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Introduction to Sobolev Spaces

People use numbers every day to quantify surrounding objects. In mathematics, the absolute value $|a-b|$ is used to measure the difference between two numbers a and b. Functions are used to describe physical states. For example, temperature is a function of time and place. Very often, we use a sequence of approximate solutions to approach a real one; and how close these solutions are to the real one depends on how we measure them (i.e., which metric we are choosing). Hence, not only must we develop suitable metrics to measure different states (functions), but we must also study relationships among different metrics. For these purposes, the Sobolev Spaces were introduced. They have many applications in various branches of mathematics, in particular, in the theory of partial differential equations.

The role of Sobolev Spaces in the analysis of PDEs is somewhat similar to the role of Euclidean Spaces in the study of geometry. The fundamental research on the relations among various Sobolev Spaces (Sobolev norms) was first carried out by G. Hardy and J. Littlewood in the 1910s and then by S. Sobolev in the 1930s. More recently, many well known mathematicians, such as H. Brezis, L. Caffarelli, A. Chang, E. Lieb, L. Nirenberg, J. Serrin, and

E. Stein have worked in this area. The main objectives are to determine if and how the norms dominate each other, what the sharp estimates are, which functions achieve these sharp estimates, and which functions are 'critically' related to these sharp estimates.

To find the existence of weak solutions for partial differential equations, especially for nonlinear partial differential equations, the method of functional analysis, in particular, the calculus of variations, has seen increasing application.

To roughly illustrate this kind of application, let's start with a simple example. Let Ω be a bounded domain in R^n and consider the Dirichlet problem associated with the Laplace equation:

$$\begin{cases} -\triangle u = f(x), & x \in \Omega \\ u = 0, & x \in \partial\Omega \end{cases} \tag{1.1}$$

To prove the existence of solutions, one may view $-\triangle$ as an operator acting on a proper linear space and then apply some known principles of functional analysis, such as the 'fixed point theory' or 'the degree theory,' to derive the existence. One may also consider the corresponding variational functional

$$J(u) = \frac{1}{2}\int_\Omega |\bigtriangledown u|^2 \, dx - \int_\Omega f(x)\, u \, dx \tag{1.2}$$

in a proper linear space and seek critical points of the functional in that space. This kind of variational approach is particularly powerful in dealing with nonlinear equations. For example, in equation (1.1), instead of $f(x)$, we consider $f(x,u)$. Then it becomes a semi-linear equation. Correspondingly, we have the functional

$$J(u) = \frac{1}{2}\int_\Omega |\bigtriangledown u|^2 \, dx - \int_\Omega F(x,u) \, dx, \tag{1.3}$$

where

$$F(x,u) = \int_0^u f(x,s)\, dx$$

is an anti-derivative of $f(x,\cdot)$. From the definition of the functional in either (1.2) or (1.3), one can see that the function u in the space need not be second order differentiable as is required by classical solutions of (1.1). Hence the critical points of the functional are solutions of the problem only in the 'weak' sense. However, by an appropriate regularity argument, one may recover the differentiability of the solutions so that they can still satisfy equation (1.1) in the classic sense.

In general, given a PDE problem, our intention is to view it as an operator A acting on some proper linear spaces X and Y of functions and to write the equation symbolically as

$$Au = f \tag{1.4}$$

We can then apply the general and elegant principles of linear or nonlinear functional analysis to study the solvability of various equations involving A, the result of which can then be applied to a broad class of partial differential equations. We may also associate this operator with a functional $J(\cdot)$, whose critical points are the solutions of the equation (1.4). In this process, the key is to find an appropriate operator 'A' and appropriate spaces 'X' and 'Y'. As we shall see later, the Sobolev spaces are designed precisely for this purpose and will work out properly.

In solving a partial differential equation, in many cases it is natural to first find a sequence of approximate solutions and proceed to investigate the convergence of the sequence. The limit function of a convergent sequence of approximate solutions represents the desired, exact solution to the equation. As we shall see in the next few chapters, there are two basic stages in showing convergence:

i) In a reflexive Banach space, every bounded sequence has a weakly convergent subsequence, and then

ii) By the compact embedding from a "stronger" Sobolev space into a "weaker" one, the weak convergent sequence in the "stronger" space becomes a strong convergent sequence in the "weaker" space.

Before going into the details of this chapter, the reader may take a glance at the introduction of the next chapter to gain motivation for studying Sobolev spaces.

In Section 1.1, we will introduce the *distributions*, mainly the notion of the *weak derivatives*, which are the elements of the Sobolev spaces.

We then define Sobolev spaces in Section 1.2.

In deriving many useful properties in Sobolev spaces, it is inconvenient to work directly with weak derivatives. Hence, in Section 1.3, we show that these weak derivatives can be approximated by smooth functions. The three sections that follow then focus on smooth functions in establishing a series of important inequalities.

1.1 Distributions

As we saw in the introduction, the functional $J(u)$ in (1.2) or (1.3) involved only the first derivatives of u rather than the second derivatives as is required for classical second order equations; moreover, these first derivatives need not be continuous nor even defined everywhere. Therefore, by using a functional analysis approach one can substantially weaken the notion of partial derivatives. The advantage is that it divides the task of finding "suitable" smooth solutions for a PDE into two major steps:

Step 1. Existence of Weak Solutions. One seeks solutions that are less differentiable but easier to obtain. It is very common to use "energy" minimization or conservation, or sometimes to use finite dimensional approximation, to show the existence of such weak solutions.

Step 2. Regularity Lifting. One uses various analysis tools to boost the differentiability of the known weak solutions and try to show that they are actually classical solutions.

Both the existence of weak solutions and regularity lifting have become two major branches of today's PDE analysis. Various function spaces and related embedding theories are basic tools in both analyses, among which Sobolev spaces are the most frequently used.

In this section, we introduce the notion of 'weak derivatives,' which will be the elements of the Sobolev spaces.

Let R^n be the n-dimensional Euclidean space and Ω be an open connected subset in R^n. Let $D(\Omega) = C_0^\infty(\Omega)$ be the linear space of infinitely differentiable functions with compact support in Ω. This is called the space of test functions on Ω.

Example 1.1.1 *Assume*

$$B_R(x^o) := \{x \in R^n \mid |x - x^o| < R\} \subset \Omega,$$

then for any $r < R$, the following function

$$f(x) = \begin{cases} \exp\{\frac{1}{|x-x^o|^2 - r^2}\} & \text{for } |x - x^o| < r \\ 0 & \text{elsewhere} \end{cases}$$

is in $C_0^\infty(\Omega)$.

Example 1.1.2 *Assume $\rho \in C_0^\infty(R^n)$, $u \in L^p(\Omega)$, and $\operatorname{supp} u \subset K \subset\subset \Omega$. Let*

$$u_\epsilon(x) := \rho_\epsilon * u := \int_{R^n} \frac{1}{\epsilon^n} \rho(\frac{x-y}{\epsilon}) u(y) dy.$$

Then $u_\epsilon \in C_0^\infty(\Omega)$ for ϵ sufficiently small.

Now, let $L^p_{\text{loc}}(\Omega)$ be the space of p^{th}-power locally summable functions for $1 \leq p \leq \infty$. Such functions are Lebesgue measurable functions f defined on Ω and with the property that

$$\|f\|_{L^p(K)} := \left(\int_K |f(x)|^p dx \right)^{1/p} < \infty$$

for every compact subset K in Ω.

Assume that u is a C^1 function in Ω and $\phi \in D(\Omega)$. Through integration by parts, we have, for $i = 1, 2, \cdots, n$,

$$\int_{\Omega} \frac{\partial u}{\partial x_i} \phi(x) dx = - \int_{\Omega} u(x) \frac{\partial \phi}{\partial x_i} dx. \tag{1.5}$$

Now if u is not in $C^1(\Omega)$, then $\frac{\partial u}{\partial x_i}$ does not exist. However, the integral on the right hand side of (1.5) still makes sense if u is a locally L^1 summable function. For this reason, we define the first derivative $\frac{\partial u}{\partial x_i}$ weakly as the function $v(x)$ that satisfies

$$\int_{\Omega} v(x) \phi(x) dx = - \int_{\Omega} u \frac{\partial \phi}{\partial x_i} dx$$

for all functions $\phi \in D(\Omega)$.

The same idea works for higher partial derivatives. Let $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ be a multi-index of order

$$k := |\alpha| := \alpha_1 + \alpha_2 + \cdots + \alpha_n.$$

For $u \in C^k(\Omega)$, the regular αth partial derivative of u is

$$D^{\alpha} u = \frac{\partial^{\alpha_1} \partial^{\alpha_2} \cdots \partial^{\alpha_n} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}.$$

Given any test function $\phi \in D(\Omega)$, through a straightforward integration by parts k times, we arrive at

$$\int_{\Omega} D^{\alpha} u \, \phi(x) \, d\, x = (-1)^{|\alpha|} \int_{\Omega} u \, D^{\alpha} \phi \, d\, x. \tag{1.6}$$

There is no boundary term because ϕ vanishes near the boundary.

Now if u is not k times differentiable, the left hand side of (1.6) makes no sense. However the right hand side is valid for functions u with much weaker differentiability, i.e., u only need to be locally L^1 summable. Thus it is natural to choose those functions v that satisfy (1.6) as the weak representatives of $D^{\alpha} u$.

Definition 1.1.1 For $u, v \in L^1_{loc}(\Omega)$, we say that v is the αth weak derivative of u, written

$$v = D^{\alpha} u$$

provided

$$\int_{\Omega} v(x) \, \phi(x) \, d\, x = (-1)^{|\alpha|} \int_{\Omega} u \, D^{\alpha} \phi \, d\, x$$

for all test functions $\phi \in D(\Omega)$.

Example 1.1.3 *For $n = 1$ and $\Omega = (-\pi, 1)$, let*

$$u(x) = \begin{cases} \cos x & \textit{if } -\pi < x \leq 0 \\ 1 - x & \textit{if } 0 < x < 1. \end{cases}$$

Then its weak derivative $u'(x)$ can be represented by

$$v(x) = \begin{cases} -\sin x & \textit{if } -\pi < x \leq 0 \\ -1 & \textit{if } 0 < x < 1. \end{cases}$$

To see this, we verify, for any $\phi \in D(\Omega)$, that

$$\int_{-\pi}^{1} u(x)\phi'(x)dx = -\int_{-\pi}^{1} v(x)\phi(x)dx. \tag{1.7}$$

In fact, through integration by parts, we have

$$\begin{aligned} \int_{-\pi}^{1} u(x)\phi'(x)dx &= \int_{-\pi}^{0} \cos x\, \phi'(x)dx + \int_{0}^{1} (1-x)\phi'(x)dx \\ &= \int_{-\pi}^{0} \sin x\, \phi(x)dx + \phi(0) - \phi(0) + \int_{0}^{1} \phi(x)dx \\ &= -\int_{-\pi}^{1} v(x)\phi(x)dx. \end{aligned}$$

In this example, one can see that, in the classical sense, the function u is not differentiable at $x = 0$. Since the weak derivative is defined by integrals, one may alter the values of the weak derivative $v(x)$ in a set of measure zero, and (1.7) still holds. However, it should be unique up to a set of measure zero.

Lemma 1.1.1 *(Uniqueness of Weak Derivatives). If v and w are the weak αth partial derivatives of u, $D^\alpha u$, then $v(x) = w(x)$ almost everywhere.*

Proof. By the definition of the weak derivatives, we have

$$\int_\Omega v(x)\phi(x)dx = (-1)^{|\alpha|} \int_\Omega u(x) D^\alpha \phi dx = \int_\Omega w(x)\phi(x)dx$$

for any $\phi \in D(\Omega)$. It follows that

$$\int_\Omega (v(x) - w(x))\phi(x)dx = 0 \quad \forall \phi \in D(\Omega).$$

Therefore, by a standard argument (as an exercise), we must have

$$v(x) = w(x) \quad \text{almost everywhere in } \Omega.$$

□

Exercise 1.1.1 *Assume that* $f \in L^1_{loc}(\Omega)$ *and*

$$\int_\Omega f(x)\phi(x)dx = 0, \;\; \forall\, \phi \in D(\Omega);$$

then

$$f(x) = 0, \quad \textit{almost everywhere in } \Omega.$$

From the definition, we can view a weak derivative as a linear functional acting on the space of test functions $D(\Omega)$, and we call it a *distribution.* More generally, we have the following definition:

Definition 1.1.2 A distribution is a continuous linear functional on $D(\Omega)$. The linear space of distributions or the generalized functions on Ω, denoted by $D'(\Omega)$, is the collection of all continuous linear functionals on $D(\Omega)$.

