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A NOTE ON A CLASS OF HIGHER ORDER CONFORMALLY COVARIANT EQUATIONS

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Abstract. In this paper, we study the higher order conformally covariant equation

$$(-\triangle)^{\frac{n}{2}} w = (n-1)! e^{nw} \quad x \in R^n$$

for all even dimensions n.

Let

$$\alpha = \frac{1}{|S^n|} \int_{R^n} e^{nw} dx.$$

We prove, for every $0 < \alpha < 1$, the existence of at least one solution. In particular, for $n = 4$, we obtain the existence of radial solutions.

1. **Introduction.** Let $\triangle$ be the Laplace operator in the Euclidean space R^n of even dimension n. We consider the following nth order equation

$$(-\triangle)^{\frac{n}{2}} w = (n-1)! e^{nw} \quad x \in R^n. \tag{1}$$

This equation naturally arise from the study of conformal geometry. Let π be the stereographic projection from the standard sphere S^n to R^n, then the pull back of the operator $(-\triangle)^{\frac{n}{2}}$ under π is the well-known Peneitz operator P_n on S^n.

On a compact manifold M with metric g, a metrically defined operator A is said to be conformally covariant if under the conformal change of metric $g_w = e^{2w} g$, the pair of corresponding operators A and A_w are related by

$$A_w(\phi) = e^{-bw} A(e^{aw}\phi), \quad \forall \phi \in C^\infty(M),$$

for some constants a and b.

Basic examples of such second order operators are the Laplacian $\triangle$ for n = 2 and the conformal Laplacian $-\frac{4(n-1)}{n-2}\triangle + R$ for $n \geq 3$, where R is the scalar curvature of the metric.

On compact 4-manifolds, Paneitz [8] discovered an interesting 4th order operator P_4:

$$P_4\phi = \triangle^2\phi + \delta(\frac{2}{3}Rg - 2Ric)d\phi,$$

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where δ denotes the divergence, d the differential, and Ric the Ricci tensor of the metric g. Under the conformal change $g_w = e^{2w}g$, P_4 covariants as:

$$(P_4)_w = e^{-4w}P_4.$$

On general compact manifolds of even dimensions n, the existence of such an operator P_n with $(P_n)_w = e^{-nw}P_n$ was verified in [6]. The explicit formula for P_n on S^n has appeared in Branson [2] and Beckner [1] as follows:

$$P_n = \begin{cases} \Pi_{k=0}^{\frac{n-2}{n}}(-\triangle + k(n-k-1)) & \text{for n even,} \\ (-\triangle + (\frac{n-1}{2})^2)^{1/2}\Pi_{k=0}^{\frac{n-2}{2}}(-\triangle + k(n-k-1)) & \text{for n odd .} \end{cases}$$

When n = 3 or 4, there exists some natural curvature invariant Q_n of order n which, under the conformal change of metric $g_w = e^{2w}g$ is related to $P_n w$ by the following differential equation:

$$-P_n w + (Q_n)_w e^{nw} = Q_n \quad \text{on M .} \tag{2}$$

On standard sphere S^n, when the metric g_w is isometric to the standard one, equation (2) becomes

$$-P_n w + (n-1)!e^{nw} = (n-1)!.$$

Now, through a stereographic projection π, we arrive at equation (1) in R^n, which we will discuss throughout this paper.

For n = 2, assuming that $\int_{R^2} e^{2w}dx < \infty$, Chen and Li [3] showed that all the solutions of (1) are radially symmetric with respect to some point $x_o \in R^n$, and there exists some $\lambda > 0$ such that

$$w(x) = \ln \frac{2\lambda}{\lambda^2 + |x - x_o|^2} \quad \text{for all} \quad x \in R^n. \tag{3}$$

For higher dimensions n, under the growth restriction on the solution

$$w(x) = \ln \frac{2}{1+|x|^2} + \xi(\pi^{-1}(x))$$

with some smooth function ξ on S^n, Chang and Yang [4] established the same classification result as in (3).

