold of Ω, then we need to extend the vector field ω on Ω to a vector field u on M_1 such that u is tangent to ∂M_1 (it is not necessary for u to be divergence-free).

A connected component $\Gamma \subset \partial\Omega$ is called a boundary component of $\partial\Omega$ if Γ has an open neighborhood $A \subset M_1$ (see Figure 6) such that $\overline{A}$ is retractable to Γ in M_1. For each boundary component Γ of $\partial\Omega$, we extend the vector field w in two different cases:

Case a). There are no Ω–boundary saddles on Γ. Since w is tangent to Γ, Γ consists of singular points and orbits of w. We call Γ an orbit curve. An orbit curve

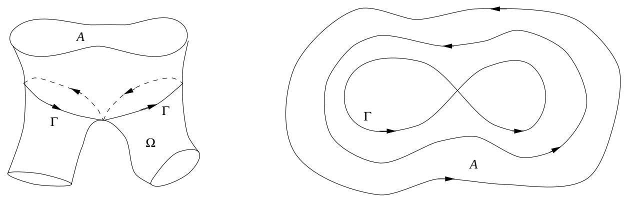

FIGURE 7

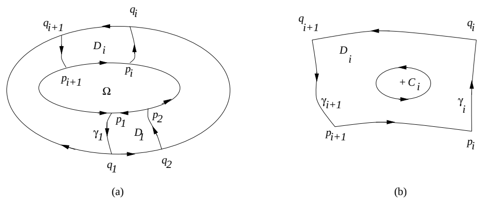

FIGURE 8

Γ is said to be co-directional, if the singular points $p_1, \cdots, p_n \in \Gamma$ divide Γ into m orbits $\gamma_1, \cdots, \gamma_m$ in order such that the limit sets of the orbits satisfy

$$\omega(\gamma_i) = \alpha(\gamma_{i+1}), \quad 1 \leq i \leq m-1,$$

or $i \in Z_m$ for Γ is a closed curve. Here $Z_m = Z/m+1$ is the mod m group.

Obviously, in this case, Γ is a closed co-directional orbit curve. It implies that w can be extended to the closed domain $A \subset M_1$ corresponding to Γ given in Step 1 by setting up tubular flows in A such that A becomes circle bands of the extended vector field, see Figure 7 below.

Case b). Since Ω–boundary saddles appear in pairs, let there be $2n$ ($n \geq 1$) Ω–boundary saddles $\{p_i \mid 1 \leq i \leq 2n\}$ on Γ. Let A be the corresponding closed domain constructed in Step 1 with boundary $\partial A = \cup_{j=1}^{k} \sigma^j \cup \sigma^0$ such that

1). each σ^j ($j = 1, \cdots k$) is homeomorphic to S^1,
2). σ^0 is homeomorphic to Γ with homeomorphism $h : \sigma^0 \to \Gamma$, and
3). A is retractable to σ^0 .

Without loss of generality, we assume that $\sigma^0 = \Gamma$ and $h = id$. Then we choose $2n$ points $\{q_i \mid 1 \leq i \leq 2n\}$ on $\cup_{j=1}^{k} \sigma^j$ and $2n$ simple curves $\{\gamma_i \mid 1 \leq i \leq 2n\}$, corresponding to $\{p_i \mid 1 \leq i \leq 2n\}$ as in Figure 8(a). Meanwhile we obtain cells D_i with ∂D_i consisting of four arc segments, γ_i, γ_{i+1}, $\widetilde{p_i p}_{i+1} \in \Gamma$ and $\widetilde{q_i q}_{i+1} \in \sum$. We prescribe flow orbits on γ_i and on $\widetilde{q_i q}_{i+1}$ as in Figure 8(b) such that $\alpha(\gamma_i) = \omega(\widetilde{p_i p}_{i+1}) = p_i$, $\omega(\gamma_i) = \alpha(\widetilde{q_i q}_{i+1}) = q_i$, $\alpha(\gamma_{i+1}) = \omega(\widetilde{q_i q}_{i+1}) = q_{i+1}$, and $\omega(\gamma_{i+1}) = \alpha(\widetilde{p_i p}_{i+1}) = p_{i+1}$. Hence ∂D_i is a co-directional orbit curve. Then we can define a vector field u_i on $\overline{D}_i$ such that the orbits of u_i on ∂D_i are as prescribed, and D_i is a circle cell of u_i, see Figure 8(b). Thus we obtain the extension u of w on $\overline{A}$ such that $u = u_i$ for $x \in \overline{D}_i$ ($1 \leq i \leq 2m$).