Here, the continuity of a functional T on $D(\Omega)$ means that, for any sequence $\{\phi_k\} \subset D(\Omega)$ with $\phi_k {\to} \phi$ in $D(\Omega)$, we have

$$T(\phi_k){\to}T(\phi), \quad \text{as } k{\to}\infty;$$

and we say that $\phi_k{\to}\phi$ in $D(\Omega)$ if

a) there exists $K \subset\subset \Omega$ such that $\text{supp}\phi_k$, $\text{supp}\phi \subset K$, and

b) for any α, $D^\alpha \phi_k {\to} D^\alpha \phi$ uniformly as $k{\to}\infty$.

The most important and most commonly used distributions are locally summable functions. In fact, for any $f \in L^p_{\text{loc}}(\Omega)$ with $1 \leq p \leq \infty$, consider

$$T_f(\phi) = \int_\Omega f(x)\phi(x)dx.$$

It is easy to verify that $T_f(\cdot)$ is a continuous linear functional on $D(\Omega)$ and is hence a distribution.

For any distribution μ, if there is an $f \in L^1_{\text{loc}}(\Omega)$ such that

$$\mu(\phi) = T_f(\phi), \;\; \forall \phi \in D(\Omega),$$

then we say that μ is (or can be realized as) a locally summable function and identify it as f.

An interesting example of a distribution that is not a locally summable function is the well-known Dirac delta function. Let x^o be a point in Ω. For any $\phi \in D(\Omega)$, the delta function at x^o can be defined as

$$\delta_{x^o}(\phi) = \phi(x^o).$$

Hence it is a distribution. However, one can show that such a delta function is not locally summable. It is not a function at all. This kind of "function" has been used widely and so successfully by physicists and engineers, who often simply view δ_{x^o} as

$$\delta_{x^o}(x) = \begin{cases} 0, & \text{for } x \neq x^o \\ \infty, & \text{for } x = x^o. \end{cases}$$

Surprisingly, such a delta function is the derivative of some function in the following distributional sense. To explain this, let $\Omega = (-1, 1)$, and let

$$f(x) = \begin{cases} 0, & \text{for } x < 0 \\ 1, & \text{for } x \geq 0. \end{cases}$$

Then, we have

$$-\int_{-1}^{1} f(x)\phi'(x)dx = -\int_{0}^{1} \phi'(x)dx = \phi(0) = \delta_0(\phi).$$

Comparing this with the definition of weak derivatives, we may regard $\delta_0(x)$ as $f'(x)$ in the sense of distributions.

1.2 Sobolev Spaces

Given a function $f \in L^p(\Omega)$, we would like to solve the partial differential equation

$$\Delta u = f(x)$$

in the sense of weak derivatives. Naturally, we would seek solutions u, such that Δu are in $L^p(\Omega)$. More generally, we would start from the collections of all distributions whose second weak derivatives are in $L^p(\Omega)$.

In a variational approach, as we have seen in the introduction, to seek critical points of the functional

$$\frac{1}{2}\int_{\Omega} |\bigtriangledown u|^2 dx - \int_{\Omega} F(x, u)dx,$$

a natural set of functions to start with is the collection of distributions whose first weak derivatives are in $L^2(\Omega)$. More generally, we have:

Definition 1.2.1 The Sobolev space $W^{k,p}(\Omega)$ ($k \geq 0$ and $p \geq 1$) is the collection of all distributions u on Ω such that for all multi-index α with $|\alpha| \leq k$, $D^\alpha u$ can be realized as an L^p function on Ω. Furthermore, $W^{k,p}(\Omega)$ is a Banach space with the norm

$$||u|| := \left(\sum_{|\alpha| \leq k} \int_{\Omega} |D^\alpha u(x)|^p dx \right)^{1/p}.$$

In the special case when $p = 2$, it is also a Hilbert space and we usually denote it by $H^k(\Omega)$.

Definition 1.2.2 $W_0^{k,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{k,p}(\Omega)$.

Roughly speaking, $W_0^{k,p}(\Omega)$ is the space of functions whose up to $(k-1)th$ order derivatives vanish on the boundary.

Example 1.2.1 *Let $\Omega = (-1, 1)$. Consider the function $f(x) = |x|^\beta$. For $0 < \beta < 1$, it is obviously not differentiable at the origin. However for any $1 \leq p < \dfrac{1}{1-\beta}$, it is in the Sobolev space $W^{1,p}(\Omega)$. More generally, let Ω be an open unit ball centered at the origin in R^n; then the function $|x|^\beta$ is in $W^{1,p}(\Omega)$ if and only if*

$$\beta > 1 - \frac{n}{p}. \tag{1.8}$$

To see this, we first calculate

$$f_{x_i}(x) = \frac{\beta x_i}{|x|^{2-\beta}}, \quad \textit{for } x \neq 0,$$

and hence

$$|\bigtriangledown f(x)| = \frac{\beta}{|x|^{1-\beta}}. \tag{1.9}$$

Fix a small $\epsilon > 0$. Then for any $\phi \in D(\Omega)$, by integration by parts, we have

$$\int_{\Omega \setminus B_\epsilon(0)} f_{x_i}(x)\phi(x)dx = -\int_{\Omega \setminus B_\epsilon(0)} f(x)\phi_{x_i}(x)dx + \int_{\partial B_\epsilon(0)} f\phi\nu_i dS,$$

where $\nu = (\nu_1, \nu_2, \cdots, \nu_n)$ is an inward-normal vector on $\partial B_\epsilon(0)$.

Now, under the condition that $\beta > 1 - n/p$, f_{x_i} is in $L^p(\Omega) \subset L^1(\Omega)$, and

$$|\int_{\partial B_\epsilon(0)} f\phi\nu_i dS| \leq C\epsilon^{n-1+\beta} {\rightarrow} 0, \quad \textit{as } \epsilon {\rightarrow} 0.$$

It follows that

$$\int_\Omega |x|^\beta \phi_{x_i}(x)dx = -\int_\Omega \frac{\beta x_i}{|x|^{2-\beta}}\phi(x)dx.$$

Therefore, the weak first partial derivatives of $|x|^\beta$ are

$$\frac{\beta x_i}{|x|^{2-\beta}}\ , \quad i = 1, 2, \cdots, n.$$

Moreover, from (1.9) one can see that $|\bigtriangledown f|$ is in $L^p(\Omega)$ if and only if $\beta > 1 - \frac{n}{p}$.

1.3 Approximation by Smooth Functions

While working in Sobolev spaces, for instance, in proving inequalities, it may feel inconvenient and cumbersome to manage a weak derivative directly based on its definition. To get around this, we will show in this section that any function in a Sobolev space can be approached by a sequence of smooth functions. In other words, the smooth functions are dense in Sobolev spaces. Based on this, when deriving many useful properties of Sobolev spaces, we can just work on smooth functions and then take limits.

At the end of the section, for more convenient application of the approximation results, we prove an Operator Completion Theorem and an Extension Theorem. Both theorems will be used frequently in the next few sections.

The idea in approximation is based on mollifiers . Let

$$j(x) = \begin{cases} c\, e^{\frac{1}{|x|^2-1}} & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1. \end{cases}$$

One can verify that

$$j(x) \in C_0^\infty(B_1(0)).$$

Choose the constant c, such that

$$\int_{R^n} j(x)dx = 1.$$

For each $\epsilon > 0$, define

$$j_\epsilon(x) = \frac{1}{\epsilon^n} j(\frac{x}{\epsilon}).$$

Obviously,

$$\int_{R^n} j_\epsilon(x)dx = 1, \quad \forall \epsilon > 0.$$

One can also verify that $j_\epsilon \in C_0^\infty(B_\epsilon(0))$, and

$$\lim_{\epsilon \to 0} j_\epsilon(x) = \begin{cases} 0 & \text{for } x \neq 0 \\ \infty & \text{for } x = 0. \end{cases}$$

The above observations suggest that the limit of $j_\epsilon(x)$ may be viewed as a delta function, and from the well-known property of the delta function, we would naturally expect that for any continuous function $f(x)$ and for any point $x \in \Omega$,

$$(J_\epsilon f)(x) := \int_\Omega j_\epsilon(x-y)f(y)dy \to f(x), \quad \text{as } \epsilon \to 0. \tag{1.10}$$

More generally, if $f(x)$ is only in $L^p(\Omega)$, we would expect (1.10) to hold almost everywhere. We call $j_\epsilon(x)$ a mollifier and $(J_\epsilon f)(x)$ the mollification of $f(x)$. We will show that for each $\epsilon > 0$, $J_\epsilon f$ is a C^∞ function, and as $\epsilon \to 0$,

$$J_\epsilon f \to f \quad \text{in } W^{k,p}.$$

Actually, notice that

$$(J_\epsilon f)(x) = \int_{B_\epsilon(x)\cap\Omega} j_\epsilon(x-y)f(y)dy,$$

and hence in order for $J_\epsilon f(x)$ to approximate $f(x)$ well, we need $B_\epsilon(x)$ to be completely contained in Ω to ensure that

$$\int_{B_\epsilon(x)\cap\Omega} j_\epsilon(x-y)dy = 1$$

(an important property of the delta function). Equivalently, we need x to be in the interior of Ω. For this reason, we first prove a local approximation theorem.

Theorem 1.3.1 *(Local Approximation by Smooth Functions).*

For any $f \in W^{k,p}(\Omega)$, $J_\epsilon f \in C^\infty(R^n)$ and $J_\epsilon f \to f$ in $W^{k,p}_{loc}(\Omega)$ as $\epsilon \to 0$.

To extend this result to the entire Ω, we will choose infinitely many open sets O_i, $i = 1, 2, \cdots$, each of which has a positive distance to the boundary of Ω and whose union is the whole Ω. Based on the above theorem, we are able to approximate a $W^{k,p}(\Omega)$ function on each O_i by a sequence of smooth functions. Combining this with a partition of unity, and a cut-off function if Ω is unbounded, we will then prove:

Theorem 1.3.2 *(Global Approximation by Smooth Functions).*

For any $f \in W^{k,p}(\Omega)$, there exists a sequence of functions $\{f_m\} \subset C^\infty(\Omega) \cap W^{k,p}(\Omega)$ such that $f_m \to f$ in $W^{k,p}(\Omega)$ as $m \to \infty$.

Theorem 1.3.3 *(Global Approximation by Smooth Functions up to the Boundary).*

Assume that Ω is bounded with C^1 boundary $\partial\Omega$. Then for any $f \in W^{k,p}(\Omega)$, there exists a sequence of functions $\{f_m\} \subset C^\infty(\overline{\Omega}) = C^\infty(\overline{\Omega}) \cap W^{k,p}(\Omega)$ such that $f_m \to f$ in $W^{k,p}(\Omega)$ as $m \to \infty$.

When Ω is the entire space R^n, the approximation by C^∞ or by C_0^∞ functions are essentially the same. We have

Theorem 1.3.4 $W^{k,p}(R^n) = W_0^{k,p}(R^n)$. *In other words, for any $f \in W^{k,p}(R^n)$, there exists a sequence of functions $\{f_m\} \subset C_0^\infty(R^n)$, such that*

$$f_m \to f, \quad \text{as } m \to \infty; \quad \text{in } W^{k,p}(R^n).$$

Proof of Theorem 1.3.1

We prove the theorem in three steps.

In step 1, we show that $J_\epsilon f \in C^\infty(R^n)$ and

$$\|J_\epsilon f\|_{L^p(\Omega)} \leq \|f\|_{L^p(\Omega)}.$$

From the definition of $J_\epsilon f(x)$, we can see that it is well defined for all $x \in R^n$, and it vanishes if x is of ϵ distance away from Ω. Here and in the following, for simplicity of argument, we extend f to be zero outside of Ω.

In step 2, we prove that if f is in $L^p(\Omega)$,

$$(J_\epsilon f) \to f \quad \text{in } L^p_{loc}(\Omega).$$

We first verify this for continuous functions and then approximate L^p functions by continuous functions.