Let

$$\alpha = \frac{1}{|S^n|}\int_{R^n} e^{nw}dx.$$

In the special case of four dimensions, Lin obtained the symmetry of solutions under more relaxed conditions:

$$\alpha < \infty \text{ and } w(x) = o(|x|^2) \text{ at } \infty.$$

Then Wei and Xu [9] generalized this result to all even dimensions.

In [7], Lin also prove the following interesting result in four dimensions:

i) If $\alpha < \infty$, then $\alpha \leq 1$; and

ii) $\alpha = 1$ if and only if the solutions assume the form of (3).

Based on the previous results, there is uniqueness of the solutions in R^2 and on S^n, that is, all the solutions assume the standard form as described in (3). Then one may naturally wonder that

In R^n for n = 4, 6, 8, $\cdots$, do we have the same uniqueness result?

Surprisingly, we found the contrary to be true. In this paper, we prove

Theorem 1. *Let n be any even number other than 2. Then for every $0 < \alpha < 1$, equation (1) has at least one corresponding solution.*

Theorem 2. *When the dimension $n = 4$, equation (1) has radially symmetric solutions with $\alpha < 1$. Moreover, those solutions assume the asymptotic behavior $-a|x|^2$ at infinity.*

Remark 1.1. *i) According to Lin, the solutions in our Theorem 1 and 2 are obviously non-standard ones.*

In section 2, we prove Theorem 1 by a variational approach. Actually, we established the existence of solutions of the following more general equation

$$(-\triangle)^{\frac{n}{2}} w = K(x) e^{nw}, \quad x \in R^n \tag{4}$$

where K(x) is positive somewhere and bounded by $\frac{1}{|x|^s}$ near infinity for some $s > 0$.

In section 3, we prove Theorem 2 by an ODE method.

2. **The Existence of Solutions in Even Dimensions.** In this section, we consider the existence of solutions of the following more general equation

$$(-\triangle)^{\frac{n}{2}} w = K(x) e^{nw} \quad x \in R^n. \tag{5}$$

Theorem 2.1. *Assume that, for some $s > 0$,*

$$K(x) \text{ is positive somewhere, and } K(x) = O(\frac{1}{|x|^s}) \quad \text{near infinity}. \tag{6}$$

Then equation (5) has at least one solution.

Proof.

To show the existence of a solution, we use a variational approach in a Sobolev space defined on a conical singular manifold. For convenience, we identify each point in R^n with a point on S^n via the stereographic projection.

Step 1. Introducing a conical singular metric on S^n and the corresponding Sobolev space.

Let

$$w_o(x) = \ln \frac{2}{1+|x|^2}, \tag{7}$$

which is one of the standard solution of the equation

$$(-\triangle)^{\frac{n}{2}} w = (n-1)! e^{nw}. \tag{8}$$

with the corresponding $\alpha = 1$.

Then the standard S^n metric can be written as

$$g_o(x) = e^{2w_o} |dx|^2.$$

Let $v = (1-\mu) w_o$, where $0 < \mu < 1$ will be chosen later according to the value of s. Obviously, v satisfies

$$(-\triangle)^{\frac{n}{2}} v = K_\mu(x) e^{nv}. \tag{9}$$

with

$$K_\mu(x) = (1-\mu)(n-1)! e^{\mu n w_o}.$$

Introduce the conical singular metric on S^n:

$$g_\mu(x) = e^{2v} |dx|^2.$$

The corresponding volume element is $dV = e^{nv}dx$. Let P_μ be the conformal covariant operator on S^n with singular metric g_μ. Then P_o is the corresponding operator on standard S^n, and the conformal covariance implies that

$$P_o = e^{-nw_o}(-\triangle)^{\frac{n}{2}} \text{ and } P_\mu = e^{-n\mu w_o}P_o. \tag{10}$$

Let $W_\mu^{\frac{n}{2},2}$ be the Sobolev space equipped with the norm

$$\|w\|^2 = \int [P_\mu(w)w + w^2]dV.$$

Step 2. The Variational Approach.