Step 4. Proof of (5). By the generalized Poincare-Hopf index formula for vector fields with singularities on the boundary, Theorem 2.1 in ([6]), we have

$$\sum_{j=1}^{N} \text{ind}(u, x_j) = \chi(M_1) = \chi(\Omega), \tag{9}$$

where $x_j \in M_1$ $(1 \leq j \leq N)$ are all singular points of u. Here if $p \in \partial M$ is an isolated singular point of v, then the index of v at p is defined by $\text{ind}(v,p) = \text{ind}(\widetilde{v},p)/2$, where $\widetilde{v}$ is the reflective extension of v.

Now we examine each term on the left hand side of (9). First if $x_j \in \partial\Omega$ is an Ω–exterior saddle, then x_j is a degenerate singular point of u, which is connected only by two orbits, and we have

$$\text{ind}(u, x_j) = 0 \quad \text{for any } \Omega\text{–exterior saddle } x_j \in \partial\Omega. \tag{10}$$

An Ω–interior saddle $x_j \in \partial\Omega$ is also a saddle point of u and

$$\text{ind}(u, x_j) = -1 \quad \text{for any } \Omega\text{–interior saddle } x_j \in \partial\Omega. \tag{11}$$

If $x_j \in \partial\Omega$ is an Ω–boundary saddle, then by the extension in case b) above, x_j is a saddle point of u on $\overset{\circ}{M}_1$. Hence

$$\text{ind}(u, x_j) = -1 \quad \text{for any } \Omega\text{–boundary saddle } x_j \in \partial\Omega. \tag{12}$$

Since Ω is an ergodic set of v, there are no centers on Ω. Therefore we have

$$\sum_{\substack{j=1,\cdots,N \\ x_j \in \overset{\circ}{\Omega}}} \text{ind}(u, x_j) = - \text{ number of saddles of } v \text{ in } \overset{\circ}{\Omega}\,. \tag{13}$$

It follows from (9–13) that

$$\sum_{j=1}^{N} \text{ind}(u, x_j) = -s - 2B + \sum_{\substack{j=1,\cdots,N \\ x_j \in \partial M_1}} \text{ind}(u, x_j) + \sum_{\substack{j=1,\cdots,N \\ x_j \in \overset{\circ}{A}_i, i=1,\cdots,m}} \text{ind}(u, x_j) \tag{14}$$

where s is the number of Ω–interior saddles of v, and $b = 2B$ is the number of Ω–boundary saddles of v.

By the extension of w, when the boundary component Γ of $\partial\Omega$ has no Ω–boundary saddles, there are no singular points in $\overset{\circ}{A}$ corresponding to Γ and on the boundary $\partial M_1 \cap \partial A$. When Γ has $2n$ $(m \geq 1)$ boundary saddles, there are $2n$ centers in A, and $2n$ singular points on $\partial M_1 \cap \partial A$. Obviously the $2n$ singular points of u on $\partial M_1 \cap \partial A$ are boundary saddle points of u on ∂M_1, by the definition of an index of a boundary singular point, we have

$$\text{ind}(u, y) = -\frac{1}{2}, \quad \text{for } y \in \partial M_1 \text{ being a boundary saddle point of } u. \tag{15}$$

Hence

$$\sum_{\substack{j=1,\cdots,N \\ x_j \in \overset{\circ}{A}_i, i=1,\cdots,m}} \text{ind}(u, x_j) = 2B, \qquad \sum_{\substack{j=1,\cdots,N \\ x_j \in \partial M_1}} \text{ind}(u, x_j) = -B. \tag{16}$$

Then (5) follows from (9), (14) and (16). The proof is complete.