In step 3, we reach the conclusion of the Theorem. For each $f \in W^{k,p}(\Omega)$ and $|\alpha| \leq k$, $D^\alpha f$ is in $L^p(\Omega)$. Then from the result in Step 2, we have

$$J_\epsilon(D^\alpha f) \to D^\alpha f \quad \text{in } L^p_{loc}(\Omega).$$

Hence, what we need to verify is

$$D^\alpha(J_\epsilon f)(x) = J_\epsilon(D^\alpha f)(x).$$

As the reader will notice, the arguments in the last two steps only work in any compact subset of Ω.

Step 1.

Let $e_i = (0, \cdots, 0, 1, 0, \cdots, 0)$ be the unit vector in the x_i direction.

Fix $\epsilon > 0$ and $x \in R^n$. By the definition of $J_\epsilon f$ we have, for $|h| < \epsilon$,

$$\frac{(J_\epsilon f)(x + he_i) - (J_\epsilon f)(x)}{h} = \int_{B_{2\epsilon}(x) \cap \Omega} \frac{j_\epsilon(x + he_i - y) - j_\epsilon(x - y)}{h} f(y) dy. \tag{1.11}$$

Since as $h \to 0$,

$$\frac{j_\epsilon(x + he_i - y) - j_\epsilon(x - y)}{h} \to \frac{\partial j_\epsilon(x - y)}{\partial x_i}$$

uniformly for all $y \in B_{2\epsilon}(x) \cap \Omega$, we can pass the limit through the integral sign in (1.11) to obtain

$$\frac{\partial (J_\epsilon f)(x)}{\partial x_i} = \int_\Omega \frac{\partial j_\epsilon(x - y)}{\partial x_i} f(y) dy.$$

Similarly, we have

$$D^\alpha(J_\epsilon f)(x) = \int_\Omega D_x^\alpha j_\epsilon(x-y)f(y)dy.$$

Noticing that $j_\epsilon(\cdot)$ is infinitely differentiable, we conclude that $J_\epsilon f$ is also infinitely differentiable.

Then we derive

$$\|J_\epsilon f\|_{L^p(\Omega)} \leq \|f\|_{L^p(\Omega)}. \tag{1.12}$$

By the Hölder inequality, we have

$$\begin{aligned}
|(J_\epsilon f)(x)| &= |\int_\Omega j_\epsilon^{\frac{p-1}{p}}(x-y) j_\epsilon^{\frac{1}{p}}(x-y)f(y)dy| \\
&\leq \left(\int_\Omega j_\epsilon(x-y)dy\right)^{\frac{p-1}{p}} \left(\int_\Omega j_\epsilon(x-y)|f(y)|^p dy\right)^{\frac{1}{p}} \\
&\leq \left(\int_{B_\epsilon(x)} j_\epsilon(x-y)|f(y)|^p dy\right)^{\frac{1}{p}}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\int_\Omega |(J_\epsilon f)(x)|^p dx &\leq \int_\Omega \left(\int_{B_\epsilon(x)} j_\epsilon(x-y)|f(y)|^p dy\right) dx \\
&\leq \int_{R^n} |f(y)|^p \left(\int_{R^n} j_\epsilon(x-y)dx\right) dy \\
&= \int_\Omega |f(y)|^p dy.
\end{aligned}$$

Notice that here we have extended f to be zero outside Ω.

This verifies (1.12).

Step 2.

We prove that for any compact subset K of Ω,

$$\|J_\epsilon f - f\|_{L^p(K)} \to 0, \quad \text{as } \epsilon \to 0. \tag{1.13}$$

We first show this for a continuous function f. By writing

$$(J_\epsilon f)(x) - f(x) = \int_{B_\epsilon(0)} j_\epsilon(y)[f(x-y) - f(x)]dy,$$

we have

$$\begin{aligned}
|(J_\epsilon f)(x) - f(x)| &\leq \max_{x \in K, |y| < \epsilon} |f(x-y) - f(x)| \int_{B_\epsilon(0)} j_\epsilon(y)dy \\
&\leq \max_{x \in K, |y| < \epsilon} |f(x-y) - f(x)|.
\end{aligned}$$

Due to the continuity of f and the compactness of K, the last term in the above inequality tends to zero uniformly as $\epsilon \to 0$. This verifies (1.13) for continuous functions.

For any f in $L^p(\Omega)$, and given any $\delta > 0$, choose a a continuous function g, such that

$$\|f - g\|_{L^p(\Omega)} < \frac{\delta}{3}. \tag{1.14}$$

This can be derived from the well-known fact that any L^p function can be approximated by a simple function of the form $\sum_{j=1}^{k} a_j \chi_j(x)$, where χ_j is the characteristic function of some measurable set A_j; and each simple function can be approximated by a continuous function.

For the continuous function g, (1.13) implies that for sufficiently small ϵ, we have

$$\|J_\epsilon g - g\|_{L^p(K)} < \frac{\delta}{3}.$$

It follows from this and (1.14) that

$$\begin{aligned}
\|J_\epsilon f - f\|_{L^p(K)} &\leq \|J_\epsilon f - J_\epsilon g\|_{L^p(K)} + \|J_\epsilon g - g\|_{L^p(K)} + \|g - f\|_{L^p(K)} \\
&\leq 2\|f - g\|_{L^p(K)} + \|J_\epsilon g - g\|_{L^p(K)} \\
&\leq 2 \cdot \frac{\delta}{3} + \frac{\delta}{3} \\
&= \delta.
\end{aligned}$$

This proves (1.13).

Step 3.

Now assume that $f \in W^{k,p}(\Omega)$. Then for any α with $|\alpha| \leq k$, we have $D^\alpha f \in L^p(\Omega)$. We show that

$$D^\alpha (J_\epsilon f) \to D^\alpha f \quad \text{in } L^p(K). \tag{1.15}$$

By the result in Step 2, we have

$$J_\epsilon (D^\alpha f) \to D^\alpha f \quad \text{in } L^p(K).$$

What remains to verify is that for sufficiently small ϵ,

$$D^\alpha (J_\epsilon f)(x) = J_\epsilon (D^\alpha f)(x), \ \ \forall x \in K.$$

To see this, we fix a point x in K. Choose $\epsilon < \frac{1}{2} dist(K, \partial\Omega)$ so that any point y in the ball $B_\epsilon(x)$ is then in the interior of Ω. Consequently,

$$\begin{aligned}
D^\alpha(J_\epsilon f)(x) &= \int_\Omega D_x^\alpha j_\epsilon(x-y)f(y)dy \\
&= \int_{B_\epsilon(x)} D_x^\alpha j_\epsilon(x-y)f(y)dy \\
&= (-1)^{|\alpha|}\int_{B_\epsilon(x)} D_y^\alpha j_\epsilon(x-y)f(y)dy \\
&= \int_{B_\epsilon(x)} j_\epsilon(x-y)D^\alpha f(y)dy \\
&= \int_\Omega j_\epsilon(x-y)D^\alpha f(y)dy.
\end{aligned}$$

Here we have used the fact that $j_\epsilon(x-y)$ and all its derivatives are supported in $B_\epsilon(x) \subset \Omega$.

This completes the proof of the Theorem 1.3.1. □

The Proof of Theorem 1.3.2

Step 1. For a Bounded Region Ω.

Let

$$\Omega_i = \{x \in \Omega \mid dist(x, \partial\Omega) > \frac{1}{i}\}, \quad i = 1, 2, 3, \cdots$$

Write $O_i = \Omega_{i+3} \setminus \bar{\Omega}_{i+1}$. Choose some open set $O_0 \subset\subset \Omega$ so that $\Omega = \cup_{i=0}^\infty O_i$. Choose a smooth partition of unity $\{\eta_i\}_{i=0}^\infty$ associated with the open sets $\{O_i\}_{i=0}^\infty$,

$$\begin{cases} 0 \leq \eta_i(x) \leq 1, \quad \eta_i \in C_0^\infty(O_i) \\ \sum_{i=0}^\infty \eta_i(x) = 1, \, x \in \Omega. \end{cases}$$

Given any function $f \in W^{k,p}(\Omega)$, obviously $\eta_i f \in W^{k,p}(\Omega)$ and $\text{supp}(\eta_i f) \subset O_i$.

Fix a $\delta > 0$. Choose $\epsilon_i > 0$ so small that $f_i := J_{\epsilon_i}(\eta_i f)$ satisfies

$$\begin{cases} \|f_i - \eta_i f\|_{W^{k,p}(\Omega)} \leq \frac{\delta}{2^{i+1}}, \, i = 0, 1, \cdots, \\ suppf_i \subset (\Omega_{i+4} \setminus \bar{\Omega}_i), \qquad i = 1, 2, \cdots. \end{cases} \tag{1.16}$$

Set $g = \sum_{i=0}^\infty f_i$. Then $g \in C^\infty(\Omega)$ because each f_i is in $C^\infty(\Omega)$, and for each open set $O \subset\subset \Omega$, there are at most finitely many nonzero terms in the sum. To see that $g \in W^{k,p}(\Omega)$, we write

$$g - f = \sum_{i=0}^\infty (f_i - \eta_i f).$$

It follows from (1.16) that

$$\sum_{i=0}^{\infty} \|f_i - \eta_i f\|_{W^{k,p}(\Omega)} \leq \delta \sum_{i=0}^{\infty} \frac{1}{2^{i+1}} = \delta.$$

From here we can see that the series $\sum_{i=0}^{\infty}(f_i - \eta_i f)$ converges in $W^{k,p}(\Omega)$ (see Exercise 1.3.1 below); hence, $(g-f) \in W^{k,p}(\Omega)$ and therefore $g \in W^{k,p}(\Omega)$. Moreover, from the above inequality, we have

$$\|g - f\|_{W^{k,p}(\Omega)} \leq \delta.$$

Since $\delta > 0$ is any number, we complete *Step 1.*

Remark 1.3.1 In the above argument, neither $\sum f_i$ nor $\sum \eta_i f$ converge, but their difference converges. Although each partial sum of $\sum f_i$ is in $C_0^{\infty}(\Omega)$, the infinite series $\sum f_i$ does not converge to g in $W^{k,p}(\Omega)$. Then how did we prove that $g \in W^{k,p}(\Omega)$? We showed that the difference $(g-f)$ is in $W^{k,p}(\Omega)$.

Exercise 1.3.1 .

Let X be a Banach space and $v_i \in X$. Show that the series $\sum_i^{\infty} v_i$ converges in X if $\sum_i^{\infty} \|v_i\| < \infty$.

Hint: Show that the partial sum is a Cauchy sequence.

Step 2. For Unbounded Region Ω.

Given any $\delta > 0$, since $f \in W^{k,p}(\Omega)$, there exist $R > 0$, such that

$$\|f\|_{W^{k,p}(\Omega \setminus B_{R-2}(0))} \leq \delta. \tag{1.17}$$

Choose a cut-off function $\phi \in C^{\infty}(R^n)$ satisfying

$$\phi(x) = \begin{cases} 1, \, x \in B_{R-2}(0), \\ 0, \, x \in R^n \setminus B_R(0); \end{cases} \quad \text{and } |D^{\alpha}\phi(x)| \leq 1 \ \ \forall x \in R^n, \forall |\alpha| \leq k.$$

Then by (1.17), it is easy to verify that there exists a constant C, such that

$$\|\phi f - f\|_{W^{k,p}(\Omega)} \leq C\delta. \tag{1.18}$$

Now in the bounded domain $\Omega \cap B_R(0)$, by the argument in *Step 1*, there is a function $g \in C^{\infty}(\Omega \cap B_R(0))$, such that

$$\|g - f\|_{W^{k,p}(\Omega \cap B_R(0))} \leq \delta. \tag{1.19}$$

Obviously, the function ϕg is in $C^{\infty}(\Omega)$, and by (1.18) and (1.19),

$$\begin{aligned} \|\phi g - f\|_{W^{k,p}(\Omega)} &\leq \|\phi g - \phi f\|_{W^{k,p}(\Omega)} + \|\phi f - f\|_{W^{k,p}(\Omega)} \\ &\leq C_1 \|g - f\|_{W^{k,p}(\Omega \cap B_R(0))} + C\delta \\ &\leq (C_1 + C)\delta. \end{aligned}$$

This completes the proof of the Theorem 1.3.2. □

The Proof of Theorem 1.3.3

We will continue to use mollifiers to approximate a function. As compared to the proofs of the previous theorem, the main difficulty here is that for a point on the boundary, there is no room to mollify. To circumvent this, we cover Ω with finitely many open sets, and on each set that covers the boundary layer, we will translate the function a little bit inward so that there is room to mollify within Ω. Then we will again use the partition of unity to complete the proof.