Let

$$H = \{w \in W_\mu^{\frac{n}{2},2} \mid \int K(x)e^{nw}dV > 0\}.$$

In H, consider the functional

$$J(w) = \int [P_\mu(w)w + 2K_\mu(x)w]dV - \frac{2(1-\mu)\lambda_o}{n} \ln \int K(x)e^{nw}dV,$$

where

$$\lambda_o = \int_{R^n} (n-1)!e^{nw_o}dx = (n-1)!|S^n|. \tag{11}$$

It is easy to verify that

$$J(w+c) = J(w), \quad \text{for any constant } c. \tag{12}$$

This enable us to choose a minimizing sequence $\{w_k\}$ such that

$$\int K_\mu w_k dV = 0.$$

We show that $\{w_k\}$ is bounded. For a given value of s, choose

$$0 < \mu < \min\{\frac{s}{2n}, 1\},$$

then one can easily verify that

$$|K(x)e^{nv}| \leq Ce^{nw_o}$$

Applying the well-known inequality on standard S^n (Cf [5])

$$\frac{2\lambda_o}{n} \ln \frac{1}{|S^n|} \int e^{nw}dV_o \leq \int P_o(w)wdV_o + C,$$

and by the conformal invariance of the integral:

$$\int P_\mu(w)wdV = \int P_o(w)wdV_o,$$

we conclude that there exists a $\delta > 0$ such that

$$J(w_k) \geq \delta \int P_\mu(w_k)w_k dV - C.$$

Now it follows from the Pointcare inequality that $\{w_k\}$ is bounded and hence possesses a subsequence converging weakly to a minimizer u in H

Obviously the minimizer u satisfies

$$P_\mu(u) + K_\mu = \frac{(1-\mu)\lambda_o K(x)e^{nu}}{\int K(x)e^{nu}dV}.$$

Choose a suitable constant C such that $\tilde{u} = u + C$ satisfies

$$\int K(x)e^{n\tilde{u}}dV = \mu\lambda_o.$$

It follows that,

$$P_\mu(\tilde{u}) + K_\mu(x) = K(x)e^{n\tilde{u}}.$$

Finally, let $w = \tilde{u} + v$, then w is the desired solution of (5) in R^n.
This completes the proof of the Theorem.

The Proof of Theorem 1.

To show that for each $0 < \alpha < 1$, there exists a solution of (1), we choose $K(x) = (n-1)!e^{-an|x|^2}$, with some $a > 0$. Obviously, K(x) satisfies the condition in Theorem 2.1 for any positive value of s. Let $\tilde{w}$ be the corresponding solution of equation (5), then $w(x) = \tilde{w}(x) - a|x|^2$ is the desired solution of (1) with $\alpha < 1$. This can be seen from the proof of Theorem 2.1, in which $w = \tilde{u} + v - a|x|^2$, and consequently,

$$(n-1)!\int_{R^n} e^{nw}dx = \int K(x)e^{n\tilde{u}}dV = (1-\mu)\lambda_o = (n-1)!(1-\mu)|S^n|.$$

Now one can easily choose s properly to make $1 - \mu = \alpha$. This completes the proof of the Theorem.

3. **The Existence of Radial Solutions.** In this section, we establish the existence of radially symmetric solutions for equation (1) in dimension 4 and thus prove Theorem 2. The solutions v(r) we obtained has asymptotic behavior $-ar^2$ near infinity. Here $r = |x|$.

Again let

$$w_o(r) = \ln\frac{2}{1+r^2}.$$

be a standard solution with $\alpha = 1$. Then one can easily see that

$$w_o(0) = \ln 2,\ w_o'(0) = 0,\ w_o''(0) = -2, \text{ and } w_o'''(0) = 0. \tag{13}$$

Solve the ODE problem

$$\begin{cases} \triangle^2 v = 6e^{4v(r)} & r \in [0,\infty) \\ v(0) = \ln 2 & (= w_o(0)) \\ v'(0) = 0 & (= w_o'(0)) \\ v''(0) = -3 & (= w_o''(0) - 1) \\ v'''(0) = 0 & (= w_o'''(0)) \end{cases} \tag{14}$$

By the standard ODE theory, the solution of (14) exists for small r.