5. **Structure of Ergodic Sets on Tori.** When M is a torus with or without boundary, the topological structure of an ergodic set of M is simpler. We denote by $\bigvee_m S^1 \subset \mathbb{T}^2$ the connected topological set which consists of m circle S^1 with exactly $m-1$ common points between them, and $\bigvee_m S^1$ encloses m closed disks in $\mathbb{T}^2$.

Theorem 5.1. *Let M be a torus with or without boundary, and $v \in D_0^r(TM)$ $(r \geq 1)$ be regular. Let $\Omega \subset M$ be an ergodic set of v. Then*

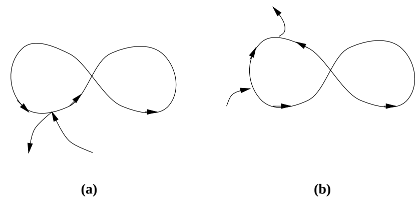

FIGURE 9

1). *Ω is a pseudo-torus (genus $g = 1$) with or without boundary;*
2). *each connected component of $\partial\Omega$ is homeomorphic to $\bigvee_m S^1$ $(m \geq 1)$, and has exactly two saddle orbits in $\mathring{\Omega}$; and*
3). *$\mathring{\Omega}$ does not contain saddle points of v.*

Proof. The assertion 1) is obvious; we only have to prove 2) and 3). We divide the proof into a few steps.

Step 1. For any simple closed curve $\gamma \in \mathbb{T}^2$, if γ is not homological to zero in the homology $H_1(\mathbb{T}^2)$, then $\mathbb{T}^2 - \gamma$ is homeomorphic to an open sub-manifold of S^2. By the Poincaré–Bendixson theorem, $\partial\Omega$ must not contain a closed curve which is not homological to zero in $H_1(M, \partial M)$. It means that each simple closed curve in $\partial\Omega$ must enclose a closed disk in $\mathbb{T}^2$. Again, by the Poincaré–Bendixson theorem, each connected component Σ of $\partial\Omega$, which consists of m circles S^1, can not have more than $m - 1$ intersections between them. Hence we have that $\Sigma = \bigvee_m S^1$.

Step 2. Assume that $\partial\Omega$ has n connected components, i.e.

$$\partial\Omega = \sum_{i=1}^{n} \Gamma_i.$$

In step 1, we showed that Γ_i is homeomorphic to $\bigvee_{m_i} S^1$. Then it is easy to see that

$$\chi(\Omega) = -\sum_{i=1}^{n} m_i.$$

Step 3. The number of intersection points in Γ_i is $m_i - 1$. Hence the total number of intersection points on $\partial\Omega$ is

$$s_1 = \sum_{i=1}^{n}(m_i - 1 = -\chi(\Omega) - n = s + \frac{b}{2} - n.$$

Hence

$$s - s_1 + \frac{b}{2} = n, \tag{17}$$

where n is the number of connected components of $\partial\Omega$.

Step 4. Claim: For each $i = 1, \cdots, n$, there are at least two orbits in $\mathring{\Omega}$ connected to Γ_i. First there is at least one orbit connected to Γ_i in $\mathring{\Omega}$; otherwise, by Lemma 2.4, there exist closed orbits in $\mathring{\Omega}$, a contradiction. Second, it is obvious to see that such orbits in $\mathring{\Omega}$ appear in pairs and are connected to saddle points on Γ_i, see Figure 9.

Then by (17) there are exactly two orbits connected to Γ_i in $\mathring{\Omega}$, and there are no saddle points in $\mathring{\Omega}$.

The proof is complete. □

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