Step 1. Approximating in a Small Set Covering $\partial\Omega$.

Let x^o be any point on $\partial\Omega$. Since $\partial\Omega$ is C^1, we can make a C^1 change of coordinates locally, so that in the new coordinates system $(x_1, \cdots, x_n)$, we can express, for a sufficiently small $r > 0$,

$$B_r(x^o) \cap \Omega = \{x \in B_r(x^o) \mid x_n > \phi(x_1, \cdots x_{n-1})\}$$

with some C^1 function ϕ.

Set

$$D = \Omega \cap B_{r/2}(x^o).$$

Shift every point $x \in D$ in x_n direction $a\epsilon$ units, define

$$x^\epsilon = x + a\epsilon e_n.$$

and

$$D^\epsilon = \{x^\epsilon \mid x \in D\}.$$

This D^ϵ is obtained by shifting D toward the inside of Ω by $a\epsilon$ units. Choose a sufficiently large, so that the ball $B_\epsilon(x)$ lies in $\Omega \cap B_r(x^o)$ for all $x \in D^\epsilon$ and for all small $\epsilon > 0$. There is now room to mollify a given $W^{k,p}$ function f on D^ϵ within Ω. More precisely, we first translate f a distance ϵ in the x_n direction to become $f^\epsilon(x) = f(x^\epsilon)$, then mollify it. We claim that

$$J_\epsilon f^\epsilon \to f \quad \text{ in } W^{k,p}(D).$$

Actually, for any multi-index $|\alpha| \leq k$, we have

$$\|D^\alpha(J_\epsilon f^\epsilon) - D^\alpha f\|_{L^p(D)} \leq \|D^\alpha(J_\epsilon f^\epsilon) - D^\alpha f^\epsilon\|_{L^p(D)} + \|D^\alpha f^\epsilon - D^\alpha f\|_{L^p(D)}.$$

A similar argument as in the proof of Theorem 1.3.1 implies that the first term on the right hand side goes to zero as $\epsilon \to 0$, while the second term also vanishes in the process due to the continuity of the translation in the L^p norm.

Step 2. Applying the Partition of Unity.

Since $\partial\Omega$ is compact, we can find finitely many such sets D, call them D_i, $i = 1, 2, \cdots, N$, the union of which covers $\partial\Omega$. Given $\delta > 0$, from the argument in *Step 1*, for each D_i, there exists $g_i \in C^\infty(\bar{D}_i)$, such that

$$\|g_i - f\|_{W^{k,p}(D_i)} \leq \delta. \tag{1.20}$$

Choose an open set $D_0 \subset\subset \Omega$ such that $\Omega \subset \cup_{i=0}^N D_i$, and select a function $g_0 \in C^\infty(\bar{D}_0)$ such that

$$\|g_0 - f\|_{W^{k,p}(D_0)} \leq \delta. \tag{1.21}$$

Let $\{\eta_i\}$ be a smooth partition of unity subordinated to the open sets $\{D_i\}_{i=0}^N$. Define

$$g = \sum_{i=0}^N \eta_i g_i.$$

Then obviously $g \in C^\infty(\bar{\Omega})$, and $f = \sum_{i=0}^N \eta_i f$. Similar to the proof of Theorem 1.3.1, it follows from (1.20) and (1.21) that, for each $|\alpha| \leq k$

$$\begin{aligned} \|D^\alpha g - D^\alpha f\|_{L^p(\Omega)} &\leq \sum_{i=0}^N \|D^\alpha(\eta_i g_i) - D^\alpha(\eta_i f)\|_{L^p(D_i)} \\ &\leq C \sum_{i=0}^N \|g_i - f\|_{W^{k,p}(D_i)} = C(N+1)\delta. \end{aligned}$$

This completes the proof of the Theorem 1.3.3. □

The Proof of Theorem 1.3.4

Let $\phi(r)$ be a C_0^∞ cut-off function such that

$$\phi(r) = \begin{cases} 1\ , & \text{for } 0 \leq r \leq 1; \\ \text{between 0 and 1}\ , & \text{for } 1 < r < 2; \\ 0\ , & \text{for } r \geq 2. \end{cases}$$

Then by a direct computation, one can show that

$$\|\phi(\frac{|x|}{R}) f(x) - f(x)\|_{W^{k,p}(R^n)} \to 0, \quad \text{as } R \to \infty. \tag{1.22}$$

Thus, there exists a sequence of numbers $\{R_m\}$ with $R_m \to \infty$, such that for

$$g_m(x) := \phi(\frac{|x|}{R_m}) f(x),$$

we have

$$\|g_m - f\|_{W^{k,p}(R^n)} \leq \frac{1}{m}. \tag{1.23}$$

On the other hand, from the Approximation Theorem 1.3.3, for each fixed m,

$$J_\epsilon(g_m) \to g_m, \quad \text{as } \epsilon \to 0, \quad \text{in } W^{k,p}(R^n).$$

Hence there exist ϵ_m, such that for

$$f_m := J_{\epsilon_m}(g_m),$$

we have

$$\|f_m - g_m\|_{W^{k,p}(R^n)} \leq \frac{1}{m}. \tag{1.24}$$

Obviously, each function f_m is in $C_0^\infty(R^n)$, and by (1.23) and (1.24),

$$\|f_m - f\|_{W^{k,p}(R^n)} \leq \|f_m - g_m\|_{W^{k,p}(R^n)} + \|g_m - f\|_{W^{k,p}(R^n)} \leq \frac{2}{m} \to 0, \text{ as } m \to \infty.$$

This completes the proof of Theorem 1.3.4. □

We have now proved all four Approximation Theorems, which show that smooth functions are dense in Sobolev spaces $W^{k,p}$. In other words, $W^{k,p}$ is the completion of C^k under the norm $\|\cdot\|_{W^{k,p}}$. Later, particularly in the next three sections, when we derive various inequalities in Sobolev spaces, we can first work on smooth functions and then extend them to whole Sobolev spaces. In order to make such extensions more convenient (i.e., avoid going through the approximation process in each particular space), we prove the following Operator Completion Theorem in general Banach spaces.

Theorem 1.3.5 *Let D be a dense linear subspace of a normed space X. Let Y be a Banach space. Assume*

$$T : D \mapsto Y$$

is a bounded linear map. Then there exists an extension $\bar{T}$ of T from D to the whole space X, such that $\bar{T}$ is a bounded linear operator from X to Y,

$$\|\bar{T}\| = \|T\| \quad \text{and} \quad \bar{T}x = Tx \ \ \forall\, x \in D.$$

Proof. Given any element $x_o \in X$, since D is dense in X, there exists a sequence $\{x_i\} \in D$, such that

$$x_i \to x_o, \quad \text{as } i \to \infty.$$

It follows that

$$\|Tx_i - Tx_j\| \leq \|T\|\|x_i - x_j\| \to 0, \quad \text{as } i,j\to\infty.$$

This implies that $\{Tx_i\}$ is a Cauchy sequence in Y. Since Y is a Banach space, $\{Tx_i\}$ converges to some element y_o in Y. Let

$$\bar{T}x_o = y_o. \tag{1.25}$$

To see that (1.25) is well defined, suppose there is another sequence $\{x_i'\}$ that converges to x_o in X and $Tx_i' \to y_1$ in Y. Then

$$\|y_1 - y_o\| = \lim_{i\to\infty} \|Tx_i' - Tx_i\| \leq C\overline{\lim}_{i\to\infty}\|x_i' - x_i\| = 0.$$

Now, for $x \in D$, define $\bar{T}x = Tx$; and for other $x \in X$, define $\bar{T}x$ by (1.25).

Obviously, $\bar{T}$ is a linear operator. Moreover

$$\|\bar{T}x_o\| = \lim_{i\to\infty} \|Tx_i\| \leq \lim_{i\to\infty} \|T\|\|x_i\| = \|T\|\|x_o\|.$$

Hence $\bar{T}$ is also bounded. This completes the proof. □

As a direct application of this Operator Completion Theorem, we prove the following Extension Theorem.

Theorem 1.3.6 *Assume that Ω is bounded and $\partial\Omega$ is C^k. Then for any open set $O \supset \overline{\Omega}$, there exists a bounded linear operator $E : W^{k,p}(\Omega) \to W_0^{k,p}(O)$ such that $Eu = u$ a.e. in Ω.*

Proof. To define the extension operator E, we first work on functions $u \in C^k(\bar{\Omega})$. Then we can apply the Operator Completion Theorem to extend E to $W^{k,p}(\Omega)$, since $C^k(\bar{\Omega})$ is dense in $W^{k,p}(\Omega)$ (see Theorem 1.3.3). From this density, it is easy to show that, for $u \in W^{k,p}(\Omega)$,

$$E(u)(x) = u(x) \quad \text{almost everywhere on } \Omega$$

based on

$$E(u)(x) = u(x) \quad \text{on } \bar{\Omega} \;\; \forall\, u \in C^k(\bar{\Omega}).$$

Now we prove the theorem for $u \in C^k(\bar{\Omega})$. We define $E(u)$ in the following two steps.

Step 1. The special case when Ω is a half ball

$$B_r^+(0) := \{x = (x', x_n) \in R^n \mid |x| < r,\ x_n > 0\}.$$

Assume $u \in C^k(\overline{B_r^+(0)})$. We extend u to be a C^k function on the whole ball $\overline{B_r(0)}$. Define

$$\bar{u}(x', x_n) = \begin{cases} u(x', x_n)\,, & x_n \geq 0 \\ \sum_{i=0}^{k} a_i\, u(x', -\lambda_i x_n)\,, & x_n < 0. \end{cases}$$

To guarantee that all partial derivatives $\dfrac{\partial^j \bar{u}(x', x_n)}{\partial x_n^j}$ up to order k are continuous across the hyper plane $x_n = 0$, we first pick λ_i, such that

$$0 < \lambda_0 < \lambda_1 < \cdots < \lambda_k < 1.$$

We then solve the algebraic system

$$\sum_{i=0}^{k} a_i \lambda_i^j = 1\ , \quad j = 0, 1, \cdots, k,$$

to determine $a_0, a_1, \cdots, a_k$. Since the coefficient determinant $\det(\lambda_i^j)$ is not zero, the system has a unique solution. One can easily verify that, for such λ_i and a_i, the extended function $\bar{u}$ is in $C^k(\overline{B_r(0)})$.

Step 2. Reduce the general domains to the case of the half ball covered in Step 1 via domain transformations and a partition of unity.

Let O be an open set containing $\bar{\Omega}$. Given any $x^o \in \partial\Omega$, there exists a neighborhood U of x^o and a C^k transformation $\phi : U \rightarrow R^n$ which satisfies that $\phi(x^o) = 0$, $\phi(U \cap \partial\Omega)$ lies on the hyper plane $x_n = 0$, and $\phi(U \cap \Omega)$ is contained in R^n_+. Then there exists an $r_o > 0$, such that $B_{r_o}(0) \subset\subset \phi(U)$, $D_{x^o} := \phi^{-1}(B_{r_o}(0))$ is an open set containing x^o, and $\phi \in C^k(\overline{D_{x^o}})$. Choose r_o sufficiently small so that $D_{x^o} \subset O$. All such D_{x^o} and Ω forms an open covering of $\bar{\Omega}$, hence there exists a finite sub-covering

$$D_0 := \Omega,\ D_1,\ \cdots,\ D_m$$

and a corresponding partition of unity

$$\eta_0, \eta_1, \cdots, \eta_m,$$

such that

$$\eta_i \in C_0^\infty(D_i) \ \ i = 0, 1, \cdots, m$$

and

$$\sum_{i=0}^{m} \eta_i(x) = 1 \ \ \forall\, x \in \Omega.$$

Let ϕ_i be the mapping associated with D_i as described above, and let $\tilde{u}_i(y) = u(\phi_i^{-1}(y)$ be the function defined on the half ball

$$\overline{B^+_{r_i}(0)} = \phi_i(\overline{D_i} \cap \bar{\Omega})\ , \quad i = 1, 2, \cdots m.$$

From *Step 1*, each $\tilde{u}_i(y)$ can be extended as a C^k function $\bar{u}_i(y)$ onto the whole ball $\overline{B_{r_i}(0)}$.