To prove that the solution exists for all r, it suffice to show that v(r) is finite for each r. To this end, we use $w_o(r)$ and $w(r) = \ln 2 - 2r^2$ to bound the solution, i.e., we show that

$$w_o(r) \geq v(r) \geq w(r) \tag{15}$$

To obtain the desired asymptotic behavior, we also show, for large r, that

$$v(r) \leq -cr^2. \tag{16}$$

The Proof of (15) and (16).

Let $g(r) = w_o(r) - v(r)$.

Step 1.

We show that

$$\frac{d}{dr}(\triangle g(r)) > 0. \tag{17}$$

In fact, since $g'(0) = 0$ and $g''(0) = 1$, we have, for small r,

$$g'(r) > 0.$$

Consequently,

$$g(r) > 0, \quad \text{for small r} . \tag{18}$$

Noticing that w_o and v satisfy the same equation (14), we have, by (18),

$$\triangle(\triangle g) > 0, \quad \text{for small r} .$$

Integrating this on the ball $B_r(0)$, we obtain (17).

Step 2.

We show that

$$v(r) \leq -cr^2, \quad \text{for large r and for some } c > 0. \tag{19}$$

We first claim that

$$g(r) > 0, \;\; i.e. \;\; w_o(r) > v(r), \quad \text{for all r} . \tag{20}$$

From Step 1, we see that this is true for small r. If it is not true for all r, let r_o be the smallest value of r, such that

$$\begin{cases} g(r) > 0 & \text{for } r < r_o \\ g(r) = 0 & \text{for } r = r_o. \end{cases} \tag{21}$$

We will derive a contradiction.

In fact, by an elementary calculation, one has

$$\triangle g(0) = 4. \tag{22}$$

Also, from *Step 1*, one can see that (17) is true as long as $g(r) > 0$. Now by (22) and (17),

$$\triangle g(r) > 0, \quad \text{for } r < r_o.$$

Then the maximum principle for elliptic equations implies that $g(r)$ can not achieve its maximum in $B_{r_o}(0)$. This is a contradiction with (21). Hence we must have, for all $r > 0$,

$$g(r) > 0,$$

$$\frac{d}{dr}(\triangle g(r)) > 0,$$

and

$$\triangle g(r) \geq 4. \tag{23}$$

Now, taking into account of the fact that

$$\triangle w_o \rightarrow 0, \quad \text{as } |x| \rightarrow \infty,$$

we have $\triangle v \leq -3$ for r large, i.e.

$$\frac{1}{r^3}\frac{d}{dr}(r^3\frac{dv}{dr}) \leq -3, \quad \text{for r large} .$$

Integrating both sides, we arrive at (19).

Similarly, one can prove $v(r) \geq \ln 2 - 2r^2$. Therefore, we have proved (17). This guarantees the existence of at least a radially symmetric solution for equation (1). Furthermore, by Theorem 1.2 of Lin [7], we also have the asymptotic behavior

$$v(r) \sim -ar^2, \quad \text{near } \infty, \quad \text{for some } a > 0.$$

Step 3.

Finally, we show that corresponding to the solution v(r), we have $\alpha < 1$.

By a result of Lin [7], one can express

$$v(r) = w(r) - ar^2$$

where

$$w(r) = -\alpha \ln r + c + O(\frac{1}{r}).$$

Then w satisfies

$$\triangle^2 w = Q(r)e^{4w}, \quad \text{with } Q(r) = 6e^{-4ar^2}. \tag{24}$$

Applying the Pohozaev Identity (see [7]) to (24), one arrives at

$$16\pi^2\alpha + \frac{1}{32\pi^2}\int_{R^4} r \cdot \frac{dQ}{dr} e^{4w} dx = 16\pi^2\alpha^2. \tag{25}$$

Obviously, the integral in (25) is negative, which implies that

$$\alpha < 1.$$

This completes the proof of Theorem 2.

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