We can now define the extension of u from Ω to its neighborhood O as

$$E(u) = \sum_{i=1}^{m} \eta_i(x)\bar{u}_i(\phi_i(x)) + \eta_0 u(x).$$

Obviously, $E(u) \in C_0^\infty(O)$, and

$$\|E(u)\|_{W^{k,p}(O)} \leq C\|u\|_{W^{k,p}(\Omega)}.$$

Moreover, for any $x \in \bar{\Omega}$, we have

$$\begin{aligned}
E(u)(x) &= \sum_{i=1}^{m} \eta_i(x)\bar{u}_i(\phi_i(x)) + \eta_0(x)u(x) = \sum_{i=1}^{m} \eta_i(x)\tilde{u}_i(\phi_i(x)) + \eta_0(x)u(x) \\
&= \sum_{i=1}^{m} \eta_i(x)u(\phi_i^{-1}(\phi_i(x))) + \eta_0(x)u(x) = \sum_{i=1}^{m} \eta_i(x)u(x) + \eta_0(x)u(x) \\
&= \left(\sum_{i=0}^{m} \eta_i(x)\right) u(x) = u(x).
\end{aligned}$$

This completes the proof of the Extension Theorem. □

Remark 1.3.2 Notice that the Extension Theorem actually implies an improved version of the Approximation Theorem under stronger assumption on $\partial\Omega$ (be C^k instead of C^1). From the Extension Theorem, one can immediately derive:

Corollary 1.3.1 *Assume that Ω is bounded and $\partial\Omega$ is C^k. Let O be an open set containing $\bar{\Omega}$. Let*

$$O_\epsilon := \{x \in R^n \mid dist(x, O) \leq \epsilon\}.$$

Then the linear operator

$$J_\epsilon(E(u)) : W^{k,p}(\Omega) {\rightarrow} C_0^\infty(O_\epsilon)$$

is bounded, and for each $u \in W^{k,p}(\Omega)$, we have

$$\|J_\epsilon(E(u)) - u\|_{W^{k,p}(\Omega)} {\rightarrow} 0 \ , \quad as\ \epsilon{\rightarrow}0.$$

Here, as compared to the previous approximation theorems, the improvement is that one can write out the explicit form of the approximation.

1.4 Sobolev Embeddings

When we seek weak solutions of partial differential equations, we start with functions in a Sobolev space $W^{k,p}$. Naturally, we would like to know whether or not the functions in this space also automatically belong to some other spaces. The following theorem answers the question and at the same time provides inequalities among the relevant norms.

Theorem 1.4.1 *(General Sobolev Inequalities).*

Assume Ω is bounded and has a C^1 boundary. Let $u \in W^{k,p}(\Omega)$.

(i) If $k < \frac{n}{p}$, then $u \in L^q(\Omega)$ with $\frac{1}{q} = \frac{1}{p} - \frac{k}{n}$ and there exists a constant C such that

$$\|u\|_{L^q(\Omega)} \leq C\|u\|_{W^{k,p}(\Omega)}$$

(ii) If $k > \frac{n}{p}$, then $u \in C^{k-[\frac{n}{p}]-1,\gamma}(\Omega)$, and there exists a constant C, such that

$$\|u\|_{C^{k-[\frac{n}{p}]-1,\gamma}(\overline{\Omega})} \leq C\|u\|_{W^{k,p}(\Omega)},$$

where

$$\gamma = \begin{cases} 1 + \left[\frac{n}{p}\right] - \frac{n}{p}, & \text{if } \frac{n}{p} \text{ is not an integer} \\ \text{any positive number } < 1, & \text{if } \frac{n}{p} \text{ is an integer .} \end{cases}$$

Here, $[b]$ is the integer part of the number b.

The proof of the Theorem is based upon several simpler theorems. We first consider the functions in $W^{1,p}(\Omega)$. From the definition, apparently these functions belong to $L^q(\Omega)$ for $1 \leq q \leq p$. Naturally, one would expect more, and what is more meaningful is to find out how large this q can be. And to control the L^q norm for larger q by $W^{1,p}$ norm–the norm of the derivatives

$$\|Du\|_{L^p(\Omega)} = \left(\int_\Omega |Du|^p dx\right)^{\frac{1}{p}}$$

–would suppose to be more useful in practice.

For simplicity, we start with the smooth functions with compact supports in R^n. We would like to know for what value of q can we establish the inequality

$$\|u\|_{L^q(R^n)} \leq C\|Du\|_{L^p(R^n)}, \tag{1.26}$$

with constant C independent of $u \in C_0^\infty(R^n)$. Now suppose (1.26) holds. Then it must also be true for the re-scaled function of u:

$$u_\lambda(x) = u(\lambda x),$$

that is

$$\|u_\lambda\|_{L^q(R^n)} \leq C\|Du_\lambda\|_{L^p(R^n)}. \tag{1.27}$$

By substitution, we have

$$\int_{R^n} |u_\lambda(x)|^q dx = \frac{1}{\lambda^n} \int_{R^n} |u(y)|^q dy, \text{ and}$$
$$\int_{R^n} |Du_\lambda|^p dx = \frac{\lambda^p}{\lambda^n} \int_{R^n} |Du(y)|^p dy.$$

It follows from (1.27) that

$$\|u\|_{L^q(R^n)} \leq C \lambda^{1-\frac{n}{p}+\frac{n}{q}} \|Du\|_{L^p(R^n)}.$$

Therefore, for C to be independent of u, it is necessary that the power of λ here be zero, that is,

$$q = \frac{np}{n-p}.$$

It turns out that this condition is also sufficient, as will be stated in the following theorem. Here for q to be positive, we must require $p < n$.

Theorem 1.4.2 *(Gagliardo-Nirenberg-Sobolev Inequality).*

Assume that $1 \leq p < n$. Then there exists a constant $C = C(p, n)$, such that

$$\|u\|_{L^{p^*}(R^n)} \leq C \|Du\|_{L^p(R^n)}, \quad u \in C_0^1(R^n), \tag{1.28}$$

where $p^* = \frac{np}{n-p}$.

Since any function in $W^{1,p}(R^n)$ can be approached by a sequence of functions in $C_0^1(R^n)$, we derive immediately:

Corollary 1.4.1 *Inequality (1.28) holds for all functions u in $W^{1,p}(R^n)$.*

For functions in $W^{1,p}(\Omega)$, we can extend them to be $W^{1,p}(R^n)$ functions by the Extension Theorem and arrive at:

Theorem 1.4.3 *Assume that Ω is a bounded, open subset of R^n with C^1 boundary. Suppose that $1 \leq p < n$ and $u \in W^{1,p}(\Omega)$. Then u is in $L^{p*}(\Omega)$ and there exists a constant $C = C(p, n, \Omega)$, such that*

$$\|u\|_{L^{p*}(\Omega)} \leq C \|u\|_{W^{1,p}(\Omega)}. \tag{1.29}$$

In the limiting case as $p {\rightarrow} n$, $p* := \frac{np}{n-p} {\rightarrow} \infty$. Then one may suspect that $u \in L^\infty$ when $p = n$. Unfortunately, this is true only in dimension one. For $n = 1$, from

$$u(x) = \int_{-\infty}^x u'(x) dx,$$

we derive immediately that

$$|u(x)| \leq \int_{-\infty}^{\infty} |u'(x)|dx.$$

However, for $n > 1$, it is false. One counter example is $u = \log\log(1 + \frac{1}{|x|})$ on $\Omega = B_1(0)$. It belongs to $W^{1,n}(\Omega)$, but not to $L^\infty(\Omega)$. This is a delicate situation, and we will deal with it later.

Naturally, for $p > n$, one would expect $W^{1,p}$ to embed into better spaces. To get some rough idea what these spaces might be, let us first consider the simplest case when $n = 1$ and $p > 1$. Obviously, for any $x, y \in R^1$ with $x < y$, we have

$$u(y) - u(x) = \int_x^y u'(t)dt,$$

and consequently, by the Hölder inequality,

$$|u(y) - u(x)| \leq \int_x^y |u'(t)|dt \leq \left(\int_x^y |u'(t)|^p dt\right)^{\frac{1}{p}} \cdot \left(\int_x^y dt\right)^{1-\frac{1}{p}}.$$

It follows that

$$\frac{|u(y) - u(x)|}{|y - x|^{1-\frac{1}{p}}} \leq \left(\int_{-\infty}^{\infty} |u'(t)|^p dt\right)^{\frac{1}{p}}.$$

Taking the supremum over all pairs x, y in R^1, the left hand side of the above inequality is the norm in the Hölder space $C^{0,\gamma}(R^1)$ with $\gamma = 1 - \frac{1}{p}$. This is indeed true in general, and we have:

Theorem 1.4.4 *(Morrey's Inequality).*

Assume $n < p \leq \infty$. Then there exists a constant $C = C(p, n)$ such that

$$\|u\|_{C^{0,\gamma}(R^n)} \leq C\|u\|_{W^{1,p}(R^n)}, \quad u \in C^1(R^n)$$

where $\gamma = 1 - \frac{n}{p}$.

To establish an inequality like (1.28), we follow two basic principles.

First, we consider the special case where $\Omega = R^n$. Instead of dealing with functions in $W^{k,p}$, we reduce the proof to functions with enough smoothness (which is just Theorem 1.4.2).

Secondly, we deal with general domains by extending functions $u \in W^{1,p}(\Omega)$ to $W^{1,p}(R^n)$ via the Extension Theorem.

Here, one sees that Theorem 1.4.2 is the 'key,' and inequality (1.28) provides the foundation for the proof of the Sobolev embedding. Often, the proof of (1.28) is called a 'hard analysis', and the steps leading from (1.28) to (1.29) are called 'soft analysis'. In the following, we will show the 'soft' parts first and then the 'hard' ones. First, we will assume that Theorem 1.4.2 is true and derive Corollary 1.4.1 and Theorem 1.4.3. Then, we will prove Theorem 1.4.2.

The Proof of Corollary 1.4.1.

Given any function $u \in W^{1,p}(R^n)$, by the Approximation Theorem, there exists a sequence $\{u_k\} \subset C_0^\infty(R^n)$ such that $\|u - u_k\|_{W^{1,p}(R^n)} \to 0$ as $k \to \infty$.

Applying Theorem 1.4.2, we obtain

$$\|u_i - u_j\|_{L^{p^*}(R^n)} \leq C\|u_i - u_j\|_{W^{1,p}(R^n)} \to 0, \text{ as } i,j \to \infty.$$

Thus $\{u_k\}$ also converges to u in $L^{p^*}(R^n)$. Consequently, we arrive at

$$\|u\|_{L^{p^*}(R^n)} = \lim \|u_k\|_{L^{p^*}(R^n)} \leq \lim C\|u_k\|_{W^{1,p}(R^n)} = C\|u\|_{W^{1,p}(R^n)}.$$

This completes the proof of the Corollary. □

The Proof of Theorem 1.4.3.

Now for functions in $W^{1,p}(\Omega)$, to apply inequality (1.27), we first extend them to be functions with compact supports in R^n. More precisely, let O be an open set that covers Ω. By the Extension Theorem (Theorem 1.3.6), for every u in $W^{1,p}(\Omega)$, there exists a function $\tilde{u}$ in $W_0^{1,p}(O)$, such that

$$\tilde{u} = u, \quad \text{almost everywhere in } \Omega;$$

moreover, there exists a constant $C_1 = C_1(p, n, \Omega, O)$, such that

$$\|\tilde{u}\|_{W^{1,p}(O)} \leq C_1\|u\|_{W^{1,p}(\Omega)}. \tag{1.30}$$

Now we can apply the Gagliardo-Nirenberg-Sobolev inequality to $\tilde{u}$ to derive

$$\|u\|_{L^{p^*}(\Omega)} \leq \|\tilde{u}\|_{L^{p^*}(O)} \leq C\|\tilde{u}\|_{W^{1,p}(O)} \leq CC_1\|u\|_{W^{1,p}(\Omega)}.$$

This completes the proof of the Theorem. □

The Proof of Theorem 1.4.2.

We first establish the inequality for $p = 1$, i.e., we prove

$$\left(\int_{R^n} |u|^{\frac{n}{n-1}} dx\right)^{\frac{n-1}{n}} \leq \int_{R^n} |Du| dx. \tag{1.31}$$

Then we will apply (1.31) to $|u|^\gamma$ for a properly chosen $\gamma > 1$ to extend the inequality to the case when $p > 1$.

We need

Lemma 1.4.1 *(General Hölder Inequality). Assume that*

$$u_i \in L^{p_i}(\Omega) \text{ for } i = 1, 2, \cdots, m$$

and

$$\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_m} = 1.$$

Then

$$\int_\Omega |u_1 u_2 \cdots u_m| dx \leq \prod_i^m \left(\int_\Omega |u_i|^{p_i} dx \right)^{\frac{1}{p_i}}. \tag{1.32}$$

The proof can be obtained by applying induction to the usual Hölder inequality for two functions.

Now we are ready to prove the Theorem.

Step 1. The case $p = 1$.

To better illustrate the idea, we first derive inequality (1.31) for $n = 2$; that is, we prove

$$\int_{R^2} |u(x)|^2 dx \leq \left(\int_{R^2} |Du| dx \right)^2. \tag{1.33}$$

Since u has a compact support, we have

$$u(x) = \int_{-\infty}^{x_1} \frac{\partial u}{\partial y_1}(y_1, x_2) dy_1.$$

It follows that

$$|u(x)| \leq \int_{-\infty}^{\infty} |Du(y_1, x_2)| dy_1.$$

Similarly, we have

$$|u(x)| \leq \int_{-\infty}^{\infty} |Du(x_1, y_2)| dy_2.$$

The above two inequalities together imply that

$$|u(x)|^2 \leq \int_{-\infty}^{\infty} |Du(y_1, x_2)| dy_1 \cdot \int_{-\infty}^{\infty} |Du(x_1, y_2)| dy_2.$$

Now, integrating both sides of the above inequality with respect to x_1 and x_2 from $-\infty$ to ∞, we arrive at (1.33).

Then we deal with the general situation when $n > 2$. We write

$$u(x) = \int_{-\infty}^{x_i} \frac{\partial u}{\partial y_i}(x_1, \cdots, x_{i-1}, y_i, x_{i+1}, \cdots, x_n) dy_i, \quad i = 1, 2, \cdots n.$$

Consequently,

$$|u(x)| \leq \int_{-\infty}^{\infty} |Du(x_1, \cdots, y_i, \cdots, x_n)| dy_i, \quad i = 1, 2, \cdots n.$$

And it follows that

$$|u(x)|^{\frac{n}{n-1}} \leq \prod_{i=1}^{n} \left(\int_{-\infty}^{\infty} |Du(x_1, \cdots, y_i, \cdots, x_n)| dy_i \right)^{\frac{1}{n-1}}.$$

Integrating both sides with respect to x_1 and applying the general Hölder inequality (1.32), we obtain

$$\int_{-\infty}^{\infty} |u|^{\frac{n}{n-1}} dx_1 \leq \left(\int_{-\infty}^{\infty} |Du| dy_1 \right)^{\frac{1}{n-1}} \int_{-\infty}^{\infty} \prod_{i=2}^{n} \left(\int_{-\infty}^{\infty} |Du| dy_i \right)^{\frac{1}{n-1}} dx_1$$
$$\leq \left(\int_{-\infty}^{\infty} |Du| dy_1 \right)^{\frac{1}{n-1}} \left(\prod_{i=2}^{n} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du| dx_1 dy_i \right)^{\frac{1}{n-1}}.$$

Then integrate the above inequality with respect to x_2. We have

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |u|^{\frac{n}{n-1}} dx_1 dx_2 \leq \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du| dx_1 dy_2 \right)^{\frac{1}{n-1}}$$
$$\times \int_{-\infty}^{\infty} \left\{ \left(\int_{-\infty}^{\infty} |Du| dy_1 \right)^{\frac{1}{n-1}} \cdot \prod_{i=3}^{n} \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du| dx_1 dy_i \right)^{\frac{1}{n-1}} \right\} dx_2.$$

Again, applying the General Hölder Inequality, we arrive at

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |u|^{\frac{n}{n-1}} dx_1 dx_2$$
$$\leq \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du| dy_1 dx_2 \right)^{\frac{1}{n-1}} \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du| dx_1 dy_2 \right)^{\frac{1}{n-1}}$$
$$\times \prod_{i=3}^{n} \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du| dx_1 dx_2 dy_i \right)^{\frac{1}{n-1}}.$$

Continuing this way by integrating with respect to $x_3, \cdots, x_{n-1}$, we deduce

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |u|^{\frac{n}{n-1}} dx_1 \cdots dx_{n-1}$$
$$\leq \left(\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |Du| dy_1 dx_2 \cdots dx_{n-1} \right)^{\frac{1}{n-1}}$$
$$\times \left(\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |Du| dx_1 dy_2 dx_3 \cdots dx_{n-1} \right)^{\frac{1}{n-1}} \cdots$$
$$\times \left(\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |Du| dx_1 \cdots dx_{n-2} dy_{n-1} \right)^{\frac{1}{n-1}} \left(\int_{R^n} |Du| dx \right)^{\frac{1}{n-1}}.$$

Finally, integrating both sides with respect to x_n and applying the general Hölder inequality, we obtain

$$\int_{R^n} |u|^{\frac{n}{n-1}} dx \leq \left(\int_{R^n} |Du| dx \right)^{\frac{n}{n-1}} .$$

This verifies (1.31).

Exercise 1.4.1 *Write your own proof with all details for the cases $n = 3$ and $n = 4$.*

Step 2. The Case $p > 1$.

Applying (1.31) to the function $|u|^\gamma$ with $\gamma > 1$ to be chosen later, and by the Hölder inequality, we have

$$\begin{aligned}\left(\int_{R^n} |u|^{\frac{\gamma n}{n-1}} dx \right)^{\frac{n-1}{n}} &\leq \int_{R^n} |D(|u|^\gamma)| dx = \gamma \int_{R^n} |u|^{\gamma-1} |Du| dx \\ &\leq \gamma \left(\int_{R^n} |u|^{\frac{\gamma n}{n-1}} dx \right)^{\frac{(\gamma-1)(n-1)}{\gamma n}} \left(\int_{R^n} |Du|^{\frac{\gamma n}{\gamma+n-1}} dx \right)^{\frac{\gamma+n-1}{\gamma n}} .\end{aligned}$$

It follows that

$$\left(\int_{R^n} |u|^{\frac{\gamma n}{n-1}} dx \right)^{\frac{n-1}{\gamma n}} \leq \gamma \left(\int_{R^n} |Du|^{\frac{\gamma n}{\gamma+n-1}} dx \right)^{\frac{\gamma+n-1}{\gamma n}} .$$

Now choose γ, so that $\dfrac{\gamma n}{\gamma + n - 1} = p$, that is

$$\gamma = \frac{p(n-1)}{n-p},$$

we obtain

$$\left(\int_{R^n} |u|^{\frac{np}{n-p}} dx \right)^{\frac{n-p}{np}} \leq \gamma \left(\int_{R^n} |Du|^p dx \right)^{\frac{1}{p}} .$$

This completes the proof of the Theorem. □

The Proof of Theorem 1.4.4 (Morrey's inequality).

We will establish two inequalities

$$\sup_{R^n} |u| \leq C \|u\|_{W^{1,p}(R^n)}, \tag{1.34}$$

and

$$\sup_{x \neq y} \frac{|u(x) - u(y)|}{|x - y|^{1-n/p}} \leq C \|Du\|_{L^p(R^n)}. \tag{1.35}$$

Both of them can be derived from the following

$$\frac{1}{|B_r(x)|}\int_{B_r(x)}|u(y)-u(x)|dy \le C\int_{B_r(x)}\frac{|Du(y)|}{|y-x|^{n-1}}dy, \tag{1.36}$$

where $|B_r(x)|$ is the volume of $B_r(x)$. We will carry the proof out in three steps. In *Step 1*, we prove (1.36), and in *Step 2* and *Step 3*, we verify (1.34) and (1.35), respectively.

Step 1. We start from

$$u(y)-u(x)=\int_0^1\frac{d}{dt}u(x+t(y-x))dt=\int_0^s Du(x+\tau\omega)\cdot\omega d\tau,$$

where

$$\omega=\frac{y-x}{|x-y|},\ s=|x-y|, \text{ and hence } y=x+s\omega.$$

It follows that

$$|u(x+s\omega)-u(x)|\le\int_0^s|Du(x+\tau\omega)|d\tau.$$

Integrating both sides with respect to ω on the unit sphere $\partial B_1(0)$, then converting the integral on the right hand side from polar to rectangular coordinates, we obtain

$$\begin{aligned}\int_{\partial B_1(0)}|u(x+s\omega)-u(x)|d\sigma &\le \int_0^s\int_{\partial B_1(0)}|Du(x+\tau\omega)|d\sigma d\tau\\ &=\int_0^s\int_{\partial B_1(0)}|Du(x+\tau\omega)|\frac{\tau^{n-1}}{\tau^{n-1}}d\sigma d\tau\\ &=\int_{B_s(x)}\frac{|Du(z)|}{|x-z|^{n-1}}dz.\end{aligned}$$

Multiplying both sides by s^{n-1}, integrating with respect to s from 0 to r, and taking into account that the integrand on the right hand side is non-negative, we arrive at

$$\int_{B_r(x)}|u(y)-u(x)|dy\le\frac{r^n}{n}\int_{B_r(x)}\frac{|Du(y)|}{|x-y|^{n-1}}dy.$$

This verifies (1.36).

Step 2. For each fixed $x\in R^n$, we have

$$\begin{aligned}|u(x)| &= \frac{1}{|B_1(x)|}\int_{B_1(x)}|u(x)|dy\\ &\le\frac{1}{|B_1(x)|}\int_{B_1(x)}|u(x)-u(y)|dy+\frac{1}{|B_1(x)|}\int_{B_1(x)}|u(y)|dy\\ &= I_1+I_2.\end{aligned} \tag{1.37}$$

By (1.36) and the Hölder inequality, we deduce

$$\begin{aligned}
I_1 &\leq C \int_{B_1(x)} \frac{|Du(y)|}{|x-y|^{n-1}} dy \\
&\leq C \left(\int_{R^n} |Du|^p dy \right)^{1/p} \left(\int_{B_1(x)} \frac{dy}{|x-y|^{\frac{(n-1)p}{p-1}}} dy \right)^{\frac{p-1}{p}} \\
&\leq C_1 \left(\int_{R^n} |Du|^p dy \right)^{1/p}. \qquad (1.38)
\end{aligned}$$

Here we have used the condition that $p > n$, so that $\frac{(n-1)p}{p-1} < n$, and hence the integral

$$\int_{B_1(x)} \frac{dy}{|x-y|^{\frac{(n-1)p}{p-1}}}$$

is finite.

Also it is obvious that

$$I_2 \leq C\|u\|_{L^p(R^n)}. \qquad (1.39)$$

Now (1.34) is an immediate consequence of (1.37), (1.38), and (1.39).

Step 3. For any pair of fixed points x and y in R^n, let $r = |x-y|$ and $D = B_r(x) \cap B_r(y)$. Then

$$|u(x) - u(y)| \leq \frac{1}{|D|} \int_D |u(x) - u(z)| dz + \frac{1}{|D|} \int_D |u(z) - u(y)| dz. \qquad (1.40)$$

Again by (1.36), we have

$$\begin{aligned}
\frac{1}{|D|} \int_D |u(x) - u(z)| dz &\leq \frac{C}{|B_r(x)|} \int_{B_r(x)} |u(x) - u(z)| dz \\
&\leq C \left(\int_{R^n} |Du|^p dz \right)^{1/p} \left(\int_{B_r(x)} \frac{dz}{|x-z|^{\frac{(n-1)p}{p-1}}} \right)^{\frac{p-1}{p}} \\
&= C r^{1-n/p} \|Du\|_{L^p(R^n)}. \qquad (1.41)
\end{aligned}$$

Similarly,

$$\frac{1}{|D|} \int_D |u(z) - u(y)| dz \leq C r^{1-n/p} \|Du\|_{L^p(R^n)}. \qquad (1.42)$$

Now (1.40), (1.41), and (1.42) yield

$$|u(x) - u(y)| \leq C|x-y|^{1-n/p} \|Du\|_{L^p(R^n)}.$$

This implies (1.35) and thus completes the proof of the Theorem. □

The Proof of Theorem 1.4.1 (the General Sobolev Inequality).

(i) Assume that $u \in W^{k,p}(\Omega)$ with $k < \dfrac{n}{p}$. We want to show that $u \in L^q(\Omega)$ and

$$\|u\|_{L^q(\Omega)} \leq C\|u\|_{W^{k,p}(\Omega)} \tag{1.43}$$

with $\frac{1}{q} = \frac{1}{p} - \frac{k}{n}$. This can be done by applying Theorem 1.4.3 successively on the integer k. Again denote $p^* = \frac{np}{n-p}$, then $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$. For $|\alpha| \leq k-1$, $D^\alpha u \in W^{1,p}(\Omega)$. By Theorem 1.4.3, we have $D^\alpha u \in L^{p^*}(\Omega)$, thus $u \in W^{k-1,p^*}(\Omega)$ and

$$\|u\|_{W^{k-1,p^*}(\Omega)} \leq C_1\|u\|_{W^{k,p}(\Omega)}.$$

Applying Theorem 1.4.3 again to $W^{k-1,p^*}(\Omega)$, we have $u \in W^{k-2,p^{**}}(\Omega)$ and

$$\|u\|_{W^{k-2,p^{**}}(\Omega)} \leq C_2\|u\|_{W^{k-1,p^*}(\Omega)} \leq C_2C_1\|u\|_{W^{k,p}(\Omega)},$$

where $p^{**} = \frac{np*}{n-p^*}$, or

$$\frac{1}{p^{**}} = \frac{1}{p^*} - \frac{1}{n} = (\frac{1}{p} - \frac{1}{n}) - \frac{1}{n} = \frac{1}{p} - \frac{2}{n}.$$

Continuing this way k times, we arrive at (1.43).

(ii) Now assume that $k > \dfrac{n}{p}$. Recall that in the Morrey's inequality

$$\|u\|_{C^{0,\gamma}(\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)}, \tag{1.44}$$

we require that $p > n$. However, in our situation, this condition is not necessarily met. To remedy this, we can use the result in the previous step. We can first decrease the order of differentiation to increase the power of summability. More precisely, we will try to find a smallest integer m, such that

$$W^{k,p}(\Omega) \hookrightarrow W^{k-m,q}(\Omega),$$

with $q > n$. That is, we want

$$q = \frac{np}{n-mp} > n,$$

and equivalently,

$$m > \frac{n}{p} - 1.$$

Obviously, the smallest such integer m is

$$m = \left[\frac{n}{p}\right].$$

For this choice of m, we can apply Morrey's inequality (1.44) to $D^\alpha u$, with any $|\alpha| \leq k-m-1$ to obtain

$$\|D^\alpha u\|_{C^{0,\gamma}(\Omega)} \leq C_1\|u\|_{W^{k-m,q}(\Omega)} \leq C_1C_2\|u\|_{W^{k,p}(\Omega)}.$$

Or equivalently,

$$\|u\|_{C^{k-\left[\frac{n}{p}\right]-1,\gamma}(\Omega)} \leq C\|u\|_{W^{k,p}(\Omega)}.$$

Here, when $\frac{n}{p}$ is not an integer,

$$\gamma = 1 - \frac{n}{q} = 1 + \left[\frac{n}{p}\right] - \frac{n}{p}.$$

While $\frac{n}{p}$ is an integer, we have $m = \frac{n}{p}$, and in this case, q can be any number $> n$, which implies that γ can be any positive number < 1.

This completes the proof of the Theorem. $\square$

1.5 Compact Embedding

In the previous section, we proved that $W^{1,p}(\Omega)$ is embedded into $L^{p^*}(\Omega)$ with $p^* = \frac{np}{n-p}$. Obviously, for $q < p^*$, the embedding of $W^{1,p}(\Omega)$ into $L^q(\Omega)$ is still true, if the region Ω is bounded. Actually, due to the strict inequality on the exponent, one can expect more, as stated below.

Theorem 1.5.1 *(Rellich-Kondrachov Compact Embedding).*

Assume that Ω is a bounded open subset in R^n with C^1 boundary $\partial\Omega$. Suppose $1 \leq p < n$. Then for each $1 \leq q < p^*$, $W^{1,p}(\Omega)$ is compactly embedded into $L^q(\Omega)$:

$$W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega),$$

in the sense that

i) there is a constant C, such that

$$\|u\|_{L^q(\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)}\ , \quad \forall\, u \in W^{1,p}(\Omega); \tag{1.45}$$

and

ii) every bounded sequence in $W^{1,p}(\Omega)$ possesses a convergent subsequence in $L^q(\Omega)$.

Proof. The first part–inequality (1.45)– is included in the general Embedding Theorem. What we need to show is that if $\{u_k\}$ is a bounded sequence in $W^{1,p}(\Omega)$, then it possesses a convergent subsequence in $L^q(\Omega)$. This can be derived immediately from the following:

Lemma 1.5.1 *Every bounded sequence in $W^{1,1}(\Omega)$ possesses a convergent subsequence in $L^1(\Omega)$.*

We postpone the proof of the Lemma for a moment and see how it implies the Theorem.

In fact, assume that $\{u_k\}$ is a bounded sequence in $W^{1,p}(\Omega)$. Then there exists a subsequence (still denoted by $\{u_k\}$) which converges weakly to an element u_o in $W^{1,p}(\Omega)$. By the Sobolev embedding, $\{u_k\}$ is bounded in $L^{p*}(\Omega)$. On the other hand, it is also bounded in $W^{1,1}(\Omega)$, since $p \geq 1$ and Ω is bounded. Now, by Lemma 1.5.1, there is a subsequence (still denoted by $\{u_k\}$) that converges strongly to u_o in $L^1(\Omega)$. Applying the Hölder inequality

$$\|u_k - u_o\|_{L^q(\Omega)} \leq \|u_k - u_o\|^{\theta}_{L^1(\Omega)} \|u_k - u_o\|^{1-\theta}_{L^{p*}(\Omega)},$$

we conclude immediately that $\{u_k\}$ converges strongly to u_o in $L^q(\Omega)$. This proves the Theorem.

We now come back to prove the Lemma. Let $\{u_k\}$ be a bounded sequence in $W^{1,1}(\Omega)$. We will show the strong convergence of this sequence in three steps with the help of a family of mollifiers

$$u_k^{\epsilon}(x) = \int_{\Omega} j_{\epsilon}(y) u_k(x-y) dy.$$

First, we show that

$$u_k^{\epsilon} \to u_k \quad \text{in } L^1(\Omega) \text{ as } \epsilon \to 0\,, \text{ uniformly in k.} \tag{1.46}$$

Then, for each fixed $\epsilon > 0$, we prove that

$$\text{there is a subsequence of } \{u_k^{\epsilon}\} \text{ which converges uniformly .} \tag{1.47}$$

Finally, corresponding to the above convergent sequence $\{u_k^{\epsilon}\}$, we extract diagonally a subsequence of $\{u_k\}$ which converges strongly in $L^1(\Omega)$.

Based on the Extension Theorem, we may assume, without loss of generality, that $\Omega = R^n$, the sequence of functions $\{u_k\}$ all have compact support in a bounded open set $G \subset R^n$, and

$$\|u_k\|_{W^{1,1}(G)} \leq C < \infty\,, \text{ for all } k = 1, 2, \cdots .$$

Since every $W^{1,1}$ function can be approached by a sequence of smooth functions, we may also assume that each u_k is smooth.

Step 1. From the property of mollifiers, we have

$$
\begin{aligned}
u_k^\epsilon(x) - u_k(x) &= \int_{B_1(0)} j(y)[u_k(x-\epsilon y) - u_k(x)]dy \\
&= \int_{B_1(0)} j(y) \int_0^1 \frac{d}{dt} u_k(x-\epsilon t y) dt\, dy \\
&= -\epsilon \int_{B_1(0)} j(y) \int_0^1 D u_k(x - \epsilon t y) dt\, y\, dy.
\end{aligned}
$$

Integrating with respect to x and changing the order of integration, we obtain

$$
\begin{aligned}
\|u_k^\epsilon - u_k\|_{L^1(G)} &\leq \epsilon \int_{B_1(0)} j(y) \int_0^1 \int_G |Du_k(x-\epsilon t y)| dx\, dt\, dy \\
&\leq \epsilon \int_G |Du_k(z)| dz \leq \epsilon \|u_k\|_{W^{1,1}(G)}. \qquad (1.48)
\end{aligned}
$$

It follows that

$$
\|u_k^\epsilon - u_k\|_{L^1(G)} \to 0\,, \text{ as } \epsilon \to 0\,, \text{ uniformly in } k. \qquad (1.49)
$$

Step 2. Now fix an $\epsilon > 0$. Then for all $x \in R^n$ and for all $k = 1, 2, \cdots$, we have

$$
\begin{aligned}
|u_k^\epsilon(x)| &\leq \int_{B_\epsilon(x)} j_\epsilon(x-y)|u_k(y)| dy \\
&\leq \|j_\epsilon\|_{L^\infty(R^n)} \|u_k\|_{L^1(G)} \leq \frac{C}{\epsilon^n} < \infty. \qquad (1.50)
\end{aligned}
$$

Similarly,

$$
|Du_k^\epsilon(x)| \leq \frac{C}{\epsilon^{n+1}} < \infty. \qquad (1.51)
$$

(1.50) and (1.51) imply that, for each fixed $\epsilon > 0$, the sequence $\{u_k^\epsilon\}$ is uniformly bounded and equi-continuous. Therefore, by the Arzela-Ascoli Theorem (see the Appendix), it possesses a subsequence (still denoted by $\{u_k^\epsilon\}$) which converges uniformly on G, in particular

$$
\lim_{j,i \to \infty} \|u_j^\epsilon - u_i^\epsilon\|_{L^1(G)} = 0. \qquad (1.52)
$$

Step 3. Now choose ϵ to be $1, 1/2, 1/3, \cdots, 1/k, \cdots$ successively, and denote the corresponding subsequence that satisfies (1.52) by

$$
u_{k1}^{1/k}, u_{k2}^{1/k}, u_{k3}^{1/k} \cdots,
$$

for $k = 1, 2, 3, \cdots$. Pick the diagonal subsequence from the above:

$$\{u_{ii}^{1/i}\} \subset \{u_k^\epsilon\}.$$

Then select the corresponding subsequence from $\{u_k\}$:

$$u_{11}, u_{22}, u_{33}, \cdots .$$

This is our desired subsequence, because as $i, j \to \infty$,

$$\|u_{ii} - u_{jj}\|_{L^1(G)} \leq \|u_{ii} - u_{ii}^{1/i}\|_{L^1(G)} + \|u_{ii}^{1/i} - u_{jj}^{1/j}\|_{L^1(G)} + \|u_{jj}^{1/j} - u_{jj}\|_{L^1(G)} \to 0 \,,$$

due to the fact that each norm on the right hand side $\to 0$.

This completes the proof of the Lemma and hence the Theorem. □

1.6 Other Basic Inequalities

1.6.1 Poincaré's Inequality

For functions that vanish on the boundary, we have

Theorem 1.6.1 *(Poincaré's Inequality I).*

Assume Ω is bounded. Suppose $u \in W_0^{1,p}(\Omega)$ for some $1 \leq p \leq \infty$. Then

$$\|u\|_{L^p(\Omega)} \leq C \|Du\|_{L^p(\Omega)}. \tag{1.53}$$

Remark 1.6.1 i) Now based on this inequality, one can take an equivalent norm of $W_0^{1,p}(\Omega)$ as $\|u\|_{W_0^{1,p}(\Omega)} = \|Du\|_{L^p(\Omega)}$.

ii) Just for this theorem, one can easily prove it by using the Sobolev inequality $\|u\|_{L^p} \leq \|\bigtriangledown u\|_{L^q}$ with $p = \frac{nq}{n-q}$ and the Hölder inequality. However, the one we present in the following is a unified proof that works for both this and the next theorem.

Proof. For convenience, we abbreviate $\|u\|_{L^p(\Omega)}$ as $\|u\|_p$. Suppose inequality (1.53) does not hold, then there exists a sequence $\{u_k\} \subset W_0^{1,p}(\Omega)$, such that

$$\|Du_k\|_p = 1 \,, \text{ while } \|u_k\|_p \to \infty \,, \text{ as } k \to \infty.$$

Since $C_0^\infty(\Omega)$ is dense in $W_0^{1,p}(\Omega)$, we may assume that $\{u_k\} \subset C_0^\infty(\Omega)$.

Let $v_k = \dfrac{u_k}{\|u_k\|_p}$. Then

$$\|v_k\|_p = 1\,, \text{ and } \|Dv_k\|_p \to 0\,.$$

Consequently, $\{v_k\}$ is bounded in $W^{1,p}(\Omega)$ and hence possesses a subsequence (still denoted by $\{v_k\}$) that converges weakly to some $v_o \in W^{1,p}(\Omega)$. From the compact embedding results in the previous section, $\{v_k\}$ converges strongly in $L^p(\Omega)$ to v_o, and therefore

$$\|v_o\|_p = 1\,. \tag{1.54}$$

On the other hand, for each $\phi \in C_0^\infty(\Omega)$,

$$\int_\Omega v\phi_{x_i} dx = \lim_{k\to\infty} \int_\Omega v_k \phi_{x_i} dx = -\lim_{k\to\infty} \int_\Omega v_{k,x_i} \phi dx = 0.$$

It follows that

$$Dv_o(x) = 0\,, a.e.$$

Thus v_o is a constant. Taking into account that $v_k \in C_0^\infty(\Omega)$, we must have $v_o \equiv 0$. This contradicts with (1.54) and therefore completes the proof of the Theorem. □

For functions in $W^{1,p}(\Omega)$, which may not be zero on the the boundary, we have another version of Poincaré's inequality.

Theorem 1.6.2 *(Poincaré Inequality II).*

Let Ω be a bounded, connected, and open subset in R^n with C^1 boundary. Let $\bar{u}$ be the average of u on Ω. Assume $1 \leq p \leq \infty$. Then there exists a constant $C = C(n, p, \Omega)$, such that

$$\|u - \bar{u}\|_p \leq C\|Du\|_p\,, \forall\, u \in W^{1,p}(\Omega). \tag{1.55}$$

The proof of this Theorem is similar to the previous one. Instead of letting $v_k = \dfrac{u_k}{\|u_k\|_p}$, we choose

$$v_k = \frac{u_k - \bar{u}_k}{\|u_k - \bar{u}_k\|_p}.$$

Remark 1.6.2 The connectedness of Ω is essential in this version of the inequality. A simple counter example is when $n = 1$, $\Omega = [0, 1] \cup [2, 3]$, and

$$u(x) = \begin{cases} -1 & \text{for } x \in [0, 1] \\ 1 & \text{for } x \in [2, 3]\,. \end{cases}$$

1.6.2 The Classical Hardy-Littlewood-Sobolev Inequality

Theorem 1.6.3 *(Hardy-Littlewood-Sobolev Inequality).*

Let $0 < \lambda < n$ *and* $s, r > 1$ *such that*

$$\frac{1}{r} + \frac{1}{s} + \frac{\lambda}{n} = 2.$$

Assume that $f \in L^r(R^n)$ *and* $g \in L^s(R^n)$*. Then*

$$\int_{R^n} \int_{R^n} f(x)|x-y|^{-\lambda} g(y) dx dy \leq C(n,s,\lambda) ||f||_r ||g||_s \tag{1.56}$$

where

$$C(n,s,\lambda) = \frac{n|B^n|^{\lambda/n}}{(n-\lambda)rs} \left(\left(\frac{\lambda/n}{1-1/r} \right)^{\lambda/n} + \left(\frac{\lambda/n}{1-1/s} \right)^{\lambda/n} \right)$$

with $|B^n|$ *being the volume of the unit ball in* R^n*, and where*

$$\|f\|_r := \|f\|_{L^r(R^n)}.$$

Proof. (Adapted from Lieb and Loss's book [LL] with minor modifications.)

Without loss of generality, we may assume that both f and g are non-negative and $\|f\|_r = 1 = \|g\|_s$.

Let

$$\chi_G(x) = \begin{cases} 1 & \text{if } x \in G \\ 0 & \text{if } x \notin G \end{cases}$$

be the characteristic function of the set G. Then one can see obviously that

$$f(x) = \int_0^\infty \chi_{\{f>a\}}(x)\, da \tag{1.57}$$

$$g(x) = \int_0^\infty \chi_{\{g>b\}}(x)\, db \tag{1.58}$$

$$|x|^{-\lambda} = \lambda \int_0^\infty c^{-\lambda-1} \chi_{\{|x|<c\}}(x)\, dc \tag{1.59}$$

To see the last identity, one may first write

$$|x|^{-\lambda} = \int_0^\infty \chi_{\{|x|^{-\lambda} > \tilde{c}\}}(x)\, d\tilde{c},$$

and then let $\tilde{c} = c^{-\lambda}$.

Substituting (1.57), (1.58), and (1.59) into the left hand side of (1.56), we have

$$I := \int_{R^n}\int_{R^n} f(x)|x-y|^{-\lambda} g(y)\,dx\,dy =$$

$$\lambda \int_0^\infty\int_0^\infty\int_0^\infty\int_{R^n}\int_{R^n} c^{-\lambda-1}\chi_{\{f>a\}}(x)\chi_{\{|x|<c\}}(x-y)\chi_{\{g>b\}}(y)\,dx\,dy\,dc\,db\,da$$

$$= \lambda \int_0^\infty\int_0^\infty\int_0^\infty c^{-\lambda-1} I(a,b,c)\,dc\,db\,da. \tag{1.60}$$

Let

$$u(c) = |B^n|c^n,$$

the volume of the ball of radius c, and let

$$v(a) = \int_{R^n}\chi_{\{f>a\}}(x)\,dx\,,\quad w(b) = \int_{R^n}\chi_{\{g>b\}}(y)\,dy,$$

the measure of the sets $\{x \mid f(x) > a\}$ and $\{y \mid g(y) > b\}$, respectively. Then we can express the norms as

$$\|f\|_r^r = r\int_0^\infty a^{r-1}v(a)\,da = 1 \text{ and } \|g\|_s^s = s\int_0^\infty b^{s-1}w(b)\,db = 1. \tag{1.61}$$

It is easy to see that

$$\begin{aligned} I(a,b,c) &\le \int_{R^n}\int_{R^n}\chi_{\{|x|<c\}}(x-y)\chi_{\{g>b\}}(y)\,dx\,dy \\ &\le \int_{R^n} u(c)\chi_{\{g>b\}}(y)\,dy = u(c)w(b) \end{aligned}$$

Similarly, one can show that $I(a,b,c)$ is bounded above by other pairs and arrive at

$$I(a,b,c) \le \min\{u(c)w(b), u(c)v(a), v(a)w(b)\}. \tag{1.62}$$

We integrate with respect to c first. By (1.62), we have

$$\begin{aligned} &\int_0^\infty c^{-\lambda-1}I(a,b,c)\,dc \\ &\le \int_{u(c)\le v(a)} c^{-\lambda-1}w(b)u(c)\,dc + \int_{u(c)>v(a)} c^{-\lambda-1}w(b)v(a)\,dc \\ &= w(b)|B^n|\int_0^{(v(a)/|B^n|)^{1/n}} c^{-\lambda-1+n}\,dc + w(b)v(a)\int_{(v(a)/|B^n|)^{1/n}}^\infty c^{-\lambda-1}\,dc \\ &= \frac{|B^n|^{\lambda/n}}{n-\lambda}w(b)v(a)^{1-\lambda/n} + \frac{|B^n|^{\lambda/n}}{\lambda}w(b)v(a)^{1-\lambda/n} \\ &= \frac{n|B^n|^{\lambda/n}}{\lambda(n-\lambda)}w(b)v(a)^{1-\lambda/n} \end{aligned} \tag{1.63}$$

Exchanging $v(a)$ with $w(b)$ in the second line of (1.63), we also obtain

$$\begin{aligned}\int_0^\infty & c^{-\lambda-1} I(a,b,c)\,dc \\ &\le \int_{u(c)\le w(b)} c^{-\lambda-1} v(a)u(c)\,dc + \int_{u(c)>w(b))} c^{-\lambda-1} w(b)v(a)\,dc \\ &\le \frac{n|B^n|^{\lambda/n}}{\lambda(n-\lambda)} v(a)w(b)^{1-\lambda/n} \end{aligned} \tag{1.64}$$

In view of (1.61), we split the b-integral into two parts, one from 0 to $a^{r/s}$ and the other from $a^{r/s}$ to ∞. By virtue of (1.60), (1.63), and (1.64), we derive

$$I \le \frac{n}{n-\lambda}|B^n|^{\lambda/n} \times \left\{ \int_0^\infty v(a) \int_0^{a^{r/s}} w(b)^{\frac{n-\lambda}{n}} dbda + \int_0^\infty v(a)^{\frac{n-\lambda}{n}} \int_{a^{r/s}}^\infty w(b)dbda \right\} \tag{1.65}$$

To estimate the first integral in (1.65), we use Hölder inequality with $m = (s-1)(1-\lambda/n)$

$$\begin{aligned}\int_0^{a^{r/s}} & w(b)^{1-\lambda/n} b^m b^{-m}\,db \\ &\le \left(\int_0^{a^{r/s}} w(b) b^{s-1}\,db \right)^{1-\lambda/n} \left(\int_0^{a^{r/s}} b^{-mn/\lambda}\,db \right)^{\lambda/n} \\ &\le \left(\int_0^{a^{r/s}} w(b) b^{s-1}\,db \right)^{1-\lambda/n} \cdot \left(\frac{\lambda}{n - s(n-\lambda)} \right)^{\lambda/n} a^{r-1}; \end{aligned} \tag{1.66}$$

because $mn/\lambda < 1$ and

$$\frac{1}{r} + \frac{1}{s} + \frac{\lambda}{n} = 2.$$

It follows that the first integral in (1.65) is bounded above by

$$\begin{aligned}\left(\frac{\lambda}{n-s(n-\lambda)} \right)^{\lambda/n} \left(\int_0^\infty v(a) a^{r-1}\,da \right) \left(\int_0^\infty w(b) b^{s-1}\,db \right)^{1-\lambda/n} \\ = \frac{1}{rs} \left(\frac{\lambda/n}{1-1/r} \right)^{\lambda/n}. \end{aligned} \tag{1.67}$$

To estimate the second integral in (1.65), we first rewrite it as

$$\int_0^\infty w(b) \int_0^{b^{s/r}} v(a)^{1-\lambda/n}\,da\,db,$$

then an analogous computation shows that it is bounded above by

$$\frac{1}{rs}\left(\frac{\lambda/n}{1-1/s}\right)^{\lambda/n}. \tag{1.68}$$

Now the desired Hardy-Littlewood-Sobolev inequality follows directly from (1.65), (1.67), and (1.68). □

Theorem 1.6.4 *(An Equivalent Form of the Hardy-Littlewood-Sobolev inequality)*

Let $g \in L^{\frac{np}{n+\alpha p}}(R^n)$ for $\frac{n}{n-\alpha} < p < \infty$. Define

$$Tg(x) = \int_{R^n} |x-y|^{\alpha-n} g(y)dy.$$

Then

$$\|Tg\|_p \leq C(n,p,\alpha)\|g\|_{\frac{np}{n+\alpha p}}. \tag{1.69}$$

Proof. By the classical Hardy-Littlewood-Sobolev inequality, we have

$$< f, Tg > = < Tf, g > \leq C(n,s,\alpha)\, \|f\|_r \, \|g\|_s \, ,$$

where $< \cdot, \cdot >$ is the L^2 inner product.

Consequently,

$$\|Tg\|_p = \sup_{\|f\|_r = 1} < f, Tg > \leq C(n,s,\alpha)\|g\|_s,$$

where

$$\begin{cases} \frac{1}{p} + \frac{1}{r} = 1 \\ \frac{1}{r} + \frac{1}{s} = \frac{n+\alpha}{n}. \end{cases}$$

Solving for s, we arrive at

$$s = \frac{np}{n+\alpha p}.$$

This completes the proof of the Theorem. □

Remark 1.6.3 To see the relation between inequality (1.69) and the Sobolev inequality, let's rewrite it as

$$\|Tg\|_{\frac{nq}{n-\alpha q}} \leq C\|g\|_q \tag{1.70}$$

with

$$1 < q := \frac{np}{n+\alpha p} < \frac{n}{\alpha}.$$

Let $u = Tg$. Then one can show that (see [CLO]),

$$(-\triangle)^{\frac{\alpha}{2}} u = g.$$

Now inequality (1.70) becomes the Sobolev one

$$\|u\|_{\frac{nq}{n-\alpha q}} \leq C \|(-\triangle)^{\frac{\alpha}{2}} u\|_q.$$